DYNAMIC HEDGING PORTFOLIOS FOR DERIVATIVE SECURITIES IN THE PRESENCE OF LARGE TRANSACTION COSTS

by

Marco Avellaneda and Antonio Parás[1]

Abstract: We introduce a new class of strategies for hedging derivative securities in the presence of transaction costs assuming lognormal continuous-time prices for the underlying asset. We do not assume necessarily that the payoff is convex as in Leland [11] or that transaction costs are small compared to the price changes between portfolio adjustments, as in Hoggard, Whalley and Wilmott [8]. The type of hedging strategy to be used depends on the value of the *Leland number* $A = \sqrt{\frac{2}{\pi}}\,\frac{k}{\sigma\sqrt{\delta t}}$, where k is the round-trip transaction cost, σ is the volatility of the underlying asset, and δt is the time-lag between transactions. If $A < 1$ it is possible to implement modified Black-Scholes delta-hedging strategies, but not otherwise. We propose new hedging strategies that can be used with $A \geq 1$ to control effectively hedging risk and transaction costs. These strategies are associated with the solution of a *nonlinear obstacle problem* for a diffusion equation with volatility $\sigma_A = \sigma\sqrt{1+A}$. In these strategies, there are periods in which rehedging takes place after each interval δt and other periods in which a static strategy is required. The solution to the obstacle problem is simple to calculate, and closed-form solutions exist for many problems of practical interest.

[1] Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, N.Y., 10012.

According to Black-Scholes theory [3],[9], the value of an option is equal to the initial cost of a dynamic portfolio of traded securities that provides a similar cash-flow at expiration (i.e., it replicates the payoff). This theory explains, to within reasonable approximation, the prices of standard options and other derivative securities over moderate periods of time in markets in which transaction costs are negligible. Under these ideal conditions, the Black-Scholes theory also provides essentially riskless strategies for hedging derivative products.

If bid-ask spreads and other transaction costs are also taken into account, each adjustment of the portfolio implies an additional cost and the replicating property of the Black-Scholes hedge no longer holds. Making frequent adjustments to maintain the theoretical hedge can increase costs considerably. On the other hand, if only few adjustments are made, the Black-Scholes exact hedge cannot be maintained due to movements in the price of the underlying asset between trades. Therefore, transaction costs cannot be ignored without incurring in risk or loss. This is an important practical problem, especially in emerging markets where roundtrip transaction costs of 1% and higher are not uncommon.

In 1985, Leland introduced a theory for pricing a call option with transaction costs [11]. Using an elegant argument, he showed that the price of a call is given by the Black-Scholes formula with an *augmented volatility*

$$\sigma_A \;=\; \sigma\,\sqrt{1+A}\,,$$

where

$$A \;=\; \sqrt{\frac{2}{\pi}}\;\cdot\;\frac{k}{\sigma\,\sqrt{\delta t}}\,.$$

Here, σ represents the volatility of the underlying security, k is the round-trip transaction cost (a percentage) and δt is the time interval between successive adjustments of the portfolio. This time interval is considered fixed and is assumed to be much smaller than the time-to-expiration. We shall refer to the parameter A as the *Leland number*. For a given level of transaction costs, the size of the Leland number depends on the frequency of adjustments which, in turn, determines the overall risk of the strategy. A large Leland number corresponds to large transaction costs or to a small interval between rehedgings (small risk), and a small Leland number to small rehedging costs or to a large time interval between rehedgings (large risk). Boyle and Vorst derived a similar option pricing formula using a binomial model [4]. Both approaches are restricted to derivative securities with convex payoffs.

In practice, there are many derivative securities of interest which have non-convex payoffs. The foremost example is a *portfolio of standard options* combining short and long positions. Recently, Hoggard, Whalley and Wilmott [15] proposed an extension of Leland's result which applies to derivative securities with arbitrary payoff functions. Their solution involves solving a nonlinear partial differential equation with a Γ-dependent volatility

$$\sigma(\Gamma) \;=\sigma\,\sqrt{1+A\,sign(\Gamma)}\,.$$

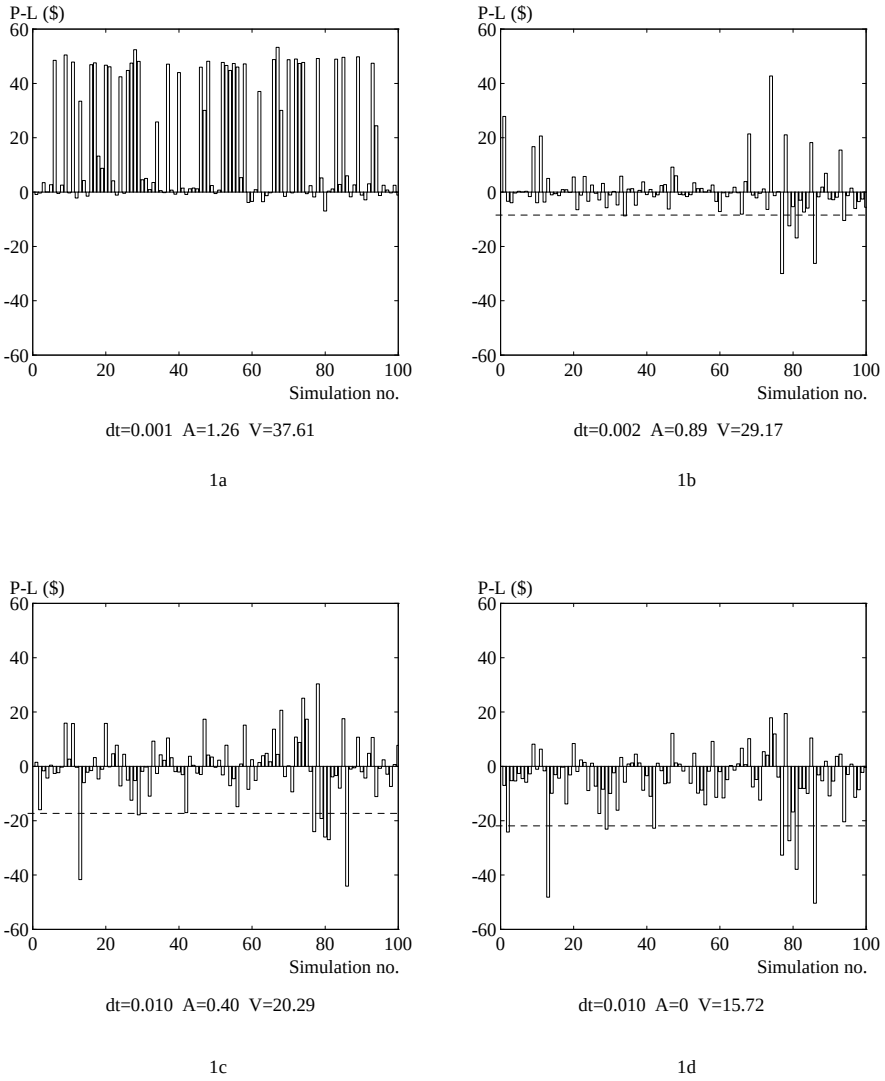


Figure 1. Results of 100 simulations for hedging the binary option. The bars indicate the total profit or loss incurred by the hedger each time. Figure 1a corresponds to using the hedging strategy "Combination of 2 & 3" (described in VI) obtained from the solution to the new obstacle problem with $A = 1.26$. Notice that this strategy shows insignificant losses. Figure 1b shows the result of using the two-volatility scheme with $A = 0.89$. The losses are more significant. The dotted line represents the break-even point if the difference between the costs of the first and the second strategies is added to the profit/loss. Notice that the number of bars crossing the dotted line and their magnitudes are significant. Figure 1c: same as 1b with $A = 0.40$. Figure 1d: same as 1b with $A = 0$ (Black-Scholes).

However, their approach is restricted to markets with relatively small transaction costs or otherwise requires fairly large intervals between adjustments. The reason for this is that the nonlinear equation that they propose is well-posed only for Leland numbers which are less than 1, i.e., when $\sigma^2(\Gamma)$ is positive. This limitation may not be suitable to avert risk in all circumstances.

In this paper, we develop a new class of valuation and hedging strategies that apply for $A \geq 1$. These schemes have the following novel features: i) they are based on dominating instead of replicating the final payoff, i.e., they deliver a final cash-flow which is *larger than or equal* to the contracted payoff; ii) the value-function is continuous but has jumps in the partial derivative with respect to the price of the underlying asset; and iii) even though δt is fixed, there may be long periods with no transactions, i.e., the adjustments to the portfolio take place after time intervals that are *multiples* of δt.

The new strategies are non-Markovian, i.e., *path-dependent*: their final cash-flow depends on the path described by the price of the underlying asset. Moreover, there can be several admissible strategies for a given payoff $f(S_T)$. These strategies are associated with the solution of a nonlinear *obstacle problem* for Black-Scholes equation with Leland's augmented volatility σ_A. The obstacle problem bears a strong similarity with the early-exercise optimal stopping problem for an American derivative security [6],[16]. However, unlike the latter, it can be solved in closed-form in many cases of interest.

To illustrate the theory, and to compare the effects of using different values of A, we carry out a detailed study of the hedging of a "cash-or-nothing" binary option [13] which is at-the-money near expiration. This is a well-known difficult problem for which standard delta-hedging using Black-Scholes yields poor results. The risk involved in hedging this security is so large, due to the large Γ at expiration, that the only reasonable rehedging schedule that will work in this case should be in the $A \geq 1$ regime. The bar graphs in Fig. 1 represent the profit/loss generated by hedging a binary option with Black-Scholes, Hoggard *et al.*, and the new strategy. To obtain these graphs we simulated 100 lognormal price histories and applied the different hedging methods to the binary option, assuming that these histories represented the paths of the price of the underlying asset. The superiority of the new solution over the ones with $A < 1$ is significant, even accounting for the differences in initial costs.

The solution of the obstacle problem has also interesting theoretical properties. It can be shown that it corresponds to the continuous-time limit of the optimal hedging strategies developed in the discrete binomial setting by Bensaid, Lesne, Pagès and Scheinkman [2]. This result will be demonstrated in a forthcoming companion paper [1].

