(4) Linear constant-coefficient systems of ODEs (homogeneous)

$$\vec{y}\,' = \overleftrightarrow{A}\,\vec{y}$$

constant matrix

Following scalar case, consider a trial (guess) solution:

$$\vec{y} = e^{\lambda t}\cdot\vec{x}\,, \quad \lambda \in \mathbb{C}$$

vector of constants

$$\vec{y}\,' = \lambda e^{\lambda t}\vec{x} = \overleftrightarrow{A} e^{\lambda t}\vec{x}$$

Since $e^{\lambda t} \neq 0$ we can divide

$$\boxed{Ax = \lambda x}$$

which is our familiar eigenvalue equation!

Every linearly independent eigenvector gives us a solution of the form $e^{\lambda t}x$.

(5) If we have n linearly independent eigenvectors, then

$$\phi = \left[e^{\lambda_1 t} x_1 ; e^{\lambda_2 t} x_2 ; \dots ; e^{\lambda_n t} x_n \right]$$

is a fundamental matrix.

If all eigenvalues are <u>distinct</u> then it is always the case that there are n linearly independent eigenvectors.

So if A has distinct eigenvectors we are done!

<u>Example</u>

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\lambda_1 = i : x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_2 = -i : x_2 = \begin{bmatrix} +i \\ 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\qquad$ $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

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$$\Phi = \begin{bmatrix} -e^{it}\, i & +e^{-it}\, i \\ e^{it} \cdot 1 & e^{-it} \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -i(\cos t + i \sin t) & +i(\cos t - i \sin t) \\ \cos t + i \sin t & \cos t - i \sin t \end{bmatrix}$$

$$= \begin{bmatrix} +\sin t & +\sin t \\ \cos t & \cos t \end{bmatrix} + i \begin{bmatrix} -\cos t & +\cos t \\ \sin t & -\sin t \end{bmatrix}$$

there are only two linearly independent columns here

so $\varphi_1 = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ $\quad \varphi_2 = \begin{bmatrix} +\sin t \\ \cos t \end{bmatrix}$

$$\Phi = \begin{bmatrix} \cos t & +\sin t \\ -\sin t & \cos t \end{bmatrix}$$

① **Complex eigenvalues** $\qquad y'(t) = Ay$

$$\begin{cases} A x = \lambda x & \text{and} \\ A \bar{x} = \bar{\lambda} \bar{x} \end{cases}$$

$\bar{x}$: complex conjugate

So λ and $\bar{\lambda}$ are eigenvalues
x and $\bar{x}$ are eigenvectors

$$\begin{cases} \lambda = a + ib & \bar{\lambda} = a - ib \\ x = v + iu & \bar{x} = v - iu \end{cases}$$

$$\begin{cases} y_1 = x e^{\lambda t} = (v + iu) e^{(a+ib)t} \\ y_2 = \bar{x} e^{\bar{\lambda} t} = (v - iu) e^{(a-ib)t} \end{cases}$$

$$\begin{cases} y_1 = \; + i u_1 e^{at} \cos bt - u_1 e^{at} \sin bt \\ y_2 = \; - i u_2 e^{at} \cos bt \quad u_2 e^{at} \sin bt \end{cases}$$

continued
$$\begin{cases} + v_1 e^{at} \cos bt + i v_1 e^{at} \sin bt \\ + v_2 e^{at} \cos bt - i v_2 e^{at} \sin bt \end{cases}$$

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$$\begin{cases} y_1 = (v\cos bt - u\sin bt)e^{at} \\ \qquad + i(v\sin bt + u\cos bt)e^{at} \\ y_2 = (v\cos bt - u\sin bt)e^{at} \\ \qquad - i(v\sin bt + u\cos bt)e^{at} \end{cases}$$

Therefore

$$\begin{cases} y_1 = \varphi_1 + i\varphi_2 \\ y_2 = \varphi_1 - i\varphi_2 \end{cases}$$

where
$$\begin{cases} \varphi_1 = (v\cos bt - u\sin bt)e^{at} \equiv y_1 \\ \varphi_2 = (v\sin bt + u\cos bt)e^{at} \equiv y_2 \end{cases}$$

<u>better</u> choice of linearly independent solutions for real-valued solutions

this is related to the fact that

$$A = PCP^{-1}$$

where $C = \begin{bmatrix} a & +b \\ -b & a \end{bmatrix}$

and $P = \begin{bmatrix} v & u \end{bmatrix}$

Example

(3) Going back to earlier example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$a = 0 \qquad b = 1$$

$$v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\varphi_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin t$$

$$= \begin{bmatrix} +\sin t \\ \cos t \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos t =$$

$$= \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

$$\phi = \left[\begin{array}{c|c} +\sin t & -\cos t \\ \hline \cos t & \sin t \end{array}\right]$$ as before