1

Autonomous Systems

Phase-Space Analysis

$\vec{y}'(t) = g(\vec{y})$ ← no t in equation

$y \in \mathbb{R}^n$

g is some real-valued function

Note that if $y(t)$ is a solution, so is $y(t+t_0)$.

[Think of a pendulum rocking now versus rocking 5 mins later]

This means autonomous systems are invariant under translations of the independent variable.

The curve traced by a solution $y(t)$ as t goes from $t=0$ to $t \to \infty$ is called an

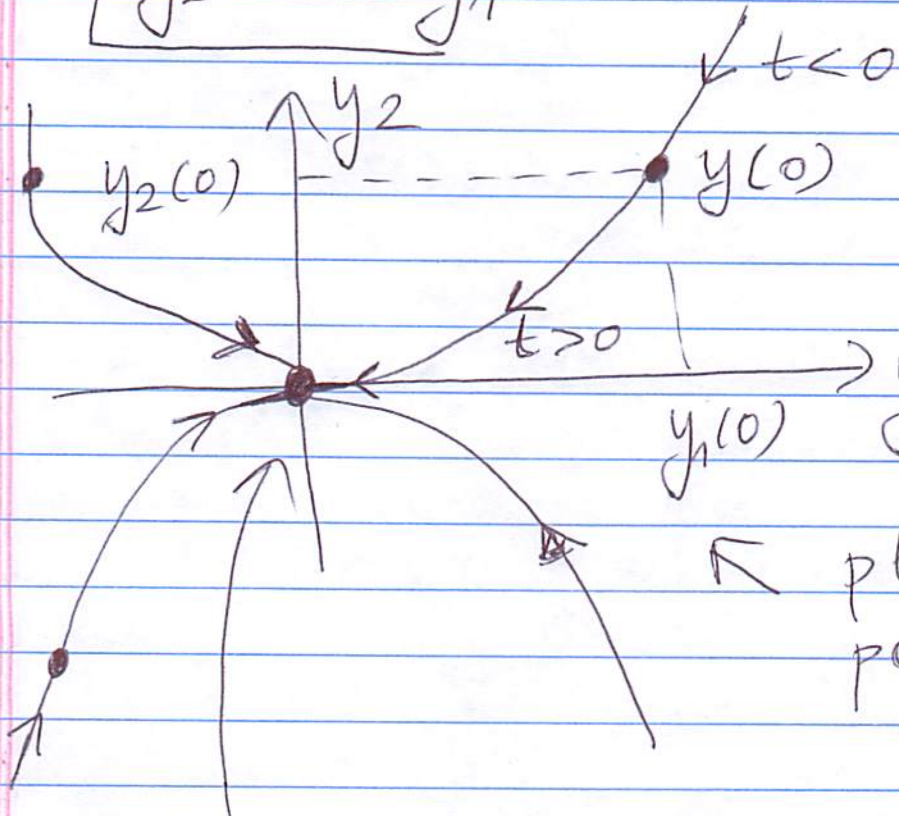
<u>orbit</u>, <u>trajectory</u>, or <u>path</u>

② *E.g.*

$$\begin{cases} y_1' = -2y_1 \\ y_2' = -3y_2 \end{cases} \Rightarrow \begin{cases} y_1 = e^{-2t} \cdot y_1(0) \\ y_2 = e^{-3t} \cdot y_2(0) \end{cases}$$

$$\begin{cases} y_1 = c_1 e^{-2t} \Rightarrow e^{-t} = \left(\frac{y_1}{c_1}\right)^{1/2} \\ y_2 = c_2 e^{-3t} \Rightarrow y_2 = c_2 \left(\frac{y_1}{c_1}\right)^{3/2} \end{cases}$$

$$\boxed{y_2 = c \cdot y_1^{3/2}}$$



↖ phase portrait

All trajectories terminate (stop) at the origin, which is the so-called fixed-point of the system

(3)

A <u>critical</u> or <u>fixed point</u> $\vec{a}$ is a solution of

$$g(\vec{a}) = 0$$

So if you start at $\vec{a} = y(0)$ you will stay there since

$$y'(0) = g(y(0)) = g(a) = 0$$

<u>Homogeneous constant-coefficient</u>

$$y' = Ay \quad , A = TJT^{-1}$$

Recall that we define

$$z = T^{-1}y \text{ to obtain}$$

$$z' = Jz \quad , \text{then } y = Tz$$

Multiplication by an invertible matrix T simply <u>stretches</u> and <u>rotates</u> the coordinate system. This deforms the phase portrait but does not change its qualitative character

④ Assume $\lambda = 0$ is $\underline{\underline{\text{not}}}$ an eigenvalue, i.e., A is invertible.

The only fixed point

$$Aa = 0 \Rightarrow a = 0$$

is the origin.

