(1) Constant Coefficients: General Theory

[alg > geometric multiplicity]

Assume that A is diagonalizable (always the case if eigenvalues are distinct

$$A = PDP^{-1}$$

$$P = [x_1 \mid x_2 \mid \dots \mid x_n]$$

linearly independent eigenvectors as columns

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ diagonal matrix of eigenvalues

$$y' = Ay = PDP^{-1}y$$

introduce a new set of variables (new basis)

$$z = P^{-1}y \Rightarrow z' = P^{-1}y'$$

$$z' = P^{-1}y' = (P^{-1}P)DP^{-1}y = Dz$$

(2) In the new basis

$$z' = Dz$$

diagonal matrix

which means the linear transformation P diagonalizes the system of equations.

Each z_k is independent:

$$z_k' = \lambda_k z_k \Rightarrow z_k = c_k e^{\lambda_k t}$$

$$y = Pz = \sum_{k=1}^{n} z_k x_k$$

$$y = \sum_{k=1}^{n} c_k \cdot e^{\lambda_k t} \cdot x_k$$

which is exactly what we obtained earlier

Note: If D were lower or upper triangular then one could solve $z' = Dz$ by forward / backward substitution

(4) What if A is <u>not diagonalizable</u>?

$$A = T J T^{-1}$$

for any matrix, where J is the <u>Jordan canonical form</u>

$$\underset{[8\times 8]}{J} = \left[\begin{array}{cc|ccc|c|c|c}
\lambda_1 & 1 & & & & & & \\
 & \lambda_1 & & & & & & \\
\hline
 & & \lambda_2 & 1 & & & & \\
 & & & \lambda_2 & 1 & & & \\
 & & & & \lambda_2 & & & \\
\hline
 & & & & & \lambda_2 & & \\
\hline
 & & & & & & \lambda_3 & \\
\hline
 & & & & & & & \lambda_4
\end{array}\right]$$

\# of Jordan blocks is geometric multiplicity of eigenvalue (one linearly independent eigenvector per block)

Jordan form is hard to find in <u>general</u>, must do by hand

(5) $$z = T^{-1} y$$

$$z' = J z$$

Notice that each Jordan block is decoupled (independent) from the other blocks.

So consider solving one block of site 2:

$$z' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} z = Az$$

Claim:

$$\phi = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

is a fundamental matrix.

Check: $\varphi_2 = \begin{bmatrix} t e^{\lambda t} \\ e^{\lambda t} \end{bmatrix}$

$$\varphi_2' = \begin{bmatrix} e^{\lambda t} + \lambda t e^{\lambda t} \\ \lambda e^{\lambda t} \end{bmatrix} = A \varphi_2 \Rightarrow \varphi_2 \text{ is a solution}$$

⑥

How could we obtain this ourselves?

$$z' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} z$$

$$\begin{cases} z_1' = \lambda z_1 + z_2 \\ z_2' = \lambda z_2 \end{cases}$$

solve for z_2 first, $z_2 = c_1 e^{\lambda t}$

$$z_1' = \lambda z_1 + e^{\lambda t}$$ ← non-homogeneous

$$\Rightarrow \begin{cases} z_1 = c_2 e^{\lambda t} + c_1 t e^{\lambda t} \\ z_2 = c_1 e^{\lambda t} \end{cases}$$

which is exactly $z = \phi c$

where we already obtained ϕ

and $c = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$ in this notation

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How do we obtain T if we know J?

$$A = T J T^{-1}$$

$$AT = TJ \leftarrow$$ system of $\underline{\underline{n^2}}$ linear equations for T_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$

There are better ways but we won't really use Jordan form to solve linear systems in practice

Note: One of the columns of T will be the eigenvector but doing the above procedure is the same and gives you the whole matrix

The full (complicated!) procedure for obtaining T & J is in the book by Miller & Michel bottom of page 84

(8) Elimination of variables

$$\begin{cases} z_1' = a_{11} z_1 + a_{12} z_2 \\ z_2' = a_{21} z_1 + a_{22} z_2 \end{cases}$$

$z = Az$

$a_{21} \neq 0$

Define new variable

$z_3 = z_1 + \alpha z_2$ such that

$$z_3' = z_1' + \alpha z_2' = \beta z_3$$

$$= a_{11} z_1 + \alpha a_{21} z_1$$

$$+ a_{12} z_2 + \alpha a_{22} z_2 =$$

$$= \beta z_1 + \alpha \beta z_2$$

Equating coefficients in front of z_1 and z_2

$$\begin{cases} a_{11} + \alpha a_{21} = \beta \\ a_{12} + \alpha a_{22} = \alpha \beta \end{cases}$$

Eliminate β to get a quadratic equation for α

DON'T REMEMBER BY HEART!

There could be two solutions for α or one

Turns out

$$\alpha = \frac{\lambda - a_{11}}{a_{21}} \; ; \; \beta = \lambda$$

where λ is an eigenvalue of the matrix A, which shows the relation with previous methods.

Example $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

Solve $y' = Ay$

$$|A - \lambda I| = \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow \lambda_1 = 3 = \lambda_2$$

$$\left[A - \lambda I \,\middle|\, 0 \right] \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & \end{array}\right] \leftarrow \text{two free parameters}$$

(10)

There is only <u>one</u> independent eigenvector

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and not a second one.

The Jordan form is

$$J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

DIY: Obtain T from J

An alternative is to do the elimination and figure out

$$\begin{cases} \beta = \lambda = 3 \\ \alpha = \dfrac{\lambda - a_{11}}{a_{21}} = \dfrac{3-2}{-1} = -1 \end{cases}$$

(11)

$$z_3' = \beta z_3 \Rightarrow z_3 = c_1 e^{\beta t}$$

$$z_3 = c_1 e^{3t}$$

$$\Rightarrow z_3 = c_1 e^{3t} = z_1 + \alpha z_2 =$$
$$= z_1 - z_2$$

$$z_1 = z_2 + c_1 e^{3t}$$

Recall one of the equations, say

$$z' = Az$$

first-order ODE

$$\begin{cases} z_1' = 2z_1 + z_2 = 3z_2 + 2c_1 e^{3t} \\ z_2' = -z_1 + 4z_2 = 3z_2 - c_1 e^{3t} \end{cases}$$

$$\Rightarrow \begin{cases} z_2 = \\ z_1 = \end{cases}$$

Try to solve the same system using the Jordan form of A (DIY) and verify you get the same solution