

PDE Spring 2016

A. DORNER

(1)

Lecture 14

Fourier / Orthogonal Series

In order for the method of separation of variables to work, we must be able to expand the initial condition in a Fourier series

$$0 \leq x \leq l$$

↙ basis functions

$$\Psi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

↑ Fourier sine series

(for heat equation with homogeneous Dirichlet conditions)

This opens a number of questions from (harmonic) analysis: (2)

- ① For what $\psi(x)$ is this possible?
- ② If possible, how do we find the coefficients A_n
- ③ What does the equal sign really mean, since the sum is infinite
- ④ Is this approach restricted to heat equation or general?

It turns out we will get most understanding and depth from #4

Recall that the way we got the sin functions was not arbitrary - we solved an eigenvalue BVP. So let us first generalize linear algebra to functions.

$$\mathcal{L} = \partial_{xx}$$

$$\mathcal{L}X = \lambda X$$

$$X'' = \lambda X + X(0) = X(0) = 0$$

Vector Function Spaces

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Function spaces are infinite dimensional vector spaces where the role of vectors is played by functions

One can also think of them as classes of functions closed under scalar multiplication and addition.

Examples: of vector spaces of functions

∧ (1) Space of all polynomials of degree n , P_n $\dim(P_n) = n$

∧ (2) Space of all polynomials P
 $P_n \subset P$

∧ (3) Space of all continuous functions C^0

∧ (4) Continuously differentiable functions C^1

∧ (5) Twice contin. diff. C^2

∧ (6) Infinitely differentiable C^∞

∧ (7) Analytic functions

smooth

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We consider here functions defined on interval $[a, b]$

Inner product for functions

$$(f, g)_{L_2} = \int_a^b f(x) \overline{g(x)} dx$$

$a \cdot b = \sum a_i \cdot \overline{b_i}$ \rightarrow a \rightarrow $\overline{g(x)}$ complex conjugate

Norms for functions

$$L_\infty: \|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$$

or max norm

$$L_1: \|f\|_1 = \int_a^b |f(x)| dx$$

Manhattan norm

$$L_2: \|f\|_2 = \left[\int_a^b |f(x)|^2 dx \right]^{1/2}$$

Eucledian norm

\uparrow $\sqrt{\text{Im}(f)^2 + \text{Re}(f)^2}$

Note that these norms are all distinct (unlike for vectors where they are an "arbitrary" choice)

A basis set for a linear (5) function space is a collection of functions such that

$$\forall f \in \mathcal{V}$$

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$\mathcal{V} = \mathcal{P}_N$$

N

Basis is $\{f_1(x), f_2(x), \dots\}$

Example

Space of polynomials of degree n is spanned by the monomial basis

$$\{x^0, x^1, x^2, x^3, \dots, x^n\}$$

Two functions are orthogonal if $(f_1, f_2) = 0$

An orthonormal basis satisfies

$$(f_i, f_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Essentially all of the linear algebra review we did for matrices (finite-dimensional vector spaces) generalizes to linear operators (infinite-dimensional operators).

Key differences

- ① The number of elements (vectors) in the basis sets will be countably infinite rather than finite - so there is no equivalent of $\mathbb{R}^n / \mathbb{C}^n$
- ② ~~There~~ There is no generic vector space / basis (as is \mathbb{R}^n for n-dimensional spaces), rather, there are different functional spaces and all norms are different.
- ③ Therefore, to really specify a PDE one must specify the function space / inner product / norm one is interested in.

$$\mathcal{L}u = f \quad \|\mathcal{L}u - f\| = 0$$

Theorem

So far: $f \in \mathcal{V}$
 $\mathcal{V} \equiv C^\infty(C^2)$ 7
 $\|\cdot\|_{L^\infty}$ pointwise

The Laplacian with ^{homog.} Dirichlet/Neumann BCs ~~is~~ is a Hermitian or self-adjoint operator

$$\mathcal{L}^* = \mathcal{L} \Rightarrow (f, \mathcal{L}g) = (\mathcal{L}^*f, g)$$

$$\begin{cases} \mathcal{L} = \partial_{xx} \\ \mathcal{L}f \\ f(0) = f(l) = 0 \end{cases}$$

More precisely, let us consider the space of all C^2 functions (twice continuously differentiable) that satisfy one of the conditions

$$\begin{cases} \overline{X}(0) = 0 & \text{or} & \overline{X}(l) = 0 \\ \overline{X}_x(0) = 0 & & \overline{X}_x(l) = 0 \end{cases}$$

Now consider the Laplacian linear operator acting on functions in this vector space.