I. Delta-hedging with transaction costs

We review, for the sake of completeness, the work of Leland [11] and Hoggard, Whalley and Wilmott [8], [15]. Consider a market in which a security is traded with a *bid-ask spread* $S_{t,ask} - S_{t,bid} = kS_t$. Here S_t represents the midpoint between the bid and the ask prices,

$$S_{t,bid} = S_t\,(1-(k/2)) \quad \text{and} \quad S_{t,ask} = S_t\,(1+(k/2))\,,$$

and k is a constant percentage. We assume that the price of the security satisfies

$$S_t = S_0\ e^{\sigma\,Z(t)+\mu\,t} \tag{1}$$

where $0 \le t \le T$ is time measured in years, $Z(t)$ is Brownian motion, σ is the annualized volatility and $\mu + \sigma^2/2$ is the expected annual rate of return.

We are interested in constructing hedging strategies to replicate European-style derivative securities (contingent claims with a payoff $f(S_T)$ depending only on the value of the underlying security at the expiration date T). The payoff functions that concern us are those which are piecewise linear, possibly having jump-discontinuities. They correspond to options, options spreads, binary options ("cash-or-nothing", "asset-or-nothing", etc.), and books of standard European options with same expiration date.

To fix ideas, we assume that an agent sells a derivative security with payoff $f(S_T)$ and immediately takes a position consisting of Δ_t shares of the security and of riskless bonds[2] with value B_t. Subsequently the portfolio is dynamically adjusted in a self-financing manner. Its value at time t is given by

$$V_t = \Delta_t\,S_t + B_t\,.$$

The goal is to maintain a portfolio that replicates the derivative's payoff, i.e., $V_T = f(S_T)$. *The time interval between successive rehedgings is assumed to be fixed and equal to δt.*

The change in the value of the portfolio from time t to $t+\delta t$ is

$$V_{t+\delta t} - V_t = \Delta_t\,(S_{t+\delta t} - S_t) + (B_t\,e^{r\delta t} - B_t) - (k/2)|\Delta_{t+\delta t} - \Delta_t|\,S_{t+\delta t}\,, \tag{2}$$

where r is the interest rate. The first term on the right-hand side represents the profit/loss due to the change in the value of the underlying security, the second is the interest paid or received from the bond, and the third is the *transaction cost of rehedging*, i.e. of changing amount of units of security from Δ_t to $\Delta_{t+\delta t}$.

As in the Black-Scholes theory of option pricing, the replicating portfolio is required to satisfy

$$V_t = V(S_t, t) \tag{3}$$

[2] We assume that lending and borrowing at the riskless rate does not involve significant transaction costs.

for all times, where $V(S,t)$ is a function of time and the spot price. We write the change in the value of the portfolio (2) in differential form

$$\delta V_t = \Delta_t \, \delta S_t + r \, B_t \, \delta t - (k/2) \, S_t \, |\delta \Delta_t| \, ,$$

up to an error of order $(\delta t)^{3/2}$. We also expand $V(S_t,t)$ using Ito's formula

$$\delta V(S_t,t) = \frac{\partial V(S_t,t)}{\partial S} \, \delta S_t \, + \, [\frac{\partial V(S_t,t)}{\partial t} \, + \frac{\sigma^2}{2} \, S_t^2 \, \frac{\partial^2 V(S_t,t)}{\partial S^2} \,] \, \delta t \, .$$

Equating the stochastic differentials (δS_t terms) and remaining terms ($\delta t \, , |\delta \Delta_t| \, $) in both expressions, we obtain

$$\Delta_t = \frac{\partial V(S_t,t)}{\partial S} \tag{4}$$

and

$$r \, B_t \, \delta t - \frac{k}{2} \, S_t \, |\delta \Delta_t| = [\frac{\partial V(S_t,t)}{\partial t} \, + \frac{\sigma^2}{2} \, S_t^2 \, \frac{\partial^2 V(S_t,t)}{\partial S^2} \,] \, \delta t \, . \tag{5}$$

Equation (4) shows that the amount of shares Δ_t depends only on S_t and t, but not on the past history of prices. Applying Ito's Lemma to $\Delta_t = \frac{\partial V(S_t,t)}{\partial S}$, we obtain, to leading order,

$$\delta \Delta_t = \frac{\partial^2 V(S_t,t)}{\partial S^2} \, \delta S_t \, + \text{ terms of order } \delta t \text{ or smaller.}$$

Therefore, the change in the value of the portfolio due to rehedging costs in (5) should be, to leading order,

$$\begin{aligned} \frac{1}{2} \, k \, S_t \, |\delta \Delta_t| \; &\approx \; \frac{1}{2} \, k \, S_t \, | \frac{\partial^2 V(S_t,t)}{\partial S^2} \, | \, |\delta S_t| \\ \\ &\approx \; \frac{1}{2} \, k \, \sigma \, S_t^2 | \frac{\partial^2 V(S_t,t)}{\partial S^2} \, | \, |\delta Z(t)| \\ \\ &= \frac{1}{2} (\frac{k}{\sigma \sqrt{\delta t}}) \, \sigma^2 \, S_t^2 \, | \frac{\partial^2 V(S_t,t)}{\partial S^2} \, | \, |\delta Z(t)| \, \sqrt{\delta t} \, . \end{aligned} \tag{6}$$

Since $\mathrm{E}\{ |\delta Z(t)| \, \} \; = \sqrt{\frac{2}{\pi}} \; \times \; \sqrt{\delta t}$, we conclude by the Law of Large Numbers that, to leading order,

$$\frac{1}{2} \sum_t \, k S_t \, |\delta \Delta_t| \; \approx \; \frac{1}{2} \sqrt{\frac{2}{\pi}} \; (\frac{k}{\sigma \sqrt{\delta t}}) \; \int_t^T \, | \frac{\partial^2 V(S_{t'},t')}{\partial S^2} \, | \, \sigma^2 \, S_t^2 \, dt' \, .$$

This justifies substituting in equation (6) the product $|\delta Z(t)| \sqrt{\delta t}$ by its expectation value $\mathrm{E}\{ |\delta Z(t)| \, \} \, \sqrt{\delta t} \; = \sqrt{\frac{2}{\pi}} \, \delta t$, for $\delta t \; \ll \; 1$. Therefore, equation (6) can be recast as

$$\frac{1}{2} \, k \, S_t \, |\delta \Delta_t| \; \approx \; \frac{\sigma^2 \, A}{2} \, S_t^2 \, | \frac{\partial^2 V(S_t,t)}{\partial S^2} \, | \, \delta t \, , \tag{7}$$

where

$$A = \sqrt{\frac{2}{\pi}}\,\left(\frac{k}{\sigma\sqrt{\delta t}}\right). \tag{8}$$

Substituting (7) into (5) and using the relation

$$\begin{aligned} B_t &= V_t - \Delta_t\, S_t \\ &= V(S_t,t) \;- \frac{\partial V(S_t,t)}{\partial S}\, S_t\,, \end{aligned}$$

we conclude that $V(S,t)$ should satisfy the equation:

$$\frac{\partial V(S,t)}{\partial t} + \frac{\tilde{\sigma}(\Gamma)^2}{2}\, S^2\, \frac{\partial^2 V(S,t)}{\partial S^2} + r\,\left(\, S\frac{\partial V(S,t)}{\partial S} - V(S,t)\right) \;= 0\,, \tag{9a}$$

for $0 < S < +\infty$, and $t < T$, where

$$\tilde{\sigma}^2(\Gamma) = \sigma^2\,(1 + A\,\mathrm{sign}(\Gamma)) \quad \text{and} \quad \Gamma = \frac{\partial^2 V(S,t)}{\partial S^2}\,, \tag{9b}$$

with final condition

$$V(S,T) = f(S)\,. \tag{10}$$

This formulation was first obtained by Leland [11] for pricing a call option ($f(S) = max(S-K,0)$) and later by Hoggard, Whalley and Wilmott [8] for general payoffs $f(S)$. Notice that equation (9a) is *nonlinear*, unless the function $V(S,t)$ is either convex or concave for all S, in which case (9a) reduces to the Black-Scholes equation with a modified volatility.

The Leland number, $A = \sqrt{\frac{2}{\pi}}\,\left(\frac{k}{\sigma\sqrt{\delta t}}\right)$, is a crucial quantity for measuring the influence of transaction costs. A large Leland number corresponds to either a large bid-ask spread or a small interval between trades. Conversely, a small value of the Leland number corresponds to a small bid-ask or to large intervals between transactions, with a larger exposure to risk. In this paper, we assume for simplicity that a time-interval between rehedgings is chosen, once and for all, in such a way that it corresponds to an acceptable level of exposure to hedge-slippage risk[3]. This in turn fixes the value of A.

We can summarize the theory of Leland and Hoggard *et al.* in

Proposition. *Fix an interval δt between rehedgings. Assume that the final-value problem (9)-(10) admits a solution which is twice continuously differentiable in S and once continuously differentiable in t.*[4] *Then, a dynamic portfolio consisting of $\frac{\partial V(S_t,t)}{\partial S}$ shares of the underlying security and riskless bonds with a present value of $V(S_t,t) - S_t\,\frac{\partial V(S_t,t)}{\partial S}$*

[3] More generally, to reduce this risk, an agent could choose to change the hedging schedule (reduce δt progressively) as the expiration date approaches, but we shall not discuss this here.

[4] These are common regularity requirements for the applicability of Ito's Lemma; cf.[12].

dollars replicates the payoff $f(S_T)$ at expiration, up to an approximation error of order $(T-t)\,\sigma\,\sqrt{\delta t}\,S_t$.[5]

II. Analysis of the nonlinear problem (9)-(10).

The dichotomy $A \geq 1 / A < 1$.

The above approach can be viewed as an extension of Black-Scholes theory. However, *the final-value problem (9)-(10) does not always admit a solution.* As we shall see, the hedging strategy suggested by the Proposition can be implemented only if $f(S)$ is convex or if the Leland number is sufficiently small.

From equation (9b), we distinguish two different cases. For $0 \leq A < 1$, the function $\tilde{\sigma}^2(\Gamma)$ takes two different *positive* values according to whether $V(S,t)$ is convex ($\Gamma \geq 0$) or concave ($\Gamma < 0$). On the other hand, if $A \geq 1$, $\tilde{\sigma}^2(\Gamma)$ vanishes or becomes negative for $\Gamma \leq 0$.

