Partial Differential Equations, Spring 2020
Homework VI: Separation of Variables Continued

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Due by **11am Thursday April 30th**, 2020

Total number of points is 110.

1. [15pts] Solve the PDE for $0 < x < 1$ and $t > 0$,

$$\begin{aligned} u_t &= 2u_{xx} \\ u(0,t) &= 0 \\ u(1,t) &= e^{-t} \\ u(x,0) &= e^{-x}. \end{aligned}$$

If possible, check that your solution satisfies all equations. [Hint: *Observe that the inhomogeneous BC depends on time.*]

2. [15pts] Write the solution of the PDE for $-\pi < x < \pi$ and $t > 0$,

$$\begin{aligned} u_t &= u_{xx} + g(x) \\ u(-\pi,t) &= 0 \\ u(\pi,t) &= 0 \\ u(x,0) &= 0. \end{aligned}$$

Make sure you *simplify as much as possible* by using the fact that the source term g depends only on x and not on t.
What is the solution at long time $\lim_{t\to\infty} u(x,t)$ if $g(x) = (x+\pi)(x-\pi)$? [*Hint: You can obtain this by solving a single (trivial) ODE.*]

3. [15 pts] Use the separable anzatz $u(x,t) = f(t)g(x)$ for the Sturm-Liouville problem

$$u_t = (a(x)u_x)_x - b(x)u$$

to derive two ODEs for $f(t)$ and $g(x)$. List all steps in the separation of variables procedure.

4. [10pts] Rewrite the ODE

$$u'' - 2u' + u = x^3$$

as a Sturm-Liouville problem $-(p(x)u_x)_x + q(x)u = f(x)$.

5. [15pts] Consider the second-order differential operator defined on the interval $-1 < x < 1$,

$$\mathcal{L}u = a(x)(\partial_{xx}u) + b(x)(\partial_x u) + c(x)u,$$

with boundary conditions $u(-1) = u(1) = 0$ and also $u_x(-1) = u_x(1) = 0$. Derive the adjoint operator $\mathcal{L}^\star$. [Hint: *Do the calculation without using the BCs as much as possible, and then use the BCs at the end.*]

6. [25pts] Solve the Poisson equation in a rectangle $0 < x < \pi$, $0 < y < \pi$,

$$\begin{aligned} u_{xx} + u_{yy} &= \cos(2x)\sin(5y) \\ u_x(0,y) = u_x(\pi,y) &= 0 \\ u(x,0) &= 0 \\ u(x,\pi) &= \frac{1}{2}x(x-\pi) \end{aligned}$$

7. [15 pts] Solve the heat equation in two dimensions

$$u_t = \nabla^2 u = u_{xx} + u_{yy}$$

for $u(x, y, t)$ on the whole $x - y$ plane with the initial condition

$$u(x, y, t = 0) = \exp\left[-\frac{1}{2}\left(x^2 + y^2\right)\right].$$

Hint: This problem can be solved by doing separation of variables in space only (recommended!), or, by using polar cordinates (hard!). It can also be done by using Green's functions or using Fourier integrals. Optional something to ponder: Could you solve this problem if the initial condition was $\exp\left[-\left(\frac{1}{2}x^2 + y^2\right)\right]$ instead? How about $\left(1 + x^2 + y^2\right)^{-1}$? The most powerful method is the one that can handle the most initial conditions.