There are now only $\underline{\underline{6}}$ possible distinct (qualitatively) forms for J:

i) $J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $\mu < \lambda < 0$ or $0 < \mu < \lambda$

ii) $\mu < 0 < \lambda$

iii) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda > 0$ or $\lambda < 0$

iv) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ where $\lambda > 0$ or $\lambda < 0$

v) $\begin{pmatrix} \sigma & \vartheta \\ -\vartheta & \sigma \end{pmatrix}$ vi) $\begin{pmatrix} 0 & \vartheta \\ -\vartheta & 0 \end{pmatrix}$

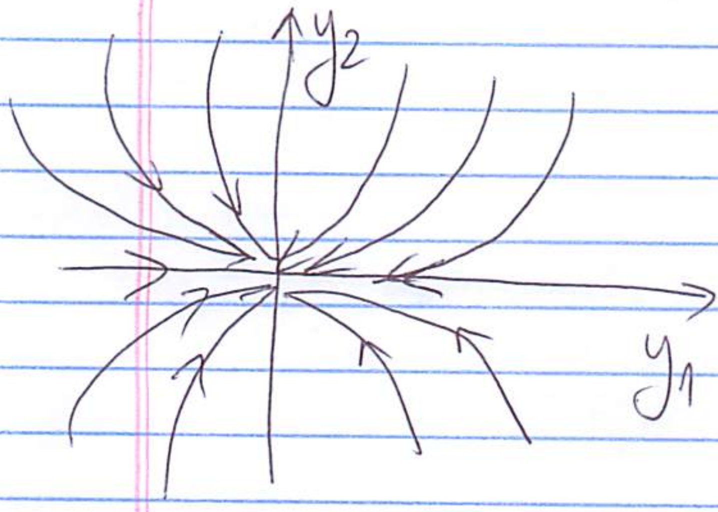
$\lambda = \sigma \pm i\vartheta$ complex eigenvalue case

(5)