The case $0 \leq A < 1$. In this regime $\tilde{\sigma}^2(\Gamma)$ is positive. It is well-known that the final-value problem (9)-(10) admits a unique solution $V(S,t)$ for any given final payoff function $f(S)$. Moreover, the solution is twice continuously differentiable in S and once continuously differentiable in t for $t < T$ [10]. Thus, the assumptions of the Proposition are satisfied and a replicating strategy exists for contingent claims with arbitrary payoffs. Notice that $V(S,t)$ is an increasing function of the Leland number A, as one might expect[6]: trading more frequently reduces risk but increases the transaction costs of the hedging strategy.

For some risky derivative securities, strategies with intervals between transactions δt that correspond to Leland numbers $A < 1$ may have too much exposure to hedge-slippage. In terms of k, the condition $A < 1$ is

$$k < (\sqrt{\pi/2})\sigma\,\sqrt{\delta t} \approx 1.25\,\sigma\,\sqrt{\delta t}\,,$$

which means that the round-trip transaction cost should not exceed the standard deviation of the price movements for a single period. This restriction is satisfied only if the bid-ask

[5] This is similar to the way in which the Black-Scholes formula works–it provides a strategy for replication which is "asymptotically" riskless, in the sense that the risk due to hedge-slippage diminishes with the size of the rehedging interval and can be made, at least in theory, arbitrarily small.

[6] This follows from the Maximum Principle satisfied by equation (9), for $A < 1$ [10].

spread is small or if δt is large[7].

The case $A \geq 1$. For convex payoff functions (put, call, etc.), $V(S,t)$ reduces to Leland's pricing formula. Notice that the modified volatility $\sigma\sqrt{1+A}$ is always positive and hence Leland's formula applies for arbitrary values of A. The situation is quite different for non-convex payoff functions. Since $\tilde{\sigma}^2(\Gamma)$ is negative or vanishes for $\Gamma \leq 0$, the final-value problem would be *mathematically ill-posed*. The evolution ("rolling back") of a non-convex datum $f(S)$ under equation (9a) leads to exponentially large modes. Accordingly, the function $V(S,t)$ develops huge oscillations or blows up for t arbitrarily close to T. Thus, *the nonlinear final-value problem (9)-(10) has no solution for generic, non-convex payoff functions* $f(S)$.

Let us analyze the dichotomy in the value of A from the point of view of an agent hedging a *short position* in the derivative security. It is important to emphasize that in this paper the function $V(S,t)$ represents the *cost of hedging* for the agent and therefore that the dollar-value of being short the derivative security is $-V(S,t)$. Hence, we make the somewhat unusual convention that a positive V represents a net short position and a negative V a net long position. For simplicity, all asset prices are taken in dollars-at-expiration, which is mathematically equivalent to assuming that the interest rate is $r = 0$.

Consider first *the classical Black-Scholes setting*. If $\Gamma > 0$ (i.e. if the agent is "short Gamma"[8]) readjusting the portfolio is necessary to avert the risk from a large movement in the stock price. Rehedging is done despite the fact that the value of the portfolio drops, since we have

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} < 0.$$

If $\Gamma < 0$ (the agent is "long Gamma") rehedging is done for a different reason; large jumps in the price of the underlying asset work in the agent's favor. But in this case, the value of the replicating portfolio *increases* with time, since

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} > 0.$$

Thus, an agent who chooses not to rehedge will not be able to claim this accrued value.

To account for transaction costs, the Black-Scholes volatility must be replaced by $\tilde{\sigma}^2(\Gamma)$. In the case where $\Gamma > 0$, the motivation for readjusting the portfolio is still risk-aversion. Even though the agent must pay the additional transaction cost, he is forced to rebalance

[7] Consider, for instance, an at-the-money binary option such that the volatility of the underlying asset is $\sigma = 0.2$ yr.$^{-1/2}$ and the bid-ask spread is $k = 0.01$. Infrequent trading can be very risky because of hedge- slippage. Hedging with $\delta t = 0.001$, which is equivalent to monitoring the position between 2 and 3 times a day, has $A \approx 1.3$. Hedging with $\delta t = 0.002$ corresponds to monitoring between once and twice a day, has $A \approx 0.9$. Finally, $\delta t = 0.01$, which corresponds to an interval of about 4 days between trades, gives $A \approx 0.4$. The first schedule correspond to a "large" Leland number and the last two to "moderate" values of A. In practice, the large value of Γ at expiration demands the use of the first schedule.

[8] The sign is consistent with the above convention.

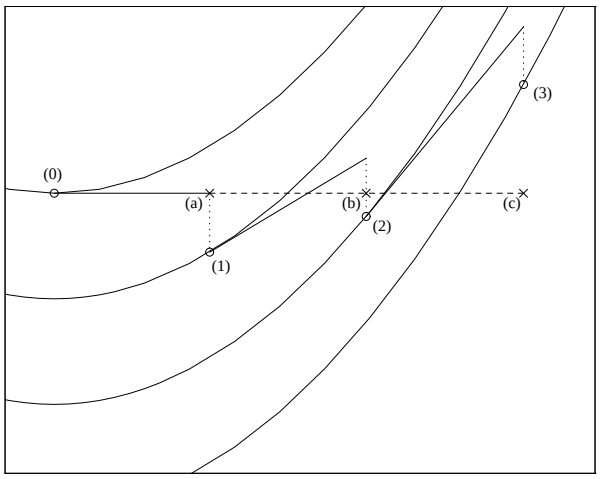


Figure 2. Delta-hedging with a convex value-function. The height of the point labeled (0) represents the initial value of the position. After an interval of duration δt, the value is represented by the height of (1) if rehedging takes place and by the height of (a) otherwise. After the second interval of duration δt, the value of the position is the height of (2) after rehedging and by the height of (b) otherwise. Similarly, points (3) and (c) represent the results of delta-hedging versus continuing not to transact. Due to the convexity of the value function, the value of the portfolio falls below the theoretical value by not transacting. (The figure shows a situation in which the price shows a tendency to increase steadily, which is when the risk of hedge-slippage is the greatest).

the portfolio as a protection against large jumps (see Fig. 2). In contrast, if $\Gamma < 0$ there is a notable difference between the cases $A < 1$ and $A \geq 1$. If $A < 1$ (see Fig. 3) the agent will clearly choose to rehedge, even after taking into account transaction costs, because the net value of the portfolio increases:

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2(1-A)S^2\frac{\partial^2 V}{\partial S^2} > 0.$$

(Thus he can claim the accrued value.) On the other hand, if $A \geq 1$ (see figure 4), the net value of the portfolio cannot be increased by rehedging, since the inequality

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2(1-A)S^2\frac{\partial^2 V}{\partial S^2} \leq 0$$

shows that the gains in the value of the portfolio are offset by the transaction cost. This means that *there is no economic incentive to readjust the position if* $\Gamma < 0$ *and* $A \geq 1$. Since there is no risk derived from using a static hedge, the agent should not adjust his or her position.

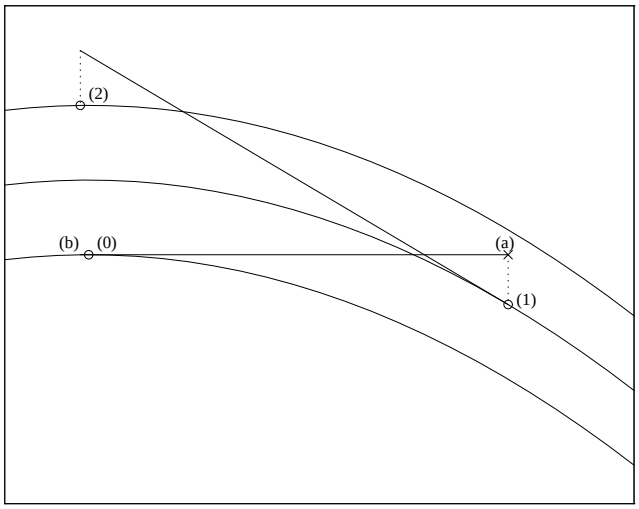


Figure 3. Delta-hedging with a concave value-function, with $A < 1$. The curves corresponding to different time-sections of the value function increase with time. The greatest risk is when then the price goes up and then goes down. If delta-hedging is used, the position goes from (0) to (1) to (2). With a static hedge, the value of the portfolio goes from (0) to (a) to (b), and hence falls below the theoretical value.

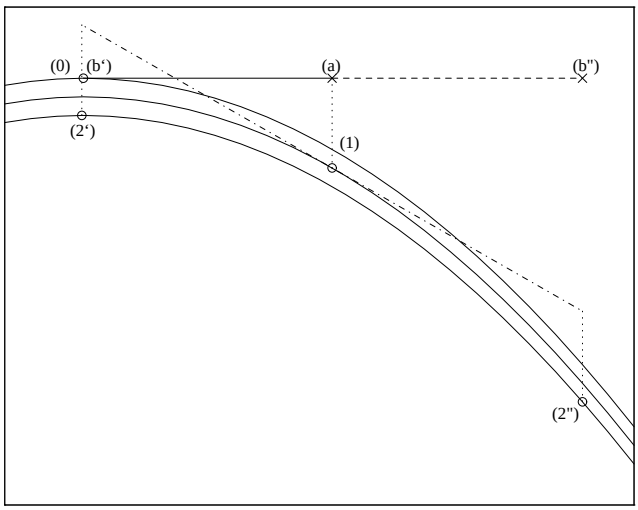


Figure 4. Delta-hedging with a concave function, with $A > 1$. The different curves now decrease with time. Delta-hedging is described by the paths $(0)-(1)-(2')$ or $(0)-(1)-(2'')$. In contrast, the corresponding paths for a static hedge are $(0)-(a)-b')$ or $(0)-(a)-(b'')$. It is therefore better not to transact.

III. The obstacle problem

Based on the previous considerations, we propose a new class of value functions and hedging strategies with $A \geq 1$. The strategies should have non-negative associated Gammas. Moreover, when $\Gamma > 0$, the value function should satisfy Leland's equation

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + r \left(S \frac{\partial V(S,t)}{\partial S} - V(S,t) \right) = 0 \tag{11}$$

with

$$\sigma_A \equiv \sigma \sqrt{1+A}$$

to avert risk. This suggests that an admissible value function $V(S,t)$ should be constructed by *"piecing together" the graphs of convex solutions of equation (11).* Before making this idea precise, several heuristic remarks are in order.

