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Review of phase-space analysis

A fixed point y_0 of an autonomous system

$$\vec{y}\,' = \vec{g}(\vec{y}) \leftarrow \text{no } t \text{ dependence}$$

is a solution to

$$\vec{g}(\vec{y}_0) = \vec{0}$$

Note that

$$\vec{y}(t) = \vec{y}_0$$

is a solution of the ODE.

Assume that y_0 is an <u>isolated</u> critical point and without loss of generality assume

$$\vec{y}_0 = \vec{0}$$

A fixed point is <u>stable</u> if any solution that starts sufficiently close to y_0 remains close to y_0 and <u>exists</u> for all time.

More precisely: for <u>any</u> $\varepsilon > 0$:

stable $\begin{cases} \text{If} & \|y(0) - y_0\| < \delta(\varepsilon) > 0 \\ \text{then} & \|y(t) - y_0\| < \varepsilon > 0 \end{cases}$

If <u>furthermore</u>

asympt. stable $\begin{cases} \lim\limits_{t\to\infty} y(t) = y_0 \\ \text{whenever} \quad \|y(0) - y_0\| < \delta_0 > 0 \end{cases}$

then y_0 is <u>asymptotically stable</u>

If y_0 is not stable it is <u>unstable</u>.

<u>Example</u>

$$y' = -y^2$$

$$\frac{dy}{dx} = -y^2 \Rightarrow \frac{dy}{y^2} = -dx$$

$$\cancel{-}\frac{1}{y} = \cancel{-}(x+c) \Rightarrow y = \frac{1}{x+c}$$

Assume $y(0) = \delta \Rightarrow y = \dfrac{\delta}{1 + \delta x}$

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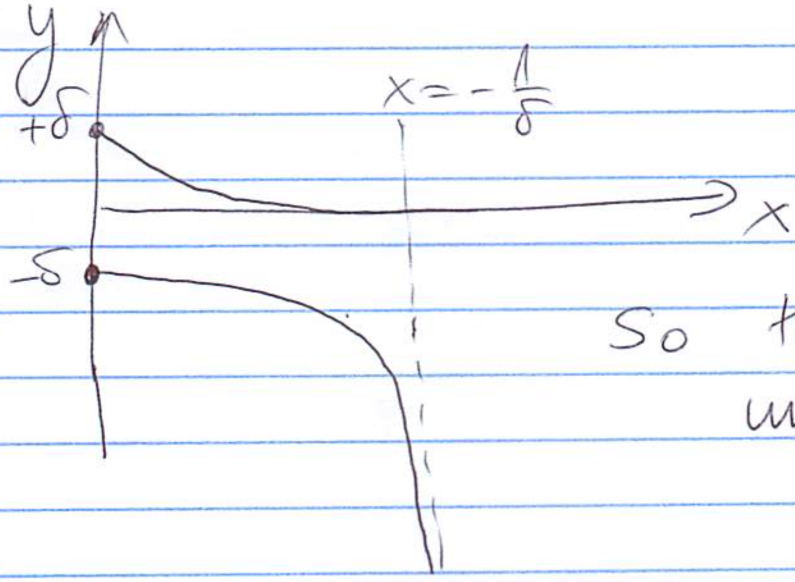
If $\delta > 0$, $y \to 0$ as $x \to \infty$

If $\delta < 0$, when

$1 + \delta x = 0 \Rightarrow x = -\frac{1}{\delta}$, $y \to -\infty$



So this is unstable

Linear Systems

$$y' = Ay \quad , y \in \mathbb{R}^n$$

Th: If $\forall \lambda$ (eigenvalues of A)

$\begin{cases} \mathrm{Re}(\lambda) < 0 \\ \text{or } \mathrm{Re}(\lambda) = 0 \text{ but } \lambda \text{ is} \\ \text{a simple (multiplicity 1) eigenvalue} \end{cases}$

then $\vec{y} = \vec{0}$ is a stable point

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Furthermore if $\forall \lambda$

$\text{Re}(\lambda) < 0$

then asymptotically stable

If $\exists \lambda$ s.t. $\text{Re}(\lambda) > 0$

$y = 0$ is unstable

Non-linear systems

$$\vec{y}\,' = \vec{F}(\vec{y})$$

Linearize the system around

$$\vec{y}_0 = 0$$

$$\vec{y}\,' = \overleftrightarrow{J}_0 \vec{y} + \vec{g}(\vec{y})$$

where $\overleftrightarrow{J}_0$ is the Jacobian matrix:

$$\overleftrightarrow{J} = \vec{\nabla} \vec{f} = \begin{bmatrix} \partial f_1/\partial y_1 & \partial f_1/\partial y_2 \\ \partial f_2/\partial y_1 & \partial f_2/\partial y_2 \end{bmatrix}$$

$$\boxed{J_0 = J(y = y_0)}$$

and $\vec{g}(\vec{y}) = o(\|\vec{y}\|)$ is small

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Theorem:

If for all eigenvalues λ of $\overleftrightarrow{J}_0$

$$Re(\lambda) < 0$$

then y_0 is an asymptotically stable point

If $\exists \lambda$ s.t. $Re(\lambda) > 0$

then y_0 is unstable

Example (Lotka-Volterra eqs)

$$\begin{cases} y_1' = 2y_1 - y_1 y_2 = f_1 \\ y_2' = -9y_2 + 3y_1 y_2 = f_2 \end{cases}$$

First, determine critical points (fixed)

$$2y_1 = y_1 y_2 \Rightarrow \begin{cases} y_2 = 2 \\ \text{or } y_2 = 0 \end{cases}$$

$$-9y_2 = 3y_1 y_2 = \begin{cases} y_1 = -3 \\ \text{or } y_1 = 0 \end{cases}$$

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So we have two isolated fixed points

$$\begin{cases} y_1 = y_2 = 0 \\ \text{and } y_1 = 3, \; y_2 = 2 \end{cases}$$

$$J = \left[\begin{array}{c|c} 2 - y_2 & -y_1 \\ \hline 3y_2 & 3y_1 - 9 \end{array}\right]$$

a) For $y_0 = (0, 0)$

$$J(y_0) = J_0 = \left[\begin{array}{c|c} 2 & 0 \\ \hline 0 & -9 \end{array}\right]$$

so $\lambda_1 = 2$, $\lambda_2 = -9$

Since $\lambda_1 > 0$ it is unstable

b) For $y_0 = (3, 2)$

$$J_0 = \left[\begin{array}{c|c} 0 & -3 \\ \hline 6 & 0 \end{array}\right]$$

$\lambda = \pm i\sqrt{18}$

we cannot say anything here