Information Theory for Neuroscience
Measuring the entropy of visual cortices

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Outline

1. Why entropy?
2. Algorithm
3. Discrete implementations
4. Continuous implementations
5. Neural data
6. Future work
Why Entropy?

What is entropy?

- Shannon entropy is a measure of “surprisal”
- Self entropy: or some random variable $X$ following a distribution $p$
  \[ H(p, p) = - \sum_x p(x) \log p(x) . \]
- Cross entropy: for some random distribution $p$ and an estimator distribution $q$
  \[ H(p, q) = - \sum_x p(x) \log q(x) . \]
Why Entropy?
Quantifying time reversal symmetry

• Kullback-Leibler divergence (KLD) - measure of distance from one probability distribution to a reference probability distribution.

\[ D_{KL}(p \parallel q) = H(p, q) - H(p) \]

• Entropy production - the measure of the distance to time-reversal symmetry

• For variable \( X' \) which is the flipped version of \( X \) and follows the distribution \( q \),

\[ D_{KL}(p \parallel q) = \text{Entropy production} \]
Algorithm
Estimate the entropy when the distribution is unknown.

- Input series
Algorithm

Estimate the entropy when the distribution is unknown.

• Input series

• Entropy estimator

\[
H = \frac{\log_2 M}{<l>}
\]
Algorithm
Pattern matching estimator

• Data compression algorithm:

Lempel-Ziv 77 Factorization (LZ77)

\[
X = \begin{array}{cccccccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
 a & b & c & a & b & a & b & a & b & a & b & a & b & a & b & b \\
\end{array}
\]

\[LZ_{77}(X) = [a,0], [b,0], [c,0], [0,2], [3,10]\]

Number of factors \( C = 5 \)

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
Algorithm
Pattern matching algorithm

\[ H \leq \text{Information required to specify the factor of a finite sequence} \]

\[ C = \text{Number of factors} \]

\[ H \leq \frac{C}{N} \log_2 N + \frac{1}{N} \sum_{i=1}^{C} \log_2 l_i \quad \text{Length of compressed data} \]

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
Algorithm
Pattern matching algorithm

\[ H \leq \text{Information required to specify the factor of a finite sequence} \]
\[ C = \text{Number of factors} \]

\[ H \leq \frac{C}{N} \log_2 N + \frac{1}{N} \sum_{i=1}^{C} \log_2 l_i \quad \text{Length of compressed data} \]

... \[ H \leq \frac{\log_2 M}{<l>} \quad \text{Pattern matching estimator} \]

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
As expected, the error of the estimator decreases as the input length grows.
Discrete Implementations
Introduce the Fast Fourier transform (FFT)

• Why Fast Fourier Transform?
  It’s faster in higher dimensions.
  A good practice before moving to the continuous case

\[
H = \frac{\log_2 M}{< l >}
\]
Discrete Implementations

Find the longest length of best matches $l_i$

$$H = \frac{\log_2 M}{<l>}$$

**Approach 1: Exact match**

**Sample**

\[ \begin{array}{cccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array} \]

**Dictionary**

\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
\end{array} \]

**Approach 2: Cross correlation**

**Sample**

\[ \begin{array}{cccccccc}
1 & 1 & -1 & 1 & -1 & -1 & 1 \\
\end{array} \]

**Padded Sample**

\[ \begin{array}{cccccccc}
1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

**Dictionary**

\[ \begin{array}{cccccccc}
1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
\end{array} \]

**Fourier Space**

\[ F = \mathcal{F}\{\text{sample}\} \quad G = \mathcal{F}\{\text{dictionary}\} \]

Cross correlation = $F \circ \overline{G}$
Continuous Implementations
Generating hard rods

• Non-overlapping constant-length rods

• Analytic entropy is known

• Discretization does not produce a good estimate
Continuous Implementations

Real-space

- Used hard rods encoded by center location
- Match distances between rod centers with some error $\varepsilon$

Sample:

\[
\ell + d_i^s
\]

\[
\ell + d_i^{s+1}
\]

\[
\ell + d_i^{s+2}
\]

\[
|d_i^s - d_j^d| < \varepsilon
\]

\[
|d_i^{s+1} - d_{i+1}^d| \neq \varepsilon
\]

