Now consider using Lax-Wendroff with diffusion ⑤

$$u_t + a u_x = d u_{xx}$$

First, figure out what the correct second-order Taylor series is:

$$u_{tt} = -a(u_t)_x + d(u_t)_{xx} =$$

$$= -a(-a u_x + d u_{xx})_x + d(-a u_x + d u_{xx})_{xx}$$

$$u_{tt} = a^2 u_{xx} - 2ad\, u_{xxx} + d^2 u_{xxxx}$$

$$\Rightarrow u^{n+1} = u^n + (-a u_x + d u_{xx})\Delta t + \frac{\Delta t^2}{2}(a^2 u_{xx} - 2ad\, u_{xxx} + d^2 u_{xxxx}) + \dots$$

It is NOT a good idea to now proceed to discretize everything using centered differences. Instead, use Lax-Wendroff only for advection and Crank-Nicolson (implicit midpoint) for diffusion. How?

First, let us consider a more general splitting framework:

$$u_t = A u + B u$$

linear operators

$$\begin{cases} A \equiv -a\,\partial_x \\ B = d\,u_{xx} \end{cases}$$

in our case

$$u_t = (A+B)u \Rightarrow$$

$$u^{n+1} = \left\{ I + (A+B)\Delta t + \frac{1}{2}(A^2 + B^2 + AB + BA)\Delta t^2 \right\} u^n + O(\Delta t^3)$$

In our case $A^2 = a^2 \partial_{xx}$,

$B^2 = d^2 u_{xxxx}$, $AB = -ad\, u_{xxx} = BA$

Note, however, that generally $AB \neq BA$ (non-constant coefficients, boundary conditions) so we should not assume this if we want a general approach.

So what Lax-Wendroff is doing (9)
is really

$$u^{n+1} = \left\{ I + A\,\Delta t + \frac{1}{2}\tilde{A}^2 \Delta t^2 \right\} u^n$$

where $\tilde{A}^2 \approx A^2$ but not equal

This is why it is $\underline{\underline{\text{NOT}}}$ an MOL scheme - MOL would only have one A!

But for purposes of $\underline{\underline{\text{error}}}$ analysis (second-order accuracy) we can treat Lax-Wendroff as doing $\tilde{A}^2 = A^2$

First try:

$$\frac{u^{n+1} - u^{n}}{\Delta t} = \text{Lax-Wendroff for advection} + \underset{\text{Crank-Nicolson}}{d}\left(\frac{u_{xx}^{n+1} + u_{xx}^{n}}{2}\right)$$

$$\Rightarrow \frac{u^{n+1} - u^{n}}{\Delta t} = \left(A + \frac{A^2 \Delta t}{2}\right) u^{n} + \frac{B}{2}\left(u^{n+1} + u^{n}\right)$$

$$\Rightarrow u^{n+1} = \left(I - \frac{B}{2}\Delta t\right)^{-1} \left\{ \left(A + \frac{A^2 \Delta t}{2}\right) u^{n} \Delta t + \left(I + \frac{B}{2}\Delta t\right) u^{n} \right\}$$

$$= \left(I - \frac{B}{2}\Delta t\right)^{-1} \left\{ I + \frac{B}{2}\Delta t + A \Delta t + \frac{A^2}{2}\Delta t^2 \right\} u^{n}$$

Expand

$$\left(I - \frac{B\Delta t}{2}\right)^{-1} = I + \frac{B\Delta t}{2} + \frac{B^2\Delta t^2}{4}$$

to get

$$u^{n+1} = \left(I + \frac{B\Delta t}{2} + \frac{B^2\Delta t^2}{4}\right)$$
$$\left(I + \frac{B\Delta t}{2} + A\Delta t + \frac{A^2\Delta t^2}{2}\right) u^n$$

$$= \Big(I + B\Delta t + A\Delta t + \frac{A^2}{2}\Delta t^2$$
$$+ \frac{B^2\Delta t^2}{2} + \frac{BA}{2}\Delta t^2 + O(\Delta t^3)\Big) u^n$$

We are <u>missing</u> $\frac{AB}{2}\Delta t^2$! NOT SECOND ORDER

How to fix this?
There are many ways up to second order:

- Time splitting (e.g. Strang)
- Predictor-corrector schemes

They are expensive because they require either two CN-solves or two Lax-Wendr. steps per time step.

Instead:

1. Solve $u^{(LW)}_t = Au + Bu^n$ ← source term
2. Solve $u_t = Bu + u^{(LW)}_t$

where the approximation of the Lax-Wendr. is now a source term.