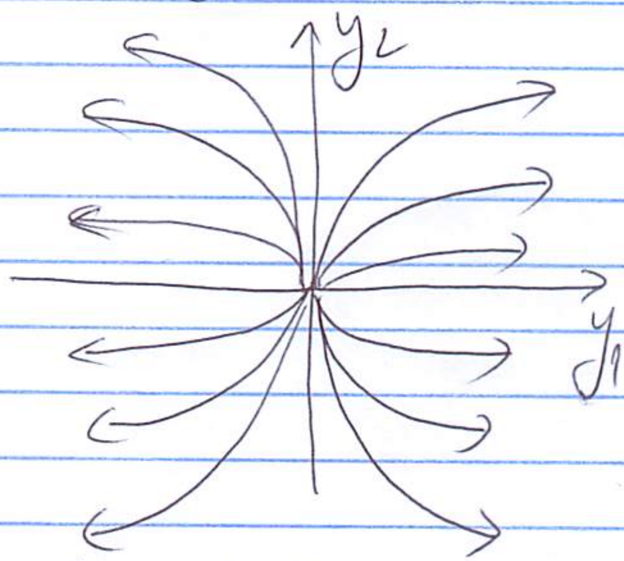
Case i

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

λ and μ both positive or negative



$\lambda, \mu < 0$

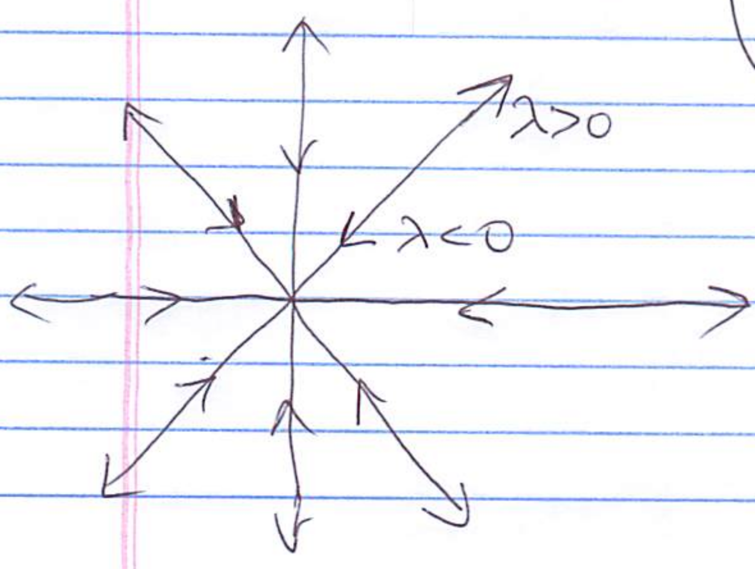


$\lambda, \mu > 0$

Origin is improper node

Case iii

$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

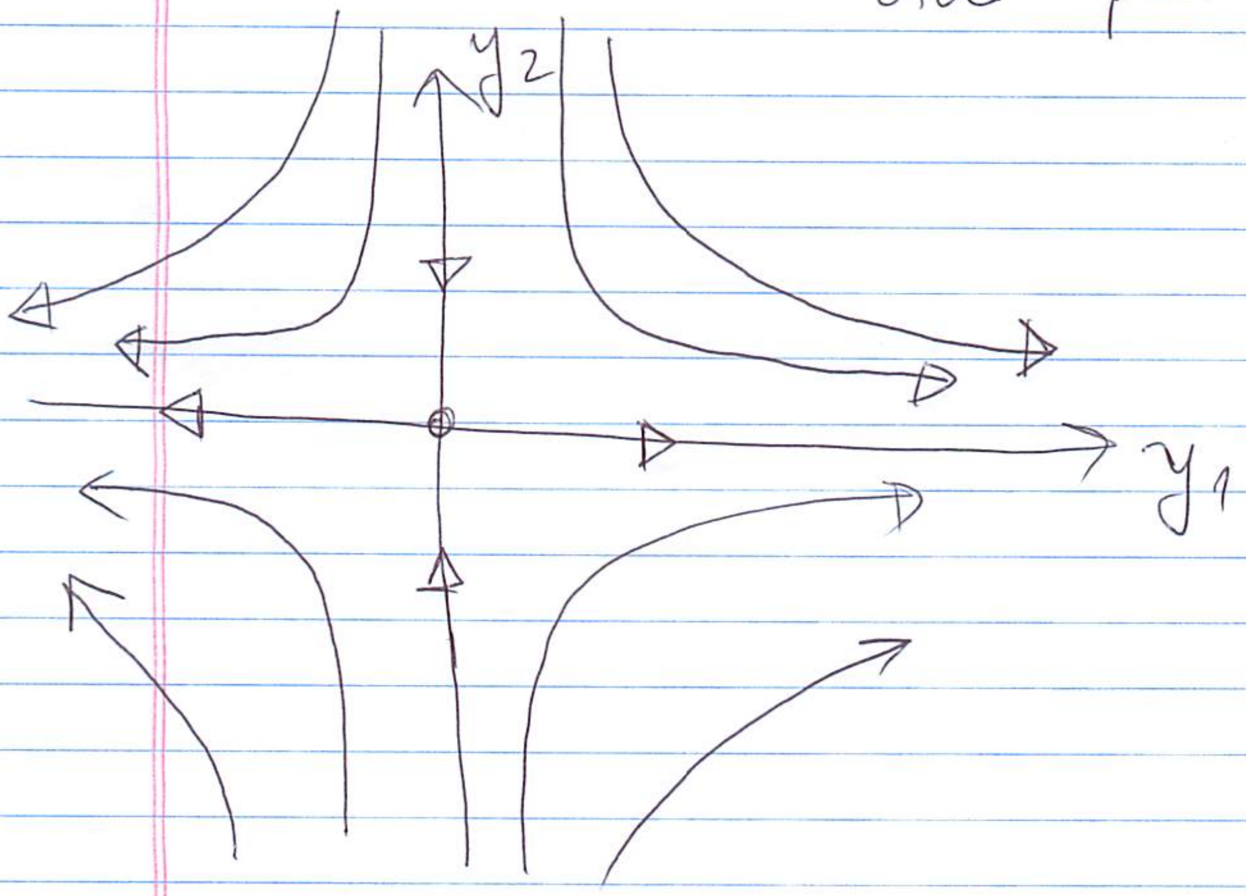


Origin is a proper node

6 <u>Case ii</u>

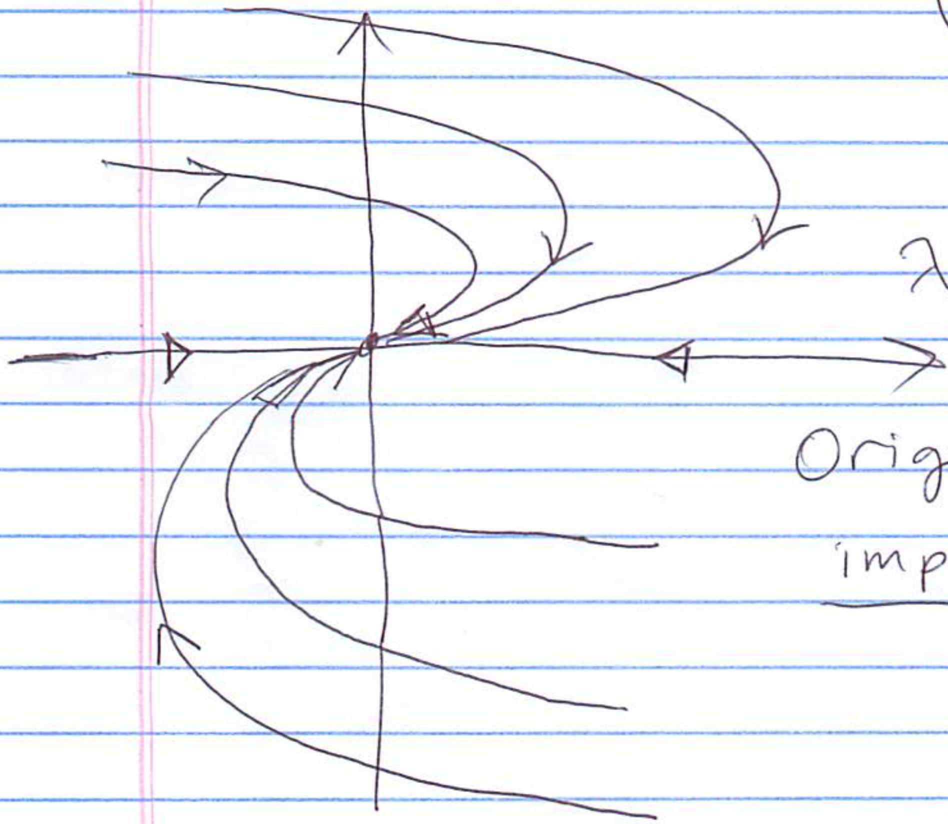
$$J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

One of λ/μ is negative, the other positive



Origin (fixed point) is a <u>saddle point</u>

(7) <u>Case iv</u> $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$



$\lambda < 0$

Origin is an <u>improper node</u>

(8) <u>Case v</u> $J = \begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix}$