Dictionary:

\[
\ell + d_j^d
\]

\[
\ell + d_{j+1}^d
\]

\[
\ell + d_{j+2}^d
\]
Continuous Implementations
Fourier-space

• Use non-uniform FFT (NUFFT)
Neural Data
Neuron activity in the visual cortex

- Spike train of neurons: recording of neuron activations.
- Virtual cortex
Neural Data

Neuron activity in the visual cortex

- Spike train of neurons: recording of neuron activations.
- Virtual cortex
- Experiment Setup

Oriented gratings

Secondary Visual Cortex (V2)
Primary Visual Cortex (V1)

Macaques
Neural Data

Quantities measured

- Hypothesis

  Neural code is more time reversible in different brain areas, revealing the computing happening in the visual cortex.
Neural Data

Quantities measured

• Hypothesis

  Neural code is more time reversible in different brain areas, revealing the computing happening in the visual cortex.

• Entropy: $H(p) = -\sum p(x)\log p(x)$.

• Cross Entropy: $H(\hat{p}, q) = -\sum p(x)\log q(x)$.

• Kullback-Leibler divergence (KLĐ):

  $$D_{KL}(p \parallel q) = H(p, q) - H(p).$$

  • $p$ - distribution of spike train data

  • $q$ - distribution of reversed spike train data

A measurement of time-reversal symmetry
Neural Data

Avg KLD vs. Firing Rate in V1

- Firing rate (Hz): $\nu = \frac{n_{sp}}{T}$
- In V1

   The KLD has a positive linear relationship with the firing rate of neurons.
Neural Data

Avg KLD vs. Firing Rate in V2

- Firing rate (Hz): \[ v = \frac{n_{sp}}{T} \]
- The same observation holds in V2:
  The KLD has a positive linear relationship with the firing rate of neurons.
Future work

- Neural data:
  - Apply algorithm to visual cortices of other animals
- Refine continuous implementation in real and Fourier space
- Extend to higher dimensions
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Why Entropy (Backup)

Cross-entropy

$$H(p, q) = - \sum_x p(x) \log q(x).$$

- Measure of bits needed when the distribution is assumed to be $q$ but is actually $p$
- Useful in machine learning context
Algorithm (Backup)
Pattern matching algorithm

- LZ77 is proved to asymptotically converge to the entropy

\[ H \leq \text{Information required to specify the factor of a finite sequence} \]

\[
H \leq \frac{C}{N} \log N + \frac{1}{N} \sum_{i=1}^{C} \log l_i
\]

\[
= \frac{C}{N} \log N + \frac{C}{N} < \log l >
\]

\[
\leq \frac{C}{N} \log N + \frac{C}{N} \log < l >
\]

\[
H \leq \frac{\log N}{< l >} + \frac{\log < l >}{< l >}
\]

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
Phase Correlation (Backup)
Non-binary input

- Cross correlation is valid only for binary input
- Phase correlation:

\[ F = \mathcal{F}\{sample\} \quad G = \mathcal{F}\{dictionary\} \]

\[
\text{Phase correlation} = \frac{F \circ \overline{G}}{|F \circ \overline{G}|}
\]
Algorithm (Backup)
Pattern matching estimator

• Data compression algorithm:
  • Lempel-Ziv 77 (LZ77) Factorization

\[ \text{LZ}^{77}(X) = [a,0], [b,0], [c,0], [0,2], [3,10] \]

Number of factors \( C = 5 \)

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
Algorithm (Backup)
Pattern matching estimator

- Data compression algorithm:
  - Lempel-Ziv 77 (LZ77) Factorization

\[ X = \text{abcabcabababab} \]

Repeat the segment starting from index 0 for the following 2 characters:

\[ \text{LZ}^{77}(X) = [a,0], [b,0], [c,0], [0,2], [3,10] \]

Examples and derivations come from Stefano’s talk at Santa Fe, “The Other Side Of Entropy”
Algorithm (Backup)
Pattern matching estimator

- Data compression algorithm:
  - Lempel-Ziv 77 (LZ77) Factorization

```
X = a b c a b a b a b a b a b a b
```

$LZ^{77}(X) = [a,0], [b,0], [c,0], [0,2], [3,10]$
Discrete Implementations (Backup)

Verification using 3-state Markov chain

As expected, the error of the estimator increases as the bias grows.