Recall definition of self-adjoint:

$$(X_1, \mathcal{L}X_2) = (\mathcal{L}X_1, X_2)$$

$$\mathcal{L} \equiv \partial_{xx}$$

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Let us work with real numbers for now (not crucial but we will see it is sufficient)

$$\begin{aligned}(X_1, X_2) &= \int_0^l X_1 X_2'' dx = \\ &= - \int_0^l X_1' X_2' dx + [X_1 X_2']_0^l \\ &= \int_0^l X_1'' X_2 dx + [X_1' X_2]_0^l \\ &\quad + [X_1 X_2']_0^l\end{aligned}$$

↑ integrate by parts
↑ integrate by parts again

Therefore

$$\int_0^l (X_1 X_2'' - X_1'' X_2) dx = [X_1 X_2' - X_1' X_2]_0^l$$

which is called Green's second identity

In operator notation

$$(X_1, \mathcal{L} X_2) - (\mathcal{L} X_1, X_2) = \left[X_1 X_2' - X_1' X_2 \right]_0^l$$

Now we see that for any combination of Dirichlet or Neumann conditions the boundary term vanishes, and so

$$(X_1, \mathcal{L} X_2) = (\mathcal{L} X_1, X_2)$$

i.e. \mathcal{L} is self-adjoint

Theorem : \Rightarrow $\left\{ \begin{array}{l} A \text{ is Hermitian } \overline{A^T} = A \in \mathbb{C}^{n \times n} \\ A \text{ was unitarily diagonalizable} \\ \text{it has } \forall \text{ orthonormal eigenvectors} \end{array} \right.$

Every self-adjoint linear operator λ real possesses a set of orthogonal eigenvectors with real eigenvalues. eigenfunctions

In fact, for the Laplacian all eigenvalues are non-positive

$$\lambda \leq 0$$

To see this, take an eigenfunction $\underline{\bar{x}}(x)$ and see

$$(\underline{\bar{x}}, \lambda \underline{\bar{x}}) = \int \underline{\bar{x}} \underline{\bar{x}}'' dx$$

integrate by parts

$$= - \int (\underline{\bar{x}}')^2 dx + [\underline{\bar{x}} \underline{\bar{x}}']_0^l$$

$$= - \int (\underline{\bar{x}}')^2 dx \leq 0$$

$$= (\underline{\bar{x}}, \lambda \underline{\bar{x}}) = \lambda (\underline{\bar{x}}, \underline{\bar{x}})$$

$$\Rightarrow \lambda = \frac{- \int (\underline{\bar{x}}')^2 dx}{\int \underline{\bar{x}}^2 dx} \leq 0$$

$\int |x|^2 dx$

Observe that all of these proofs rely only on the axioms and thus generalize from ordinary linear algebra...

$$\Psi(x) = \sum A_n \sin\left(\frac{n\pi x}{l}\right) ?$$

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We now know that the $\sin\left(\frac{n\pi x}{l}\right)$ functions are orthogonal!

This can be proven using

$$\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$$

but we don't have to prove it again!

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = 0 \text{ if } n \neq m$$

If we take the Laplacian with Neumann BCs, we get:

$$\begin{cases} \bar{X}'' = -\lambda \bar{X} \\ \bar{X}'(0) = \bar{X}'(l) = 0 \end{cases}$$

proceeding as for Dirichlet (do it on your own!), we

get

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \text{ for } n=0, 1, 2$$

Note \emptyset is here!

$$\bar{X}_n = \cos \frac{n\pi x}{l}$$

and of course these cos functions must also be orthogonal to each other.

The separation of variables method for

$$\begin{cases} u_t = k u_{xx} & u(x,0) = \psi(x) \\ u_x(0) = u_x(l) = 0 \end{cases}$$

gives

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{- (n\pi/l)^2 kt} \cos\left(\frac{n\pi x}{l}\right)$$

with A_n determined from

$$\psi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

which is called the Fourier cosine series.

(The factor $\frac{1}{2}$ will be explained later)

Assume that we have an orthogonal basis set of functions, and that

$$\Psi(x) = \sum_{n=0, \pm 1, \dots}^{\infty} A_n f_n(x)$$

Fourier
Sine
cosine
complex exponential

How do we find A_n ?

