(1) Non-homogeneous

Linear systems $y' = Ay + g$

Just as for scalar ODEs, if we know how to solve the homogeneous system we can solve the non-homogeneous case by the method of variation-of-constants

ALERT: Change of notation:

$\varphi_1(t), \ldots, \varphi_n(t)$ — linearly independent solutions of homog.

$\phi = [\varphi_1 ; \ldots ; \varphi_n]$ — fundamental solution matrix

columns

$A\phi = [A\varphi_1 ; \ldots ; A\varphi_n]$

columns of ϕ times A

$= [\varphi_1' ; \ldots ; \varphi_n']$

$$\Rightarrow \boxed{\phi' = A\phi}$$

Let $y = \phi c$ be the solution for some constants c

2 Particular solution

$$y_p' = A y_p + g$$

Try variation of constants

$$y_p = \phi\, v(t)$$

$$y_p' = \phi v' + \phi' v = \phi v' + \cancel{A\phi v} = A y_p + g = \cancel{A\phi v} + g$$

cancel

cancel

$\Rightarrow \quad \phi v' = g$ and ϕ is invertible

$$\Rightarrow \quad \boxed{v' = \phi^{-1} g}$$

$$v(t) = \int_{s=t_0}^{t} \phi^{-1}(s)\, g(s)\, ds$$

Recall $\phi^{-1} g$ means solve $\phi x = g$

NOT invert ϕ !

(3) Theorem $y' = A(t)y + g(t)$

If ϕ is a fundamental matrix of $y' = Ay$ on I, then the unique solution on I has the form

$$y = \phi c + y_p$$
$$= \phi c + \phi v(t)$$

where $v' = \phi^{-1} g$, for some vector of constants c

This is just like for scalar equations, which are a special case of the above.

The only challenge now is to solve the homogeneous equation $y' = Ay$.

Start with constant A...

(4) **Example**: (non-homogeneous linear)

$$A = \begin{bmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{bmatrix}, \quad t>0$$

$$\phi = \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \quad \text{from earlier}$$

$$g = \begin{bmatrix} t^4 \\ t^3 \end{bmatrix}$$

and $\vec{y}(2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Find $\vec{y}(t)$

Solution:

First solve

$$\phi v' = g$$

$$\begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 \end{bmatrix}$$

(5)

$$v' = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^3 \end{bmatrix}$$

$$v = \begin{bmatrix} \int 0\,dt \\ \int t^3\,dt \end{bmatrix} = \begin{bmatrix} 0 \\ t^4/4 \end{bmatrix}$$

$$y_p = \Phi v = \begin{bmatrix} t^5/4 \\ t^4/4 \end{bmatrix}$$

$$y = \Phi c + y_p \Rightarrow$$

$$\Phi c = y - y_p$$

Evaluate at initial point $t=2$

$$\Phi(t=2) = \begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix}$$

$$y_p(t=2) = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad y(t=2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Phi c = \begin{bmatrix} -7 \\ 0 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 7/4 \\ -7 \end{bmatrix}$$

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$$y = \phi c + y_p =$$

$$= \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \begin{bmatrix} 7/4 \\ -7 \end{bmatrix} + \begin{bmatrix} t^5/4 \\ t^4/4 \end{bmatrix}$$

$$= \begin{bmatrix} t^5/4 + 7/4\, t^2 \quad 7\, t \\ t^4/4 + 7/2\, t - 7 \end{bmatrix}$$