Robust Control for Quantum Systems
finding optimal control under noise

Qianyu Zhu

Supervised by G. Stadler F. Garcia

New York University, Shanghai

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Introduction to quantum system

- simplest quantum system: **qubit**
  classical bit is either 0 or 1, **qubit** can be both.

- state vector:
  \[ \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad |\psi|^2 = |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}. \]

- \( \psi \) can have \( > 2 \) energy states.

**examples:**

\[ \psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} \cos \theta \\ i \sin \theta \end{pmatrix} \]
evolution of quantum state vector $\psi(t) \in \mathbb{C}^d$:

Schrödinger’s equation

$$\dot{\psi}(t) = -i H(t; f(t)) \psi(t).$$

- **Hamiltonian** $H$: $\mathbb{C}^{d \times d}$, Hermitian, $(t; f(t))$-dependent.
- **$f(t)$**: reflects our control.

our Hamiltonian model:

$$H(t) = \omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + f(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Demonstration of "control":

\[
H(t) := f(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ start from } \psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

\[f(t) \equiv 0\]

\[f(t) \equiv \text{const}\]

\[f(t) = c \cdot (100 - t)^2\]
Control problem

given:

\[
\begin{cases}
\dot{\psi} = -iH(t; f(t))\psi, \\
\psi(0) = \psi_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{cases}
\]

we want:

\[
\psi(T) = \psi_{tg} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

loss function:

\[
L(f(t)) = 1 - |\langle \psi(T), \psi_{tg} \rangle|^2
\]

Q: when does \( L(f) = 0? \) \( \psi(T) = \psi_{tg}! \)
Noise attack

the optimization problem

\[
\min_{f:[0,T] \to \mathbb{R}} L(f)
\]

s.t. \[
\begin{cases}
\dot{\psi} = -iH(t; f(t))\psi, \\
\psi(0) = \psi_0.
\end{cases}
\]

noise model:

\[
\epsilon \sim \text{Unif}(-\delta, \delta), \quad H(\epsilon) := H + \epsilon A
\]

noise-aware loss function:

\[
L_{NA} = \mathbb{E}_\epsilon [e^{L(f; \epsilon)}] = \frac{1}{2\delta} \int_\epsilon e^{L(f; \epsilon)} d\epsilon.
\]
Why is this challenging?

updated optimization problem

$$\min_{f: [0, T] \rightarrow \mathbb{R}} \mathbb{E}^\epsilon[f^{\mu L}(f; \epsilon)]$$

$$\text{s.t. } \begin{cases} \dot{\psi} = -iH(t; f(t); \epsilon) \psi, \\ \psi(0) = \psi_0. \end{cases}$$

we use Juqbox, a Julia package!

- $f$ in infinite-dim space
- need $\partial_f L$ to update $f$
- ODE constraint
- integral $\int_\epsilon$ in loss
- approximate $f$ by finite basis
- get gradient by adjoint method
- quadrature/Monte-Carlo sampling
Test problem: 1-dim uniform noise

Hamiltonian

\[ H(\epsilon) = \omega A + f(t)B + \epsilon C \]

where

- \( \omega A \): natural evolution,
- \( f(t)B \): human control,
- \( \epsilon C \): uniform noise,
- \( A, B, C \) are fixed, Hermitian matrices \( \in \mathbb{C}^{3 \times 3} \).

Our goal

start from \( \psi(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), arrive at \( \psi(T) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \).
Comparison between NF and NA methods

- noise-free method is only accurate for $\epsilon \approx 0$;
- noise-aware method behaves robustly under large noise;
- good generalization ability.
Test problem: high-dim noise

- noise model:

\[
\begin{pmatrix}
\epsilon_1 & \epsilon_4 \\
\epsilon_4 & \epsilon_2 & \epsilon_5 \\
\epsilon_5 & \epsilon_3 \\
\end{pmatrix}
\]

\(\epsilon_i \sim \text{Unif}(\delta_i, \delta_i)\).

- curse of dimensionality:
  - MC method.
Contribution and future work

**contribution:**

- implement risk-aware loss functions
- test on 1-dim noise
- test on higher-dim noise

**future work:**

- N-qubit system
- non-uniform noise
- new loss function
- ...

Two physicists trying to control a qubit in the lab
References

- **Juqbox.**

- **Michael A. Nielsen and Isaac L. Chuang.**
  *Quantum Computation and Quantum Information: 10th Anniversary Edition.*

- **N. Anders Petersson and Fortino Garcia.**
  Optimal control of closed quantum systems via b-splines with carrier waves, 2021.
other risk aware loss functions we experimented on:

- **risk-neutral:**
  \[ F(\alpha) := \mathbb{E}^\epsilon[L(\alpha, \epsilon)] \]

- **risk-sensitive:**
  \[ F(\alpha) := \mathbb{E}^\epsilon[e^{L(f; \epsilon)}] \]

- **mean-variance:**
  \[ F(\alpha) := \mathbb{E}^\epsilon[L(\alpha, \epsilon)] + \frac{\theta}{2} \cdot \text{Var}[L(\alpha, \epsilon)] \]

- **conditional value-at-risk:**
  \[ \text{CVaR}_\beta[X] := \inf_{t \in \mathbb{R}} \left\{ t + (1 - \beta)^{-1} \mathbb{E}[(X - t)^+] \right\} \]