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Linear systems of ODEs

$$\begin{cases} y_1' = 3y_1 \\ y_2' = 5y_1 - 3y_2 + 2 \end{cases}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{y}\,'(t) = \overleftrightarrow{A}\,\vec{y} + \vec{g}$$ ← constant-coefficient first-order linear system of ODEs

More generally,

$$\vec{y}\,'(t) = \overleftrightarrow{A}(t)\,\vec{y}(t) + g(t)$$

↑ non-constant coefficient

If $g \equiv 0$ it is homogeneous

E.g.

$$\begin{cases} y_1' = y_2 + t^4 \\ y_2' = -\frac{2}{t^2} y_1 + \frac{2}{t} y_2 + t^3 \end{cases}$$

(2)

$$\Rightarrow \quad A(t) = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix} \quad g(t) = \begin{bmatrix} t^4 \\ t^3 \end{bmatrix}$$

Existence & uniqueness theorem

If $A(t)$ and $g(t)$ are continuous on I, then the linear system of ODEs

$$y' = Ay + g$$

has a unique solution for any initial condition $y(t_0) = y_0$, where $t_0 \in I$ is any point

The solution of the inhomogeneous equation can be obtained as

$$y = y_h + y_p$$

where $y_h(t)$ solves the homogeneous

$$y_h' = A y_h$$

and y_p is any particular solution of the inhomogeneous equation, 4 Homogeneous systems

$$\vec{y}'(t) = \overleftrightarrow{A}(t)\, y(t) \quad , y \in \mathbb{R}^n$$

There exists a set of n linearly-independent solutions

$$\{ y_1, y_2, \ldots, y_n \}$$

called the fundamental set of solutions, and every solution can be written as a linear combination of these,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t)$$

for a unique choice of the constants $c_1, c_2, \ldots, c_n$.

Put $\{y_1, \ldots, y_n\}$ as columns of a fundamental solution matrix

$$\overleftrightarrow{Y}(t) = [\, y_1 \mid y_2 \mid \cdots \mid y_n \,]$$

$$\Rightarrow \boxed{\vec{y}(t) = \overleftrightarrow{Y}(t)\, \vec{c}}$$

where $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is a column vector

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A matrix $Y(t)$ is a <u>fundamental matrix</u> if every column of it is a solution to the system of ODEs, and

$$\det Y(t) = |Y(t)| \neq 0$$

for all $t \in I$

This means the $y_i(t)$, $i = 1, 2, \dots, n$, are <u>linearly independent</u> solutions.

<u>Example</u>

Consider $A = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix}$, $t > 0$

and see if

$$Y = \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix}$$

is a fundamental matrix?

y_1:

$$A \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix} = \begin{bmatrix} 2t \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} t^2 \\ 2t \end{bmatrix}' = \begin{bmatrix} 2t \\ 2 \end{bmatrix}$$

so $y_1 = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ is a solution

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Similarly, $y_2 = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is a solution

Now,

$$|Y(t)| = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - 2t^2 = -t^2$$
$$\neq 0 \text{ if } t \neq 0$$

So yes, $Y(t)$ is a fundamental matrix and every solution has the form

$$y_c = \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 t^2 + c_2 t \\ 2c_1 t + c_2 \end{bmatrix}$$

for some c_1 and c_2 determined from the initial condition, e.g.,

$$y_0 = y(t_0 = 1) = \begin{bmatrix} c_1 + c_2 \\ 2c_1 + c_2 \end{bmatrix} = \begin{bmatrix} (y_0)_1 \\ (y_0)_2 \end{bmatrix}$$

Note: This system of linear equations for c_1 and c_2 always has a unique solution because $|Y_0| = |Y(t_0)| \neq 0$!

(7) Consider the second-order linear ODE

$$t^2 y'' + a_1 t y' + a_2 y = 0 \quad \boxed{\text{Euler}}$$

and define $\vec{y} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$

$$\vec{y}\,' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ -\frac{a_1}{t} y' - \frac{a_2}{t^2} y \end{bmatrix}$$

$$\Rightarrow \quad \vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -\frac{a_2}{t^2} & -\frac{a_1}{t} \end{bmatrix} \vec{y}$$

Let us assume we know two linearly-independent solutions of the ODE, y_1 and y_2.

$$\overleftrightarrow{Y} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

and $|\overleftrightarrow{Y}|$ is the Wronskian, which is not zero, so $\overleftrightarrow{Y}$ is a fundamental matrix