Follow the same procedure as for representing in a unary basis

$$x = Uy \Rightarrow y = U^{-1}x = U^*x$$

$$\Psi(x) = \sum_n A_n f_n \quad \left. \begin{array}{l} \text{Multiply} \\ \text{by } f_m \end{array} \right\}$$

$$(f_m, \Psi) = \sum_n A_n (f_m, f_n)$$

$$= \sum_n A_n \delta_{mn} (f_n, f_n) \quad \text{orthogonal}$$

$$= A_m (f_m, f_m) \Rightarrow$$

$$A_m = \frac{(f_m, \Psi)}{(f_m, f_m)} \quad \text{general!}$$

Let's apply this to the sine series

$$f_n = \sin\left(\frac{n\pi x}{l}\right) \quad n \in \mathbb{Z}^+$$

$$(f_n, f_n) = \int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx = \frac{l}{2}$$

$$\Rightarrow \left[\begin{aligned} \psi(x) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \Rightarrow \\ A_n &= \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx \end{aligned} \right]$$

Fourier sine series

Similarly, for cosine series

$$\left[\begin{aligned} \psi(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \Rightarrow \\ A_n &= \frac{2}{l} \int_0^l \psi(x) \cos\left(\frac{n\pi x}{l}\right) dx \quad n \geq 0 \end{aligned} \right]$$

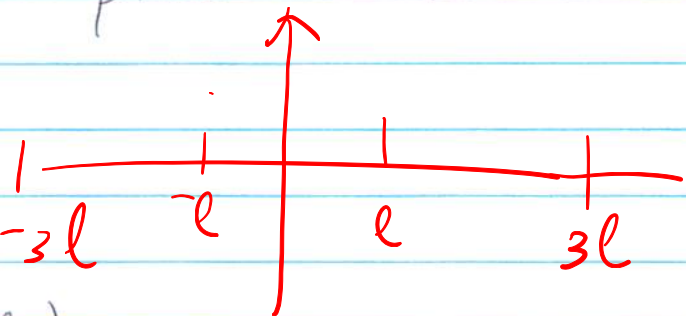
and we put the factor 1/2 in front of A_0 so this covers $n=0$

Periodic boundary conditions

As a last example consider the heat equation with periodic BCs:

$$\begin{cases} u(0) = u(l) \\ u_x(0) = u_x(l) \end{cases} \leftarrow \text{useful in physics of waves}$$

The eigenvalue problem is

$$\begin{cases} \bar{x}'' = -\lambda \bar{x} \\ \bar{x}(0) = \bar{x}(l) \\ \bar{x}'(0) = \bar{x}'(l) \end{cases}$$


$-\lambda x$

Recall $\bar{x}(x) = e^{-\lambda x}$ and we want

$$e^{-0} = 1 = e^{-\lambda l}$$

$$\lambda = a + ib$$

$$e^{-(a+ib)l} = e^{-al} e^{-ibl} = 1$$

Note $|e^{-al}| \leq 1$ and $|e^{-ibl}| = |\cos(bl) + i\sin(bl)| = 1$

$$\text{So } |e^{-al} e^{-ibl}| = |e^{-al}| |e^{-ibl}| \stackrel{!}{=} 1$$

with $= 1$ only if $a = 0$

So λ must be purely imaginary

$$e^{ibl} = \underbrace{\cos(bl)}_{\parallel 1} + i \underbrace{\sin(bl)}_{\parallel 0} = 1$$

$$\Rightarrow b = \frac{2n\pi}{l}, \quad n = \text{integer}$$

To avoid the factor of 2 usually the problem is specified on the interval $(-l, l)$

$$u(-l) = u(l)$$

$$\Rightarrow \boxed{X(x) = e^{\frac{i n \pi x}{l}}, \quad n \in \mathbb{Z}}$$

which is often simpler to work with. These are orthogonal functions

$$\begin{aligned}
 (\overline{X_n}, \overline{X_n}) &= \int_{-l}^l e^{i n \pi x / l} e^{-i n \pi x / l} dx \\
 &= \int_{-l}^l 1 dx = 2l
 \end{aligned}$$

complex conjugate

$$\Rightarrow \left[\begin{aligned}
 \psi(x) &= \sum_{n=-\infty}^{\infty} c_n e^{i n \pi x / l} \\
 c_n &= \frac{1}{2l} \int_{-l}^l \psi(x) e^{-i n \pi x / l} dx
 \end{aligned} \right] \Rightarrow$$

Complex / Periodic Fourier series

Note that one can rewrite this to use real numbers only but analytical computations are easier using the complex notation

In fact,

$$\left\{ \begin{aligned}
 \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
 \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}
 \end{aligned} \right. \text{Simplifies a lot of algebra often}$$