1

Linear Stability

We consider the autonomous system

$$\vec{y}' = \vec{f}(\vec{y}) \qquad , \vec{y} \in \mathbb{R}^n$$

Let y_0 be a fixed point or critical point

$$\vec{f}(\vec{y_0}) = \vec{0}$$

and let's assume that y_0 is an isolated critical point, which means in the neighborhood (vicinity) of y_0 there are no other critical points.

Note that without loss of generality we can assume

$$\vec{y_0} = \vec{0}$$

To see this, assume $y_0 \neq 0$ and define a new variable

$$z(t) = y(t) - y_0$$

2

Then

$$z' = y' = f(y) = f(y) - f(y_0)$$

$$z' = f(z + y_0)$$

and now $z_0 = 0$ is the transformed critical point

We are interested in studying the behavior of solutions under small perturbations.

Stability

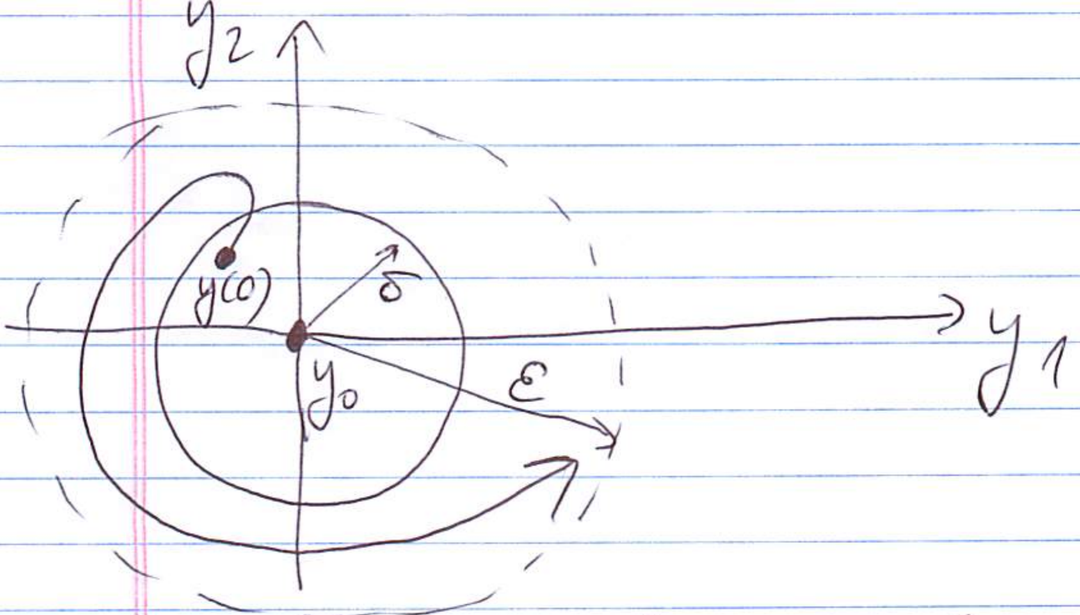
Let's first look at what the solutions do if we start from an initial condition

$$y(0) = y_0 + \delta$$

where δ is a small perturbation.

Will the solution remain near y_0 or maybe converge to y_0 as $t \to \infty$?