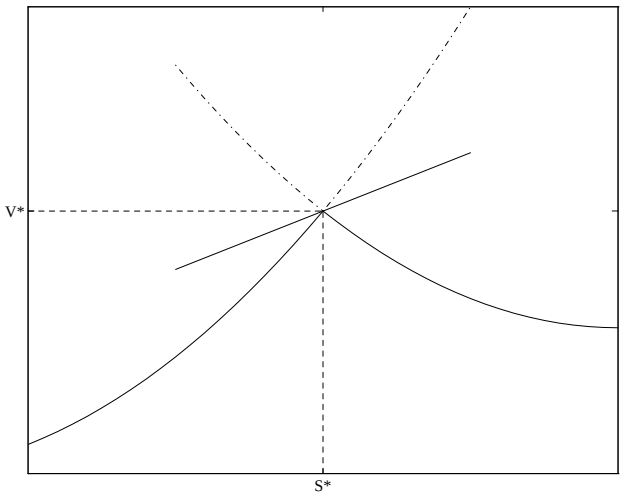


Figure 5. Schematic rendering of a function $V(S,t)$ with a "locally concave" kink. The straight line segment through the kink lies above the graph of the function. Therefore, if the price of the underlying is S^*, the initial value of a portfolio is V^*, and the number of shares is equal to the slope of of the line segment, then a static strategy yields a value of the portfolio greater than $V(S_t,t)$, if the price remains in a neighborhood of S^*.

The first remark concerns the *smoothness* of the value function. When piecing together solutions of (11), we obtain a function which will have "kinks", or jumps in $\frac{\partial V(S,t)}{\partial S}$ along curves in the (S,t)-plane where the different graphs join. This introduces a difficulty, since the hedge-ratio $\Delta_t = \frac{\partial V(S_t,t)}{\partial S}$ is undefined along these curves. Maintaining Δ_t equal to $\frac{\partial V(S_t,t)}{\partial S}$ is risky because the price may oscillate back and forth near the location of a kink.

As shown in ([9], p. 297), adjustments about the kink can lead to losses which cannot be recuperated or predicted ahead of time[9]. This risk is unavoidable if the value function is *convex* at the kink, i.e., if

$$\frac{\partial V(S-0,t)}{\partial S} \;<\; \frac{\partial V(S+0,t)}{\partial S}\,, \tag{12}$$

because it is necessary to transact each time the price crosses a (small) neighborhood around the critical point to prevent the value of the portfolio from dropping below $V(S_t,\, t)$. On the other hand, if the value function is *concave* at the kink, i.e.

$$\frac{\partial V(S-0,t)}{\partial S} \;>\; \frac{\partial V(S+0,t)}{\partial S}\,, \tag{13}$$

there is no need to transact immediately after the price hits the critical value. Instead, a static strategy can be adopted by choosing Δ in such a way that the value of the portfolio varies on a line passing through the kink and lying *above* the value function in an interval around the critical value (see Fig. 5). As long as the price S_t remains in such interval, there is no risk and hence no further transactions are needed[10]. Since we are seeking riskless strategies, we can allow value-functions which satisfy (13) along a kink, but not (12).

The second important remark is related to the *decay in time* of the value function. Since this function is almost everywhere convex, we have, in dollars-at-expiration,

$$\frac{\partial V}{\partial t} \;=\; -\,\frac{1}{2}\,\sigma_A^2\,S^2\,\frac{\partial^2 V}{\partial S^2} \;\leq\; 0\,. \tag{14}$$

Since the value function must be equal to the intrinsic value of the derivative security at expiration

$$V(S,T) \;=\; f(S)\,, \tag{15}$$

we must have

$$V(S,t) \;\geq\; f(S)$$

for all (S,t). If we take into account the time-value of money this last inequality becomes

$$V(S,t) \;\geq\; e^{-r(T-t)}\; f(\, S\, e^{r(T-t)}\,)\,. \tag{16}$$

Therefore, admissible value functions should be piecewise convex solutions of equation (11) such that inequality (13) holds along any curve where $\frac{\partial V}{\partial S}$ is discontinuous. Moreover, they should satisfy the final-value condition (15) for $t = T$ as well as inequality (16) for all values of S and t.

[9] A simple example that illustrates the nature of this risk is the "stop-loss" strategy for hedging a call option described in Hull [9]. In this strategy, the delta is kept at values of either 0 or 1 according to whether $S_t < K$ or $S_t > K$, where K is the strike price. This is like delta-hedging with a "value function" equal to the nominal value of the option, $max(\,S-K,0)$.

[10] Such holding strategies are explained in detail in Sections IV and V.

It is not hard to show that there are many functions satisfying these properties for a given payoff $f(S)$. It is natural to choose among these functions the one which is *minimal*, since it would correspond (at least within this class) to the least expensive initial portfolio leading to a riskless dynamical hedge. This minimal function corresponds mathematically to the solution of an *obstacle problem*[11].

Definition. *Given a piecewise linear payoff function $f(S)$, we say that the function $V(S,t)$ is the* solution of the obstacle problem *if*

$$\text{(i)} \quad V(S,t) \geq e^{-r(T-t)}\, f(S\, e^{r(T-t)}) \quad \textit{and}$$

$$\text{(ii)} \quad \frac{\partial V(S,t)}{\partial t} + \frac{\sigma_A^2}{2}\, S^2\, \frac{\partial^2 V(S,t)}{\partial S^2} + r\,\Big(\, S\frac{\partial V(S,t)}{\partial S} - V(S,t)\,\Big) \leq 0$$

for almost all S and all $t < T$ (in the sense of distributions; see Friedman [7]),

$$\text{(iii)} \quad \frac{\partial V(S,t)}{\partial t} + \frac{\sigma_A^2}{2}\, S^2\, \frac{\partial^2 V(S,t)}{\partial S^2} + r\,\Big(\, S\frac{\partial V(S,t)}{\partial S} - V(S,t)\,\Big) = 0$$

on the set of points (S,t) for which $V(S,t) > e^{-r(T-t)}\, f(S\, e^{r(T-t)})$; and

$$\text{(iv)} \quad V(S,T) = f(S)\,. \tag{17}$$

It is well-known that a solution to this problem exists and is unique [7]. For the special class of payoffs $f(S)$ considered here, the function $V(S,t)$ is continuous everywhere and differentiable except along a finite number of curves in the (S,t)-plane. Outside such curves, which correspond to the boundary of the set $\{(S,t) : V(S,t) = e^{-r(T-t)}\, f(S\, e^{r(T-t)})\}$, the function is either convex or linear ($\Gamma \geq 0$) and satisfies the PDE in (11). Furthermore, condition (13) holds whenever $\frac{\partial V}{\partial S}$ has a jump; this is a consequence of inequality ii) in (17). Therefore, the function $V(S,t)$ satisfies all the requirements outlined above. Finally, it can be shown that $V(S,t)$ is the function with minimum value satisfying these requirements[12].

In practice, it is convenient to solve problem (17) with $r = 0$, i.e. in dollars-at-expiration first and then obtain the general solution by discounting at the riskless rate[13]. Thus, if $V_0(S,t)$ represents the solution with $r = 0$, then $V(S,t) = e^{-r(T-t)}\, V_0(S\, e^{r(T-t)}, t)$. In the formulation with $r = 0$, the obstacle problem becomes

[11] In the sense of nonlinear partial differential equations; see for instance Friedman [7].

[12] Mathematically, the solution of the obstacle problem is the minimal supersolution of equation (11) satisfying (16); cf. [7].

[13] This reduction is valid only for *European-style* contingent claims.

$$\text{(i)} \quad V(S,t) \geq f(S) \quad \text{and}$$

$$\text{(ii)} \quad \frac{\partial V(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 V(S,t)}{\partial S^2} \leq 0$$

for all S and all $t < T$;

$$\text{(iii)} \quad \frac{\partial V(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 V(S,t)}{\partial S^2} = 0$$

on the set of points (S,t) for which $V(S,t) > f(S)$; and

$$\text{(iv)} \quad V(S,T) = f(S)\,. \tag{18}$$

If $f(S)$ is convex, the solution of the linear final-value problem for equation (11) is convex for all t and greater than $f(S)$. Hence, it also solves the obstacle problem. Thus, the solution of (17) coincides with the one of Leland and Hoggard, Whalley and Wilmott if $f(S)$ is convex.

Before describing the solution of the obstacle problem for general payoffs, it is instructive to solve a simple example for which there exists a closed-form solution. This example also serves to illustrate the new path-dependent hedging strategies associated with the obstacle problem.

IV. Example: a binary option

.

Consider the payoff function corresponding to a "cash-or-nothing" binary option,

$$f(S) = \begin{cases} 0 & \text{if } 0 < S < K, \\ \\ H & \text{if } K \leq S < +\infty\,. \end{cases} \tag{19}$$

Here, K is a strike price and H represents a cash amount payable to the holder of the option if $S_T \geq K$. The obstacle problem (18) for this payoff can be solved analytically: the solution is given by

$$V(S,t) = \begin{cases} V_1(S,t)\,, & \text{if } 0 < S < K, \\ \\ H & \text{if } K \leq S < +\infty\,, \end{cases}$$

where $V_1(S,t)$ solves the boundary-value problem

$$\begin{cases} \frac{\partial V_1(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 V_1(S,t)}{\partial S^2} = 0 \quad \text{for } 0 < S < K \text{ and } t < T, \\ V_1(0,t) = 0\,, \qquad V_1(K,t) = H\,, \\ V_1(S,T) = 0 \quad \text{for all} \quad 0 < S < H. \end{cases}$$

To see this, it suffices to notice that $V(S,t)$ is a solution of the linear PDE in (18-(iii)) outside the boundary of the set where $V = f$, which is $\{(S,t) : S = K\}$, and that the "kink" at $S = K$ is concave (i.e., (13) holds). Since $V(S,t)$ lies above $f(S)$ for all S and is equal to f at time T, the four conditions in (18) are satisfied. Notice that $V_1(S,t)$ is convex in the interval $(0,K)$.