If $\sigma > 0$

$$y(t) = e^{\sigma t} \begin{bmatrix} \eta_1 \cos \nu t + \eta_2 \sin \nu t \\ -\eta_1 \sin \nu t + \eta_2 \cos \nu t \end{bmatrix}$$

$$= e^{\sigma t} \rho \begin{bmatrix} \cos(\nu t - \alpha) \\ -\sin(\nu t - \alpha) \end{bmatrix}$$

where $\rho = \sqrt{\eta_1^2 + \eta_2^2}$ and

8

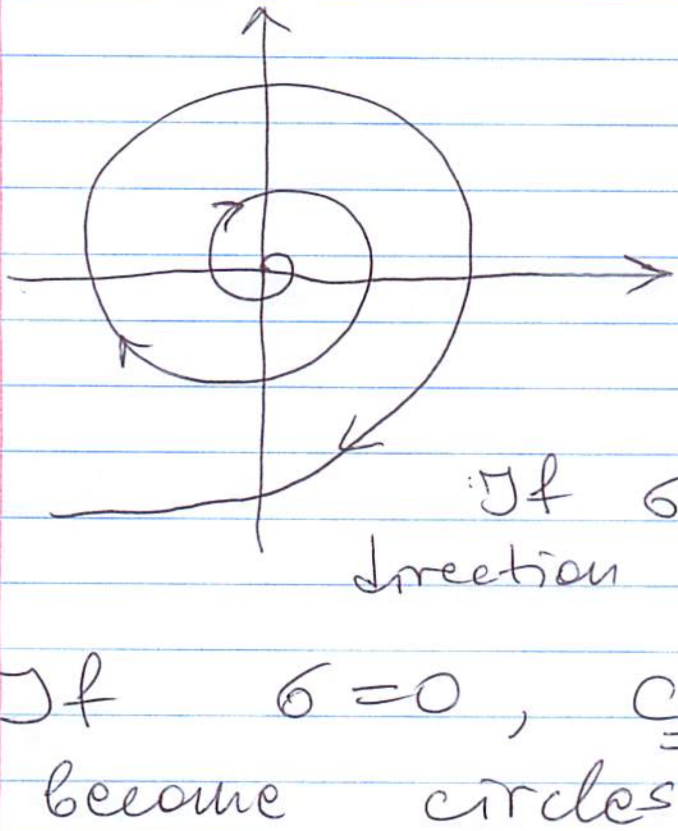
the phase $\sin\alpha = \frac{\eta_2}{\rho}$, $\cos\alpha = \frac{\eta_1}{\rho}$

Let's write this in polar coordinates

$$(y_1, y_2) \Leftrightarrow (r, \theta)$$

$$r^2 = y_1^2 + y_2^2 \Rightarrow r = \rho e^{\sigma t}$$

$$\theta = \nu t - \alpha \leftarrow \text{linear}$$



$\sigma > 0$

Origin is a spiral point

If $\sigma < 0$ just reverse direction toward origin

If $\sigma = 0$, Case vi, orbits become circles, and the origin (fixed point) is a center point