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Review of Linear Systems

$$\vec{y}'(t) = \overleftrightarrow{A}(t)\,\vec{y}(t) + \vec{g}(t) \qquad \in \mathbb{R}^n$$

Has unique solution for all initial conditions over the whole interval I if $A(t)$ and $g(t)$ are continuous

(A) Homogeneous:

$$y'(t) = A(t)\,y(t) \quad \Rightarrow$$

$$\vec{y} = \overleftrightarrow{\Phi}\,\vec{c} \leftarrow \text{integration constants}$$

fundamental matrix

where $\Phi = [\varphi_1 | \varphi_2 | \dots | \varphi_n]$

check → and each of the φ_k's is a solution and the columns
check → of Φ are linearly independent

$$\det \Phi(t) = |\Phi(t)| \neq 0$$

for all $t \in I$

In general, there is no simple way to find $\Phi(t)$.

(2) Note: $\Phi' = A\Phi$

We can determine $\vec{c}$ from the initial condition

$$y(0) = \Phi(0)\,c \leftarrow \text{Linear system for } c \text{ (has solution!)}$$

<u>Note</u>: Higher order equations can be written as a system of first-order equations

(B) <u>Non-homogeneous</u>

Solve $\overleftrightarrow{\Phi}\,\vec{v}\,' = \vec{g}$ for $\vec{v}\,' \in \mathbb{R}^n$

then integrate each entry (indefinite integral is fine) to get $\vec{v}(t)$. Then

$$y_p = \phi\, v \leftarrow \text{particular solution}$$

$$y = \Phi c + y_p = \Phi(c + v(t))$$

where the integration constants $\vec{c}$ can be determined from the initial conditions

(3) <u>Constant-coefficient</u>

$$y' = Ay$$

$$y = \phi c = \exp[At]\, c$$

$$\phi \equiv \exp[At] \quad \text{where} \quad \exp[0] = I$$

this is a definition of the <u>matrix exponential</u> (or series)

How to find ϕ?

Find eigenvalues of A by solving the polynomial equation

$$|A - \lambda I| = 0$$

roots $\lambda_1, \lambda_2, \ldots, \lambda_n$

For <u>each</u> <u>distinct</u> λ, find a set of eigenvectors (linearly independent) by solving

$$(A - \lambda I)x = 0 \quad \leftarrow \text{do it by hand}$$

[Note: will have at least one non-zero solution if λ is an eigenvalue]

(4)

The number of linearly-independent eigenvectors is the number of free parameters you get

case i) If there exist n linearly independent eigenvectors
[Note: This __will__ be the case if
$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$ are distinct]
then

$$\phi = \left[x_1 \cdot e^{\lambda_1 t} \,\middle|\, x_2 e^{\lambda_2 t} \,\middle|\, \ldots \,\middle|\, x_n e^{\lambda_n t} \right]$$

If the λ's are complex, it is possible to write a purely real $\phi(t)$. It will have terms of the form

$$\begin{cases} e^{at} \cdot \sin(bt) \text{ and} \\ e^{at} \cdot \cos(bt) \end{cases}$$

where $\lambda = a \pm ib$

[We wrote a specific formula for ~~##~~ ~~####~~ two variables]

(5)

Case ii) If there are not n linearly independent eigenvectors, we need the Jordan form:

$$A = T J T^{-1}$$

where J is a block-diagonal Jordan matrix with the λ's on the diagonal. The number of Jordan blocks is equal to the number of linearly-independent eigenvectors (one block per eigenvector)

First, solve

$$z' = J z$$

block-by-block. For each block, use backward substitution:

$$z' = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} z \quad \Rightarrow \left\{ \begin{array}{l} \dot{z}_1 = \lambda z_1 + z_2 \\ \dot{z}_2 = \lambda z_2 + z_3 \\ \dot{z}_3 = \lambda z_3 \end{array} \right.$$

Solve for z_3 first, then z_2, and finally z_1.

<u>Note</u>: The fundamental matrix for $z' = J z$ has the form

$$\phi_z = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} e^{\lambda t} \quad \text{for } n=2$$

$$\phi_z = \begin{bmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{bmatrix} e^{\lambda t} \quad \text{for } n=3$$

or in general

$$\phi_z = \begin{bmatrix} 1 & t & \cdots & t^k/k! \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{bmatrix}$$

But you do <u>not</u> need to remember this.

Finally, from $z(t) = \phi_z(t)\, c$, obtain

$$y = T z = T \phi_z c = \phi c$$

where $\phi = T \phi_z$ is the fundamental matrix

~~Note: For nxn equations~~

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$$y' = Ay \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

For a system of 2 equations, if there is only one linearly independent eigenvector, we can use a simpler procedure.

Define: $y_3 = y_1 + \alpha y_2$

such that $y_3' = \beta y_3$

<u>Note</u>: $\begin{cases} a_{11} + \alpha a_{21} = \beta \\ a_{12} + \alpha a_{22} = \alpha \beta \end{cases}$

Solve for α and β

Once you know $y_3 = c_1 e^{\beta t}$

substitute $y_1 = y_3 - \alpha y_2$ into one of the equations and solve for y_2, then $y_1 = y_3 - \alpha y_2$

Let's say you get

$$\begin{cases} y_1 = c_1 e^{\beta t} + 5c_2 t e^{\beta t} \\ y_2 = -3c_1 e^{\beta t} - 6c_2 t e^{\beta t} \end{cases}$$

$y = \Phi c$ where $\Phi = \begin{bmatrix} 1 & 5t \\ -3 & -6t \end{bmatrix} e^{\beta t}$