To obtain the solution analytically, we observe that $V_1(S,t)$ is equal to H times the probability that a geometric Brownian motion with volatility σ_A and drift $-(1/2)\sigma_A^2$, starting at S at time t will cross the barrier $S = K$ before time T. This function can be expressed in closed-form. After bringing to present value, we obtain

$$V(S,t) = H\, e^{-r(T-t)} \left[(S/K\, e^{-r(T-t)})\, N(Z_1) + N(Z_2) \right],$$

for $S < K\, e^{-r(T-t)}$, where $N(\cdot)$ is the error function, and

$$V(S,t) = H\, e^{-r(T-t)} \tag{20}$$

for $S \geq K\, e^{-r(T-t)}$. The values of Z_1 and Z_2 are

$$Z_1 = \frac{\ln(S/K\, e^{-r(T-t)})}{\sigma_A \sqrt{T-t}} + \frac{\sigma_A \sqrt{T-t}}{2}\,,$$

and

$$Z_2 = Z_1 - \sigma_A \sqrt{T-t}\,.$$

We pursue this example further and describe several riskless hedging strategies that require an initial investment of $V(S_t,t)$ (plus a small initial transaction cost of order $A\,\sigma\,\sqrt{\delta t}$). Suppose that, initially, $S_t < K\, e^{-r(T-t)}$. In this case, the writer sells the option at the price $V(S_t,t)$ and immediately purchases $\Delta_t = \frac{\partial V(S_t,t)}{\partial S}$ shares of the security, which he pays for with the premium and by borrowing $S_t \frac{\partial V(S_t,t)}{\partial S} - V(S_t,t)$ at the riskless rate. (The initial transaction cost of order $k\,\Delta_t\,S_t \approx \sigma\,A\,\Delta_t\,S_t\,\sqrt{\delta t}$ is neglected here. In practice, it should be incorporated into the premium.)

Strategy 1: "Cash-if-hit". As the price of the security changes, the delta of the portfolio is adjusted after each period δt so as to match $\frac{\partial V(S_{t'},t')}{\partial S}$. This is done at all times $t' > t$ *prior to the first hitting time* τ *of curve* $\{(S,t) : S = K\, e^{-r(T-t)}\}$. Since $V_1(S,t)$ is

convex in the variable S in the interval $(0, K)$, it follows that $V(S,t)$ is convex in the interval $(0, K\,e^{-r(T-t)})$. Therefore, according Section I, the hedging portfolio is worth approximately $V(S_{t'}, t')$ at times $t' \leq \tau$, taking into account transaction costs. If $\tau \geq T$ then we will have perfect replication of the payoff[14] at time T, because then $V_1(S_T, T) = f(S_T) = 0$. On the other hand, if $\tau < T$, the value of the portfolio at time τ is $V(S_\tau, \tau) = H\,e^{-r(T-\tau)}$, which is the present value of the cash settlement of the option. All shares of the underlying security can therefore be sold for $H\,e^{-r(T-\tau)}$ minus a transaction cost. This cost of order $\sqrt{\delta t}$ is negligible if Δ_τ is not too large. Lending $H\,e^{-r(T-t)}$ at the riskless rate gives a strategy which replicates the option payoff or even results in a profit, depending on whether the value of the underlying security finishes above or below the strike price K.

Strategy 2: " Dominating portfolio-if-hit". Closing out the position in securities and lending V_τ is not the unique strategy after the hitting time τ. Another possible solution would be to maintain a static hedge with a Δ between 0 and H/K. For instance, if after time τ one takes a position with $\Delta = H/K$, the value of the portfolio at time $t' > \tau$ will be

$$V_{t'} = \Delta \cdot S_{t'}\,,$$

neglecting the small transaction cost of taking this position. This strategy results in a final value $(H/K)\,S_T$, which again dominates the payoff of the binary option. Any static strategy for $t' > \tau$ with $0 < \Delta < H/K$ yields a dominating portfolio with final values greater than the option payoff. As in the case of the "cash-if-hit" strategy, if the stock price never hits the discounted strike price, then the final zero-payoff is approximately replicated. If $\tau < T$, the final value of the position will be $V_T = \Delta\,(S_T - K) + H$, which is greater than the option payoff.

Strategy 3: "On-and-off delta-hedging". Many other strategies are available at the initial cost $V(S_t, t)$. Here we present strategies that have the advantage of having small transaction costs at the kink. As before, delta-hedging is used up to the hitting time τ. Then the Delta of the portfolio is changed to *any* $0 \leq \Delta \leq \Delta_\tau = \frac{\partial V(K\,e^{-r(T-\tau)}, \tau)}{\partial S}$. In particular, the agent can choose not to adjust the position at time τ. Define $S^*_{t'}$ to be the solution of the equation

$$\frac{\partial V(S^*_{t'}, t')}{\partial S} = \Delta$$

for all $t' > \tau$. Due to the convexity of $V(S,t)$, this defines a unique value of $S^*_{t'}$. The trading strategy consists in using $\Delta_{t'} = \Delta$, for subsequent times up to the first time, τ_1, when $S_{\tau_1} = S^*_{\tau_1}$. Since the kink is concave, the portfolio has value *greater* than $V(S_{\tau_1}, \tau_1)$ at this time. After time τ_1 the agent goes back to delta-hedging, i.e.,

$$\Delta_{t'} = \frac{\partial V(S_{t'}, t')}{\partial S}\,,$$

until the next time, τ_2, when $S_{\tau_2} = K\,e^{-r(T-\tau_2)}$, and so forth. A key feature of this strategy is that at each time that delta-hedging is "switched on" the agent has a positive cash-flow

[14] Up to an error proportional to $\sqrt{\delta t}$, as usual.

$V_{\tau_1} - V(S_{\tau_1}, \tau_1)$, since he needs only $V(S_{\tau_1}, \tau_1)$ dollars to hedge the derivative from that moment on. Therefore, he will end up with a portfolio with a final value larger than or equal to the payoff, neglecting the small transaction costs at times when $S_{t'} = K\, e^{-r(T-t')}$ (i.e., when delta-hedging is temporarily "switched off") and the initial cost. It is easy to see that the number of switchings is always finite with probability one, *independently of the magnitude of* δt, and hence contribute only an amount of order $\sqrt{\delta t}$. This on-and-off hedging strategy can be particularly useful near expiration in cases where Gamma and Delta are very large.

Finally, we note that if the option is initially *in-the-money*, i.e. $S_t \geq K\, e^{-r(T-t)}$, then according to the obstacle solution (20), the writer must charge a premium of at least $H\, e^{-r(T-t)}$, which is the present value of the cash payment of the option[15].

V. Solution of the obstacle problem for general payoffs.

In this section presents a "constructive" approach for solving the obstacle problem (18) for arbitrary piecewise linear payoffs. This solution also yields an algorithm that can be easily implemented on the computer. The solution reduces essentially to solving several boundary-value problems for the *linear* Black-Scholes-Leland equation (11) on adjacent intervals on the line, keeping track of certain "compatibility conditions" on the boundary points[16].

Consider a piecewise linear payoff $f(S)$, having finitely many discontinuities in $f(S)$ and $f'(S)$. Let us denote the points of discontinuity of the function or of its first derivative by $S_1\,, S_2\,, ..., S_{N-1}\,$. We also set $S_0 = 0$ and $S_N = \infty$. Since $f(S)$ is piecewise linear, we can find a subset of these points $\tilde{S}_k$, $k = 0, 1, ...M$ (labeled in increasing order), such that $\tilde{S}_0 = S_0\,$, $\tilde{S}_M = S_N$, and which is the smallest subset such that $f(S)$ restricted to $(\tilde{S}_k, \tilde{S}_{k+1})$ is convex. The points $\tilde{S}_k$ correspond to values of S where either $f(S)$ is discontinuous or where $f'(S)$ is discontinuous and $f'(S-0) > f'(S+0)$.

Consider the M linear boundary-value problems

[15] At first, this price for an in-the-money binary option appears to be incorrect, since it offers no advantage over a zero-coupon bond with face value H. However, it can be shown [1] that $V(S_t, t)$ is the *minimum price* that can be charged if the writer intends to use a dynamical hedge with $A \geq 1$. This simply means that offering this option in the presence of large transaction costs is inherently risky.

[16] Problem (17) can be viewed as an "optimal stopping" problem, similar to the valuation of American contingent claims. However, the solution is simplified by the fact that we can use the formulation (18) with $r = 0$.

$$\begin{cases} \frac{\partial \tilde{V}_k(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 \tilde{V}_k(S,t)}{\partial S^2} = 0\,, \\ \\ \tilde{V}_k(S,T) = f(S)\,, \\ \\ \tilde{V}_k(\tilde{S}_{k-1},t) = f(\tilde{S}_{k-1}) \qquad \tilde{V}_k(\tilde{S}_k,t) = f(\tilde{S}_k)\,. \end{cases} \tag{21}$$

At the expiration time T (and thus immediately before) the solutions of these problems satisfy

$$\frac{\partial \tilde{V}_k(\tilde{S}_k - 0, t)}{\partial S} > \frac{\partial \tilde{V}_{k+1}(\tilde{S}_k + 0, t)}{\partial S}$$

for all k because the kink of f at $\tilde{S}_k$ is concave. Let T^* be the time nearest to expiration for which

$$\frac{\partial \tilde{V}_k(\tilde{S}_k - 0, T^*)}{\partial S} = \frac{\partial \tilde{V}_{k+1}(\tilde{S}_k + 0, T^*)}{\partial S} \tag{22}$$

for some k such that $1 \leq k \leq M-1$. If no such T^* exists, then the solution of the obstacle problem is given by

$$V(S,t) = \tilde{V}_k(S,t) \quad \text{for all } \tilde{S}_{k-1} < S < \tilde{S}_k\,, \tag{23}$$

i.e. by solving M *independent* boundary-value problems[17]. If T^* is finite, then the solution of the obstacle problem is given by (23) for all times t such that $T^* \leq t \leq T$.

Assume that T^* is finite and that (22) holds at $1 \leq k_0 \leq M-1$. Observe that $V(S,T^*)$ is now convex in the *larger interval* $(\tilde{S}_{k_0-1}\,,\, \tilde{S}_{k_0+1}\,)$ since the "kink" at $\tilde{S}_{k_0}$ has been smoothed out. To "roll-back" the solution further for times prior to T^*, continue solving the $M-2$ boundary value problems (21) with $k \neq k_0$ and k_0+1. This determines the value function outside the interval $(\tilde{S}_{k_0-1}\,,\, \tilde{S}_{k_0+1})$. The function inside the interval at times prior to T^*, is found by solving the boundary-value problem

$$\begin{cases} \frac{\partial \tilde{V}(S,t)}{\partial t} + \frac{\sigma_A^2}{2} S^2 \frac{\partial^2 \tilde{V}(S,t)}{\partial S^2} = 0\,, \\ \\ \tilde{V}(S,\, T^*) = V(S,\, T^*)\,, \\ \\ \tilde{V}(\tilde{S}_{k_0-1},t) = f(\tilde{S}_{k_0-1}) \qquad \tilde{V}(\tilde{S}_{k_0+1},t) = f(\tilde{S}_{k_0+1})\,. \end{cases}$$

This procedure gives the value of $V(S,t)$ for all times t such that $T^{**} \leq t \leq T^*$, where T^{**} is the time at which one more kink is eliminated. The solution of problem (18) can be obtained for all times $-\infty < t \leq T$ by applying this algorithm recursively. Finally, the solution for $r \neq 0$ is obtained by taking present values.