Definition: Stable point



A fixed point y_0 of $y' = f(y)$ is stable if for every

$\varepsilon > 0$

we can find a $\delta(\varepsilon)$

$\delta > 0$

such that any solution $y(t)$ starting near y_0,

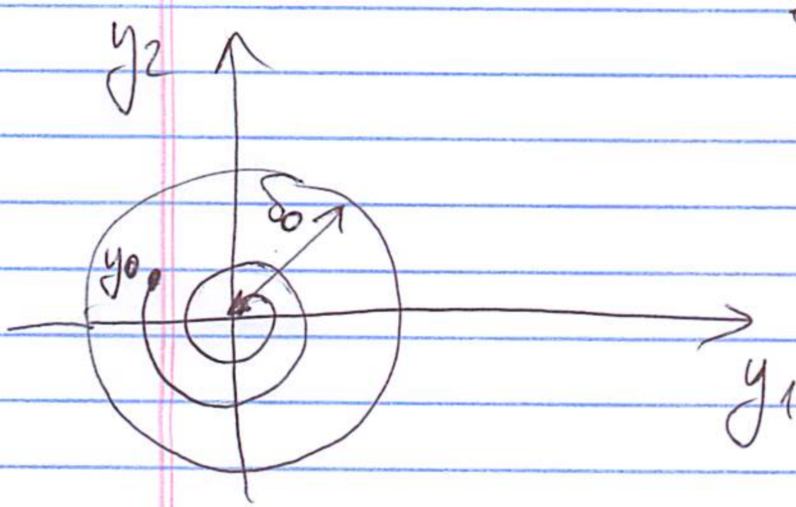
$$\| y(0) - y_0 \| < \delta$$

the solution exists for all $t \geq 0$ and remains close to y_0

$$\| y(t) - y_0 \| < \varepsilon$$

(4)

Definition: Asymptotically stable fixed point



A fixed-point y_0 is said to be asymptotically stable if it is stable and if there exists a

$$\delta_0 > 0$$

such that solutions starting closer to y_0 than a distance δ_0 will tend toward y_0:

if $\|y(0) - y_0\| < \delta_0$ then

$$\lim_{t \to \infty} y(t) = y_0$$

We call y_0 an <u>attractor</u> of the linear system

(5)

The set of initial conditions $y(0)$ such that $y(t)$ tends toward a particular stable point y_0 is called the <u>region of asymptotic stability</u> of y_0.

Linear Systems

$$y' = Ay \quad , \quad A \in \mathbb{R}^{n \times n} \text{ constant}$$

As we saw, the fundamental solutions have the form

$$\varphi_k(t) = x_k e^{\lambda_k t}$$

with possibly polynomial corrections if A is not diagonalizable.

$$\lambda_k = a + ib$$

$$e^{\lambda_k t} = e^{at}(\cos bt + i \sin bt)$$

It is clear the sign of a controls stability (decay or growth of solutions)

Theorem

If all eigenvalues of A have non positive real parts, and the purely imaginary eigenvalues have algebraic multiplicity of one (simple eigenvalues), then $y=0$ is a stable point. Furthermore, If all eigenvalues have negative real part, $y=0$ is asymptotically stable

If one or more eigenvalues have a positive real part, $y=0$ is unstable

Note: We will not consider non-simple purely imaginary eigenvalues

Example:

$$y'' + 2ky' + y = 0 \;, \quad k > 0$$

Define new variable $z \in \mathbb{R}^2$

$$z_1 = y' \;, \quad z_2 = y$$

$$z_1' = y'' = -2ky' - y = -2k z_1 - z_2$$

$$z_2' = y' = z_1$$

$$\Rightarrow \quad z' = \begin{bmatrix} -2k & -1 \\ 1 & 0 \end{bmatrix} z$$

$$A = \begin{bmatrix} -2k & -1 \\ 1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2k-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda(2k+\lambda) + 1$$

$$= +\lambda^2 + 2k\lambda + 1$$

$$\lambda = -k \pm \sqrt{k^2 - 1}$$

$$\left\{ \begin{array}{l} \text{If } k > 1, \quad \lambda_1, \lambda_2 < 0 \\ \text{If } 0 < k < 1, \quad Re(\lambda_1, \lambda_2) < 0 \end{array} \right\} \text{asymptotically } \underline{\text{stable}}$$

If $k < 0$, $Re(\lambda_1, \lambda_2) > 0$, unstable

If $k = 0$, $\lambda = \pm i$, stable

(8)

The theorem can be generalized to allow perturbations to the equation

$$\vec{y}\,' = (\overleftrightarrow{A} + \overleftrightarrow{B}(t))\vec{y}$$

where $B(t)$ is a matrix that decays:

$$\lim_{t\to\infty} B(t) = 0$$

Theorem: If $Re(\lambda) < 0$ for all eigenvalues of A the origin is a (globally) asymptotically stable

Note however that the theorem <u>does not</u> generalize to non-constant coefficient case

$$y' = A(t)\, y$$

Here the eigenvalues of $A(t)$ are not an indicator of stability (see homework 7, p3)