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Nonlinear Stability

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Mean value theorem review +

Consider a scalar function
$$f: \mathbb{R}^n \to \mathbb{R} \quad \text{(differentiable)}$$
and define the gradient of the function
$$\vec{\nabla} f(\vec{x}) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$
Also define the dot product of two vectors as
$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$$
The mean-value theorem states that there is a
$$0 \leq c \leq 1$$
such that
$$f(\vec{y}) - f(\vec{x}) = \vec{\nabla} f\left((1-c)x + cy\right) \cdot (\vec{y} - \vec{x})$$
$$= \vec{\nabla} f(\vec{z}) \cdot (\vec{y} - \vec{x})$$

where $\vec{z} = (1-c)\vec{x} + c\vec{y}$ is a point on the line segment connecting $\vec{x}$ and $\vec{y}$

A closely-related statement is that

$$f(\vec{x}+\Delta\vec{x}) - f(\vec{x}) = \vec{\nabla} f(\vec{x}) \cdot \Delta\vec{x} + g(\Delta\vec{x})$$

first term in Taylor series

where $g(\vec{x})$ is a <u>Remainder term</u> that can be shown to be continuous and

$$g(\vec{0}) = \vec{0}$$

$$\lim_{\Delta x \to 0} \frac{\| g(\Delta\vec{x}) \|}{\| \Delta\vec{x} \|} = 0$$

which means $g(\Delta\vec{x})$ is usually quadratic in Δx, ~~smaller~~ or at least sub-linear

(3) Linearized ODEs

Take the ODE

$$\vec{y}\,' = \vec{F}(\vec{y})$$

where $\vec{F}$ is continuous and has continuous first partial derivatives in the domain of interest

Let $\vec{y}_0 = \vec{0}$ be a fixed point, $\vec{F}(\vec{y}_0) = \vec{0}$

Then

$$y_k'(t) = F_k(\vec{y}) - F_k(\vec{0})$$

remainder

mean value theorem

$$\rightarrow \; = \left(\vec{\nabla} F_k(\vec{0})\right) \cdot \vec{y} + g_k(\vec{y})$$

If we collect the gradients of $F_1, F_2, \dots, F_n$ as rows of a matrix

$$J_{kk'} = \frac{\partial F_k}{\partial x_k'}$$

← Jacobian matrix

(4) e.g. $\vec{f}(y_1, y_2) = \begin{bmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{bmatrix}$

$$\overleftrightarrow{J} = \vec{\nabla}\vec{f} = \begin{bmatrix} \partial f_1/\partial y_1 & \partial f_1/\partial y_2 \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 \end{bmatrix}$$

In compact matrix notation

$$\vec{y}\,'(t) = \underbrace{\left(\vec{\nabla}\vec{F}(\vec{0})\right)}_{\overleftrightarrow{J}_0}\cdot\vec{y} + \vec{g}(\vec{y})$$
$$= \overleftrightarrow{J}_0 \cdot \vec{y} + \vec{g}(\vec{y})$$

which is of the form

$$y' = Ay + g(y)$$

with A being a constant matrix and $\vec{g}$ being a small "perturbation", meaning

$$\lim_{\|\vec{y}\| \to 0} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|} = 0$$

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Theorem: $y' = Ay + g(t,y)$

If g is small for all t and if <u>all</u> eigenvalues of A have negative real parts, and $\vec{g}$ and $\vec{\nabla}\vec{g}$ are continuous, then $\vec{y}_0 = \vec{0}$ is a stable point (solution)

Example

Swinging pendulum equation:

air friction

$$\theta'' + \gamma\theta' + \omega^2 \sin\theta = 0$$

where $\theta(t)$ is the angle of the pendulum. Define

$$\theta = y_1 \qquad \theta' = y_2$$

$$\begin{cases} y_1' = \theta' = y_2 \\ y_2' = \theta'' = -\gamma\theta' - \omega^2 \sin\theta = -\gamma y_2 - \omega^2 \sin y_1 \end{cases}$$

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So $\quad y' = F(y)$

$$\begin{cases} F_1 = y_2 \\ F_2 = -\gamma y_2 - \omega^2 \sin y_1 \end{cases}$$

$$\overleftrightarrow{J} = \vec{\nabla} \vec{F} = \begin{bmatrix} \partial F_1 / \partial y_1 & \partial F_1 / \partial y_2 \\ \partial F_2 / \partial y_1 & \partial F_2 / \partial y_2 \end{bmatrix}$$

$$\overleftrightarrow{J} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos y_1 & -\gamma \end{bmatrix}$$

$$\overleftrightarrow{J}_0 = \overleftrightarrow{J}(\vec{0}) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}$$

with eigenvalues

$$\lambda_{1/2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2}$$

$Re(\lambda_{1/2}) < 0 \quad$ if $\omega > 0$

rest

and therefore $\theta = 0$ is a stable point $(y_1 = \theta = 0, y_2 = \theta' = 0)$

The remainder terms here are

$$g_1 = 0$$

$$g_2 = \omega^2(\sin\theta - \theta) = \omega^2\left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots\right)$$

which is cubic in θ

Is the fixed point

DIY: $\boxed{\theta = \pi, \ \theta' = 0}$

stable? (inverted pendulum)

Note: there is no claim here that the phase portrait of the perturbed system looks the same as the linearization $y' = Ay$

E.g., a center can become a spiral point