[17] This is the case of the binary option treated in Section IV.

It is easy to deduce from this algorithm that, as $t \to -\infty$, the solution of the obstacle problem (18) approaches the *concave envelope* of the payoff function $f(S)$. This result has a simple financial interpretation. Neglecting the time-value of money ($r = 0$), the concave envelope corresponds to the value of the *least costly riskless static strategy* for hedging a short position in the derivative security. This is consistent with the intuitive notion that the benefit of implementing a dynamical hedge (with large transaction costs) versus a static one (without transaction costs) diminishes as the time-to-expiration increases.

Next, we discuss possible hedging strategies based on the obstacle problem. Here, it is more realistic to describe dynamical hedging in the context of the solution of problem (17), obtained from (18) by taking present values. It follows from the above construction that the S-derivative of the value function $V(S,t)$ in (17) is defined everywhere except along a set of curves in the (S,t)-plane, say, $S_1(t') < S_2(t') < S_3(t'), ... < S_j(t') < S_{j+1}(t') < ... < S_{M-1}(t')$, where

$$S_j(t') \;=\; \tilde{S}_j \, e^{-r(T-t')} \, .$$

These curves consist of points where equality in (17-(i)) and strict inequality in (17-(ii)) holds. For each j, $S_j(t')$ is a curve defined for $T_j \;\leq\; T - t' \;\leq\; T$, where T_j represents the first time at which $V(S,t')$ makes contact with the "obstacle" $e^{-r(T-t')} \, f(\, e^{r(T-t')} \, S \,)$. The value function is smooth in each of the adjacent intervals $(0, S_1(t'))$, $(S_1(t'),\, S_2(t'))$, ..., $(S_{M-1}(t'),\, \infty)$.

It is important to notice that

$$\frac{\partial V(S_j(t') - 0, t')}{\partial S} \;> \frac{\partial V(S_j(t') + 0, t')}{\partial S} \tag{24}$$

for all j. Moreover, the left partial derivative in (24) *increases* with t' and the right partial derivative in (24) *decreases* with t'. In particular, we have

$$\frac{\partial V(S_j(t') - 0, t')}{\partial S} \;> \frac{\partial V(S_j(T_j), T_j)}{\partial S} \;> \frac{\partial V(S_j(t') + 0, t')}{\partial S} \, , \tag{25}$$

for all $T_j \;< t' \;\leq T$. Denote the slope at time T_j by

$$\Delta_j \;\equiv\; \frac{\partial V(S_j(T_j), T_j)}{\partial S} \, . \tag{26}$$

Hedging is done as follows: initially, the agent will take a position consisting in $\Delta_t \;= \frac{\partial V(S_t \, , t)}{\partial S}$ shares and $V(S_t \, , t) \;- \Delta_t \, S_t$ in riskless bonds. We assume, without loss of generality, that the spot price S_t is not on one of the curves $S_j(t')$. Subsequently, the agent delta-hedges until the first hitting time τ_1 of a curve where $V(s,\, t)$ is not differentiable, i.e., when

$$S_{\tau_1} \;=\; S_j(\tau_1)$$

for some j. At this time, the Delta is set to

$$\Delta_{\tau_1} \;=\; \Delta_j \quad \, ,$$

incurring in a transaction cost of $\sqrt{\frac{2}{\pi}}\,|\Delta_{\tau_1-\delta t} - \Delta_j\,| \times S_j(\tau_1)\,A\,\sigma\,\sqrt{\delta t}$. We call this rehedging event a *nonsmooth rehedging* to distinguish it from portfolio adjustments at times when $V(S,t)$ is smooth. Define, for $t' > \tau_1$, the interval $(S^-_{t'}, S^+_{t'})$, where

$$S^-_{t'} \equiv sup\left\{S < S_j(t') : \frac{\partial V(S,t')}{\partial S} = \Delta_j\right\},$$

and

$$S^+_{t'} \equiv inf\left\{S > S_j(t') : \frac{\partial V(S,t')}{\partial S} = \Delta_j\right\}.$$

The proposed strategy is to maintain a static hedge, keeping $\Delta_{t'} = \Delta_j$, until the price $S_{t'}$ hits one of the endpoints $S^\pm_{t'}$. This has the effect of maintaining the value of the position above $V(S_{t'}, t')$ at all times (see Fig. 5). Let τ_2 represent the first time that $S_{t'} = S^\pm_{t'}$. At this time, the agent reverts to delta-hedging according to the relation $\Delta_{t'} = \frac{\partial V(S_{t'},t')}{\partial S}$. This strategy is continued in the same way until the next nonsmooth rehedging, etc., up to the expiration date T.

What is the agent's final position? By the theory in Section I, the value of the portfolio is essentially equal to $V(S_{t'}, t')$ until a kink is hit for the first time. After the kink is hit and Delta is temporarily held constant, the value of the portfolio exceeds $V(S_{t'}, t')$ (cf. Fig. 5). Thus, there will be a positive differential between the value of the hedging portfolio and the value function after delta-hedging resumes. Similarly, at each future time when the agent switches back from a static strategy to delta-hedging he will have a positive cash-flow. The final value of the hedging portfolio is thus $f(S_T)$ plus the sum of all cash-flows obtained in this fashion[18].

As seen with the example of the binary option, there is great flexibility in the choice of the amount of shares to hold immediately after hitting a kink. The Δ_j's chosen in (26) are just one of many possible choices. Alternatively, instead of setting the delta of his portfolio equal to a *fixed* amount of shares, the agent can use a different Δ_j each time the price hits a "kink", corresponding to a dominating static position. In this way, he can have better control on the amount of shares transacted each time.

[18] For this reason, the strategy *dominates* rather than replicates the payoff at expiration.

VI. Examples and Monte Carlo simulations

This section presents an empirical study of the performance of the different hedging strategies as A (or, equivalently, δt) is varied. We present Monte Carlo simulations of the profit/loss probability distributions arising from hedging a binary option with various strategies. We demonstrate that the hedge based on the obstacle problem is more effective than others when the option is at-the-money near expiration. We also investigate the effects of using different holding/delta-hedging strategies with $A \geq 1$, such as "cash-if-hit", "dominating portfolio-if-hit", etc. Finally, we give an example of a solution of (17) for which there is a kink in the value-function only close to the expiration date, the function being smooth far from expiry. The example consists of a derivative security equivalent to a "basket" of a standard call and a binary option (i.e., a "call with rebate").

The binary option revisited. We consider a security with annualized volatility $\sigma = 0.2$ and annual expected rate of return $\mu + \frac{\sigma^2}{2} = 0.06$. The annualized riskless short rate is $r = 0.02$. The round-trip transaction cost percentage is assumed to be $k = 0.01$. A binary option with the following terms is considered:

Time to expiration $= T - t = 0.1$ yr. ($\approx$ 26 trading days).

Strike price$= K = \$100$.

Payoff$= H = \$50$ if $S_T \geq K$ and 0 otherwise.

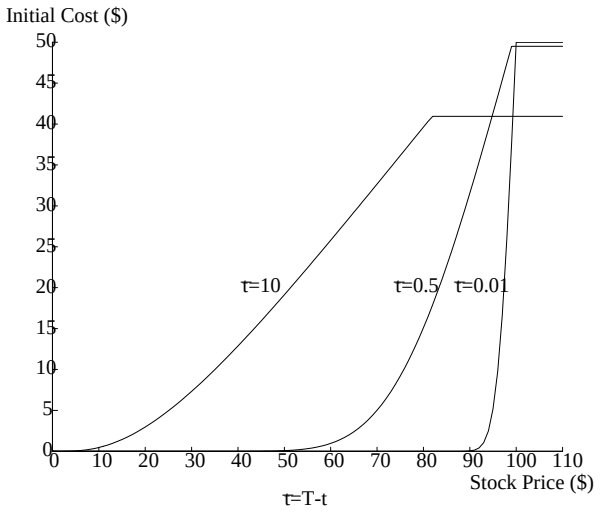


Figure 6. Solution of the obstacle problem for the binary option with payoff (18). The parameter values are $k = 0.01$, $\sigma = 0.2$,$r = 0.02$, $K = 100$ and $H = 50$. The time between rehedgings is $\delta t = 0.001$, with its corresponding $A = 1.26$, The value function is depicted at times-to-expiration $\bar{t} = 10, 0.5, 0.01$ years.

We shall compare three different rehedging schedules, namely $\delta t = 0.001$, $\delta t = 0.002$ and $\delta t = 0.01$, corresponding to Leland numbers of values $A = 1.26$, 0.89 and 0.40, respectively. In terms of rehedgings, these strategies correspond to 100, 50 and 10 adjustments over the 26-day period. Note however that, since the first strategy has $A > 1$, its total number of adjustments is actually less than 100, because there will be periods when there will be no transactions[19]. In Figure 6, we show the time-evolution of the value-function for $A = 1.26$. Notice the evolution of the kink, located at $S(t) = 100\, e^{-0.02(T-t)}$. In Figure 7, we exhibit the value functions $V(S,t)$ for the 3 values of A and the Black-Scholes solution corresponding to $A = 0$.

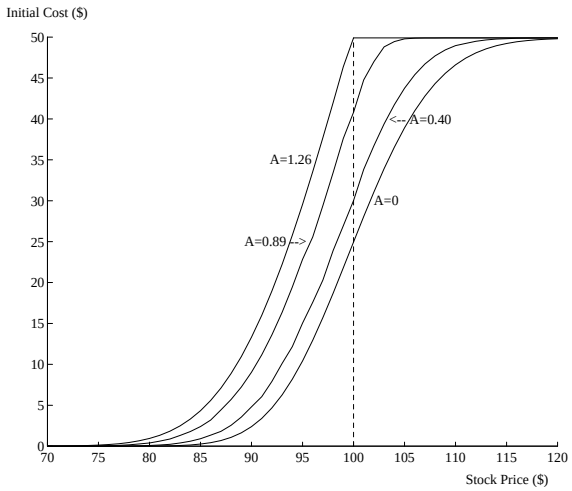


Figure 7. Comparison of different value functions for different values of the Leland number A. The time-to-expiration is 0.1 years. The values of the parameters are as in Fig. 6.