Lax-Wendroff with spatial source term (13)

$$u_t + a u_x = S(x)$$

$\nwarrow$ NOT a function of time

$$u_{tt} = -a (u_t)_x = -a(-a u_x + S)_x$$

$$= a^2 u_{xx} - a S_x$$

So

$$u_i^{n+1} = u_i^n - \frac{a \Delta t}{2 \Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + S_i \Delta t$$

$$+ \frac{a^2 \Delta t^2}{2 \Delta x^2}\left(u_{i+1}^n - 2 u_i^n + u_{i-1}^n\right)$$

$$- \frac{a \Delta t^2}{2 \Delta x}\left(S_{i+1}^n - S_{i-1}^n\right)$$

(for example)

In our case

$$S \equiv B u^n = d\, u^n_{xx}$$

$$S^n_i = d \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \right)$$

Algebraically, what we have done is

$$u^{n+1} = \left(I + A\Delta t + B\Delta t + \frac{\tilde{A}^2}{2}\Delta t^2 + \frac{AB}{2}\Delta t^2 \right) u^n$$

which is <u>almost</u> second-order accurate (now missing $\frac{BA}{2}\Delta t^2$ term)

But it would not be A-stable because diffusion is treated explicitly

Instead, do Crank-Nicolson for diffusion but treat the Lax-Wendroff update as a source term! (15)

$$\frac{u^{n+1} - u^{n}}{\Delta t} = \frac{B}{2}\left(u^{n+1} + u^{n}\right) + \Big(\underbrace{\underset{\text{centered advection}}{A\,\Delta t} + \underset{\text{centered diffusion}}{\frac{\tilde{A}^2\,\Delta t^2}{2}}}_{\text{Lax-Wendroff}} + \underset{\text{source-term second correction}}{\frac{AB}{2}\Delta t^2}\Big)u^n$$

So now

$$u^{n+1} = \left(I + \frac{B\Delta t}{2} + \frac{B^2\Delta t^2}{4}\right) + O(\Delta t^3) \leftarrow \text{SECOND ORDER!}$$

$$\times\left(I + \frac{B\Delta t}{2} + A\,\Delta t + \frac{\tilde{A}^2}{2}\Delta t^2 + \frac{AB}{2}\Delta t^2\right)u^n$$

$$= \left[I + (A+B)\Delta t + \frac{1}{2}\left(\tilde{A}^2 + B^2 + AB + BA\right)\Delta t^2\right]u^n$$

How about stability? Is the only limitation now $\frac{a \Delta t}{\Delta x} \leq C \simeq 1$?

It turns out no, to get true stability for any diffusive CFL number we need some upwinding.

So, instead of Lax-Wendroff consider

Fromm's scheme. (with diffusion)

We need source term:

$$u_t = -a u_x + S \quad , \quad a > 0$$

to Fromm's method

Extrapolate state to faces at midpoint as we did before:

$$u^{n+1/2}_{j+1/2} = u^n_j + \frac{\Delta x}{2}(u^n_x)_j + \frac{\Delta t}{2}(u^n_t)_j$$

$$= u^n_j + \frac{\Delta x}{2}(u^n_x)_j + \frac{\Delta t}{2}\left(-a(u^n_x)_j + s_j\right)$$

$$= u^n_j + \frac{1}{2}(\Delta x - a\Delta t)(u^n_x)_j \quad \Big\} \text{ as before}$$

$$+ \frac{\Delta t}{2} s^n_j \quad \Big\} \text{ new term}$$

And here $s \equiv d\, u_{xx}$ so

$$s^n_j = \frac{d}{\Delta x^2}\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right)$$

Note that this is <u>upwinded</u> since the extrapolation is done from the cell to the left! 18

Now

$$u_j^{n+1} = u_j^n - \Delta t\, a\left(\frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{\Delta x}\right) + S_j^n \cdot \Delta t$$

is Fromm's scheme with a source.

The extra term added in $u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}$:

$$-\frac{a\,d\,\Delta t^2}{2\Delta x^3}\left[(u_{j+1} - 2u_j + u_{j-1}) - (u_j - 2u_{j-1} + u_{j-2})\right]$$

$$= -\frac{a\,d\,\Delta t^2}{2\Delta x^3}\left[u_{j+1} - 3u_j + 3u_{j-1} - u_{j-2}\right]$$

$$\rightarrow -\frac{ad}{2}(u_{xxx})_j\, \Delta t^2$$

Talyor series shows this is a discretization of $AB\frac{\Delta t^2}{2} \equiv -\frac{ad}{2}(u_{xxx})_j \Delta t^2$

(19)

but here u_{xxx} is <u>upwind biased</u>.

Compare this to what Lax-Wendroff does, giving $AB\Delta t^2/2$ in the form

$$-\frac{a\,d\Delta t^2}{2\Delta x^3}\left[(u_{j+2}-2u_{j+1}+u_j)-(u_j-2u_{j-1}+u_{j-2})\right]$$

$$=-\frac{a\,d\Delta t^2}{2\Delta x^3}\left[u_{j+2}-2u_{j+1}+2u_{j-1}-u_{j-2}\right]$$

$$\rightarrow -\frac{ad}{2}(u_{xxx})_j\,\Delta t^2$$

but this is centered now.