We simulated hedging this option over 200 different 26-day periods using various strategies. To simulate price fluctuations of the underlying asset, we generated 100 independent lognormal random shocks for each period. First, we considered the $A = 1.26$ schedule, for which several trading strategies are possible (cf. IV), and assumed that the initial price of the security was $S_t = \$97$. We compared the profit and loss distributions generated by the 200 simulations for the following strategies:

1. "Cash-if-hit": use delta-hedging until the price hits the value $H\, e^{r(T-t)}$ for the first time. After this time, sell the shares and convert the entire portfolio to riskless bonds.

2. "Dominating portfolio-if-hit": do as before until the price hits the critical value. Then change the Δ of the portfolio into H/K $(=1/2)$ shares.

[19] In general, $(T-t)/\delta t$ represents the number of adjustments only for $A < 1$. For $A \geq 1$, this number corresponds to the *maximum* number of possible rehedgings.

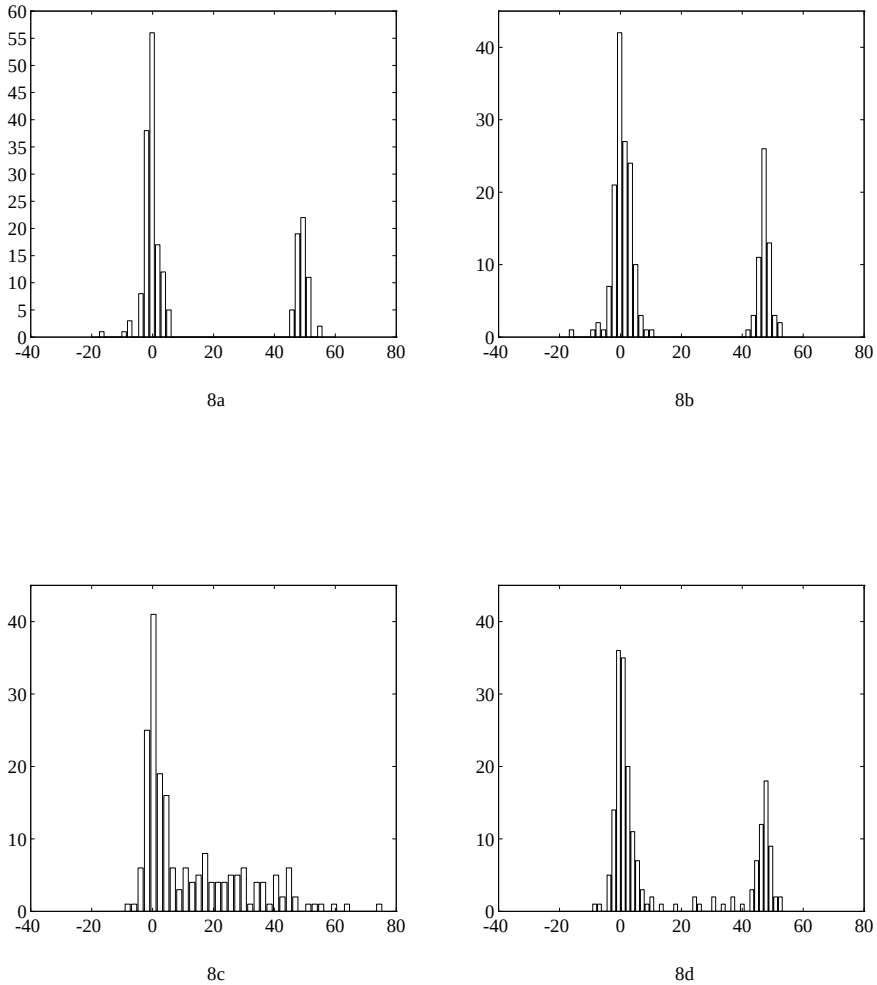


Figure 8. Histograms for 200 simulations of hedging a binary option with $A = 1.26$ with different admissible strategies. Figure 8a: "cash-if-hit"; 8b: "dominating portfolio-if-hit"; 8c: "on-and-off delta-hedging" and 8d: "combination of 2 & 3". The parameters, as well as δt and A, are the same as the ones for Fig. 6. The time-to-expiration is 0.1 years and the initial price of the underlying security is $\$97$.

3. "On-and-off delta-hedging": when the price hits the critical value for the first time, the hedger does nothing and stops transacting, "freezing" his Δ momentarily. He will resume standard delta-hedging once the price of the underlying security satisfies the inequality

$$\frac{\partial V(S'_t, t')}{\partial S} \leq \Delta$$

for the first time. This on-and-off procedure is repeated until expiration.

4. "Combination of 2 & 3". This strategy consists of using "dominate-if-hit" when the time to expiration is between 0.1 and 0.025 yr., and after that the "on-and-off" strategy. The reason for using this mixed approach is the following: far from expiration it is more convenient to "dominate if hit" because this will avoid any further adjustments and will generate a profit after delivering the payoff. However, if the critical value is not reached relatively soon, the cost for changing the delta to the value 1/2 may be quite large since the left-derivative of the $V(S,t)$ increases sharply as we approach expiration. Thus, if at the hitting time we have $\Delta = 5$, say, then the cost for changing to $\Delta = 0.5$ is approximately $0.01 \times 4.5 \times 100 = \$4.5$. Passing to the "on-and-off" strategy eliminates this sharp cost at the expense of having to make further adjustments.

In Figures 8 a, b, c, d, we present the histograms of the profit and loss (P/L) distributions for the four strategies. These are the distributions of the final value $V_T - f(S_T)$ across the 200 simulations. Notice that the frequency of losses is quite small in all four cases, whereas the strategies can generate considerable profits to the hedger, as shown in IV. For this reason, the P/L histograms show a strong skewness towards positive values. The combination of the strategies 2 and 3 generates more profits than the others and has practically no losses (See also Fig. 1a).

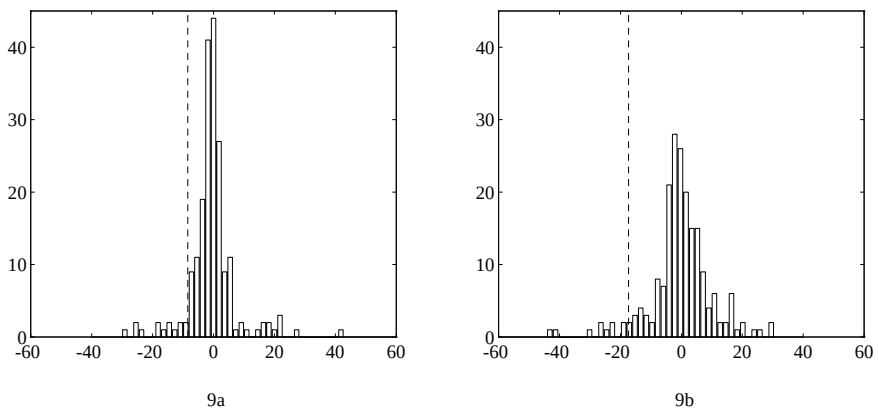
In Figures 9 a,b, we exhibit the P/L for the Hoggard *et al.* strategies for $\delta t = 0.002$ (50 adjustments) and for $\delta t = 0.01$ (10 adjustments). We also show in Figs. 10 a, b, the P/L distributions for the Black-Scholes strategies for the same values of δt (which do not take into account transaction costs). Examination of Figs. 9 a, b shows that the P/L distributions for $A < 1$ have long tails, i.e. are much riskier than for $A = 1.26$. There is not much improvement in the P/L distribution as the Leland number is increased from 0.4 to 0.89. This can be explained by the fact that the binary option has a huge hedge-slippage risk because the payoff function is discontinuous. The contribution from errors accumulated after many transactions cancels the benefits of narrowing δt.

We can also compare the performance of the two-volatility scheme of (9)-(10) with Black-Scholes. Comparing figures 9a and 10a, we notice that the two-volatility scheme works better in the sense that its P/L is less skewed towards negative values. However, for longer times between adjustments, the two strategies perform similarly (Figs. 9b and 10b).

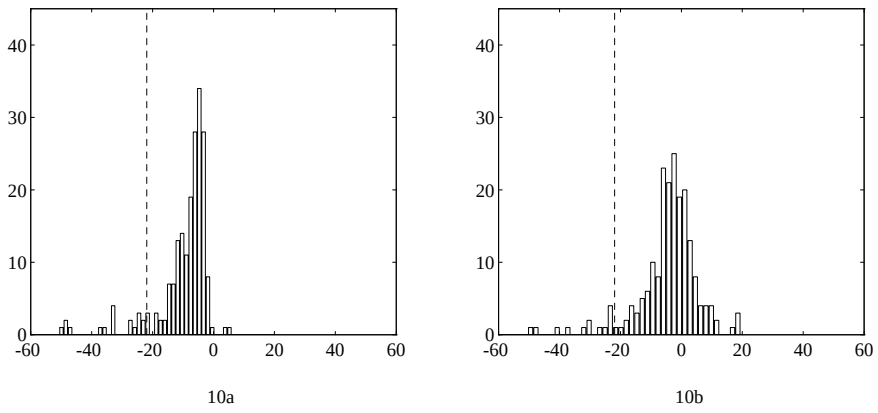
The cost of the initial portfolio increases with A. Therefore, an agent using the smaller values of A would be able, at least in theory, to charge less than one using a larger value. This suggests a different test, in which the difference between initial costs is taken into account in the P/L. If we take the $A = 1.26$ cost of \$37.61 as the reference, then the histograms in Figs. 9 and 10 have to be shifted to the right by the difference in initial costs. The location of the shifted origin is indicated by a dotted line in each graph. Notice that the losses in the strategies for $A < 1$ and Black-Scholes are substantial, even after normalizing by the initial cost.

Figure 11 shows the P/L distributions for the values $A = 0, 0.40, 0.89, 1.26$ with an initial price of \$95 .

Figure 9. Profit/loss histograms for hedging the binary option with the two-volatility scheme (9)-(10), for the same 200 samples as in Fig. 8. The parameters are the same as in Fig. 6. Figure 9a shows the results with $\delta t = 0.002$, and $A = 0.89$. The dotted line represents the break-even point if the difference between the initial costs of the strategies in Fig. 8 and the latter strategies are added to the profit/loss. Figure 9b: same as 9a with $\delta t = 0.010$, and $A = 0.40$.



9a 9b



10a 10b

Figure 10. Profit/loss histograms for hedging the binary option with the Black-Scholes strategy, for the same 200 samples as in Fig. 8. The parameters are the same as in Fig. 6. Figure 10a shows the results with $\delta t = 0.002$. The dotted line represents the break-even point if the difference between the initial costs of the strategies in Fig. 8 and this latter strategy are added to the profit/loss. Figure 10b: same as 10a with $\delta t = 0.010$.

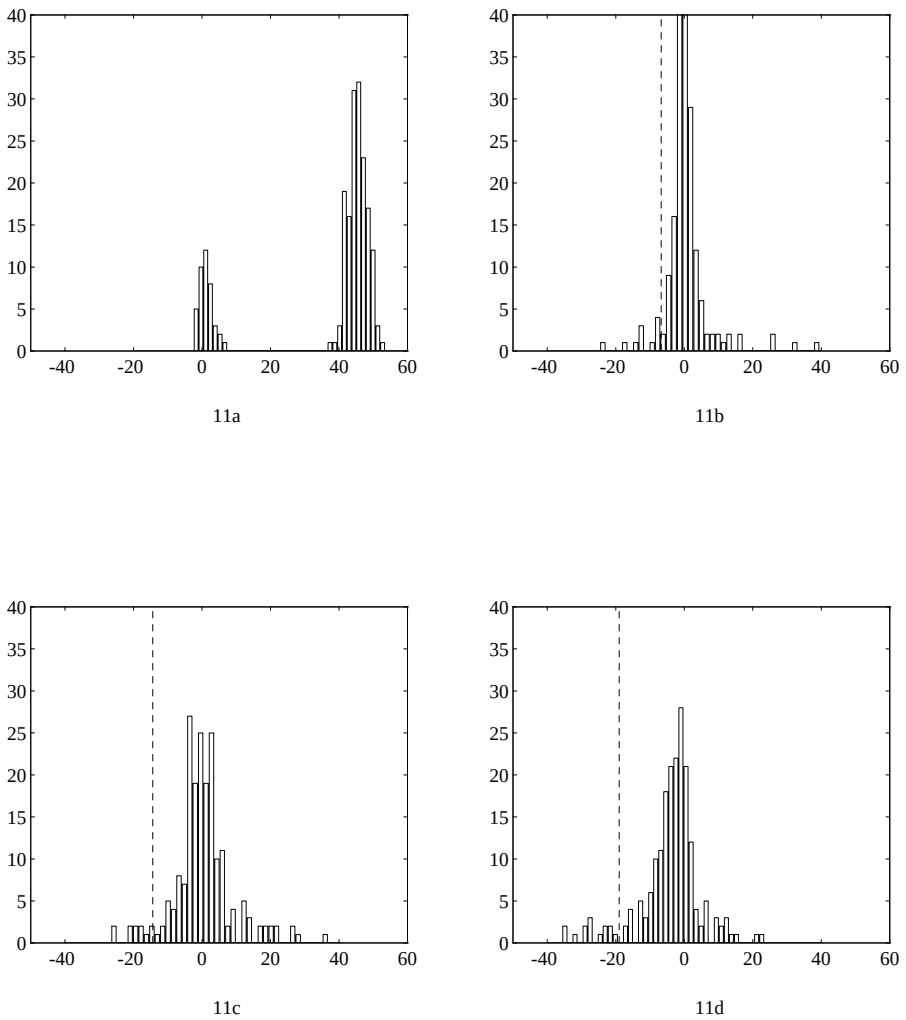


Figure 11. Histograms for the profit/loss generated with 200 simulations of hedging a binary option with strategies corresponding to different Leland numbers A.The parameters are the same as in Fig. 6. The time-to-expiration is 0.1 years and the initial price of the underlying security is $\$95$. Figure 11a: rehedging interval $\delta t = 0.001$, $A = 1.26$, admissible strategy "combination of 2 & 3". Figure 11b: $\delta t = 0.002$, and $A = 0.89$. The dotted line represents the break-even point in initial costs with respect to strategy 11a. Figure 11c: same as 11b with $\delta t = 0.010$, and $A = 0.40$. Figure 11d: same as 11b with $\delta t = 0.010$, and $A = 0$ (Black-Scholes).

The initial costs of the hedging portfolio for strategies with different values of the parameter A and of S_t are given in Table I.

Call plus rebate. This example illustrates the most general behavior of the solution to the obstacle problem, in which kinks are formed progressively in time as the expiration date approaches. Consider the payoff function for a "basket" of a long call and a long binary option:

$$f(S) = \begin{cases} 0 & , \text{if } S < K \\ \\ S - K + H & , \text{if } S \geq K \, . \end{cases}$$

For $H < K$, the obstacle problem corresponding to this payoff is more complicated than the one for the binary option. In fact, the value function is smooth in S until a time $T^* < T$. After this time, a kink develops at $S(t) = K\, e^{-r(T-t)}$. This phenomenon is illustrated in Figure 12, where we took $K = 100$ and $H = 20$. The critical time time-to-expiration is $T - T^* = .317$ yr. After time T^*, the out-of-the-money value of such a package is equal to the value of the binary option, which presents the largest hedging risk.

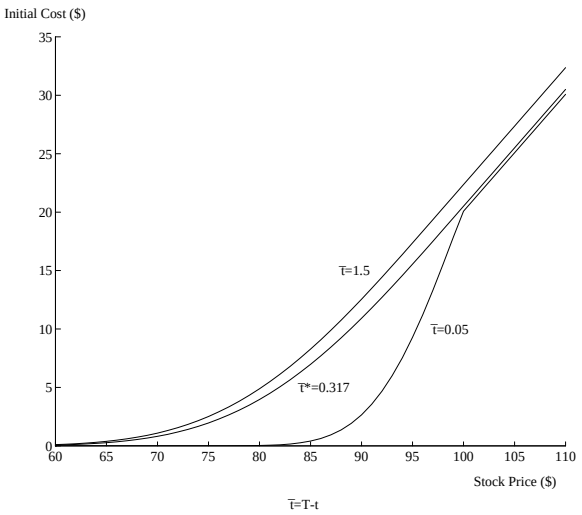


Figure 12. Solutions to the obstacle problem for the call plus rebate option. The values of the parameters are $k = 0.01$,$\sigma = 0.2$,$r = 0.02$, $K = 100$, $H = 20$, $\delta t = 0.001$, and $A = 1.26$. The value function is depicted at times-to-expiration: $\overline{(t)} = 1.5$, critical time-to-expiration $\overline{(t)}^* = 0.317$, and $\overline{(t)} = 0.05$ for which a kink exists.

VII. Conclusions

We have presented a new class of hedging strategies which can be used to hedge European derivative securities in the presence of large transaction costs or, alternatively, small time-intervals between adjustments. The use of such strategies is recommended whenever there is a need to use a strategy such that $A = \sqrt{\frac{2}{\pi}} \frac{k}{\sigma \sqrt{\delta t}} \geq 1$. Strategies with Leland numbers larger than unity are needed to limit the hedging risk for derivative securities with large values of gamma and delta near expiration. For such values of A, it is not possible to use standard delta-hedging techniques unless the payoff function $f(S)$ is convex.

The new strategies are based on solving an obstacle problem for a Black-Scholes equation with Leland volatility $\sigma_A = \sigma \sqrt{1 + A}$. Unlike the usual delta-hedging strategies, in which the hedge ratio is a function of the price of the underlying security, the new hedging strategies are non-Markovian, with a hedge-ratio depending on the path taken by the price. The new hedging strategies admit closed-form solutions in cases of interest and can be easily be implemented on a workstation. Monte-Carlo simulations for hedging a binary option near expiration clearly show that the new strategies are more efficient in reducing hedging risk in the presence of large transaction costs.

Acknowledgment: This research was partially funded by the National Science Foundation under grant NSF-DMS-92-07085 and by the Mexican Consejo Nacional de Ciencia y Tecnología (CONACYT, México).

References

[1] M. Avellaneda and A. Parás, *"Optimal Hedging Strategies for European-Style Derivative Securities in the Presence of Transaction Costs"*, in preparation.

[2] B. Bensaid, J. Lesne, H. Pagès and J. Scheinkman, *"Derivative Asset Pricing with Transaction Costs"*, Mathematical Finance, Vol.2, No.2, 63-86 (1992)

[3] F. Black and M. Scholes, *"The Pricing of Options and Corporate Liabilities"*, J. Pol. Economics, 81, 637-659 (1973)

[4] P. P. Boyle and T. Vorst, *"Option replication in discrete time with transaction costs"*, Journal of Finance, 47, 271, (1992)

[5] J. C. Cox, S. A. Ross and M. Rubinstein, *"Option Pricing: A Simplified Approach"*, Journal of Financial Econometrics, 7, 229-263 (1979)

[6] D. Duffie, *"Dynamic Asset Pricing Theory"*, Princeton University Press, Princeton 1992

[7] A. Friedman, *"Variational Principles and Free Boundary Problems"*, Robert E. Krieger Publ., (1988)

[8] T. Hoggard, A. E. Whalley and P. Wilmott, *"Hedging Option Portfolios in the Presence of Transaction Costs"*, to appear in Advances in Futures and Options Research (1993)

[9] J. Hull, *"Options, Futures, and other Derivative Securities"*, 2nd edition, Prentice-Hall, Englewood Cliffs, N.J. (1993)

[10] N. V. Krylov, *"Controlled Diffusion Processes"*, Springer-Verlag, New York, 1980.

[11] H. E. Leland, *"Option Pricing and Replication with Transaction Costs"*, J. of Finance, 40, 1283-1301 (1985)

[12] H.P. McKean, *"Stochastic Integrals"*, Academic Press, (1969)

[13] M. Rubinstein, *"Exotic Options"*, Finance Working Paper 220, Haas School of Business, University of California at Berkeley, December 1991

[14] M. Rubinstein, *"Unscrambling the Binary Code"*, RISK, October 1991

[15] A. E. Whalley and P. Wilmott, *"Counting the Costs"*, RISK, Vol. 6, 10, October 1993

[16] P. Wilmott, J. Dewynne and S. Howison, *"Option Pricing"*, Oxford Financial Press, Oxford (1993).