A sharp interface version of the immersed boundary finite element method

Charles Puelz and Boyce Griffith

April 23, 2020
**IB finite element method**

\[ U = \text{reference configuration} \]
\[ \chi(\cdot, t) = \text{motion map} \]
\[ \mathbf{X} \in U = \text{Lagrangian coordinates} \]
\[ \mathbf{x} \in \Omega = \text{current coordinates} \]

\[ \sigma(x, t) = \sigma^f(x, t) \]
\[ + \begin{cases} 
\sigma^e(x, t) & \text{for } x \in \chi(U, t) \\
0 & \text{otherwise} 
\end{cases} \]

first Piola–Kirchoff stress:

\[ P^e(\mathbf{X}, t) = J(\mathbf{X}, t) \sigma^e(\chi(\mathbf{X}, t), t) \mathbf{F}^{-T}(\mathbf{X}, t), \quad \mathbf{X} \in U \]

Figure: fluid–solid system at time \( t \)
Equations of motion

balance of momentum

\[ \rho \left( \frac{\partial u}{\partial t}(x, t) + u(x, t) \cdot \nabla u(x, t) \right) = -\nabla p(x, t) + \mu \Delta u(x, t) + f(x, t) + f_{ext}(x, t) \]

mass conservation

\[ \nabla \cdot u(x, t) = 0 \]

force density from the solid

\[ f(x, t) = \int_U \nabla x \cdot \mathbb{P}^e(X, t) \delta(x - \chi(X, t)) dX \]

\[ - \int_{\partial U} \mathbb{P}^e(X, t) \mathbf{N}(X) \delta(x - \chi(X, t)) dA(X) \]

no slip between solid and fluid

\[ \frac{\partial \chi}{\partial t}(X, t) = \int_{\Omega} u(x, t) \delta(x - \chi(X, t)) dX \]

First example: thick pre-stressed ring

\[ s \in U = [0, 2\pi R] \times [0, w], \]

\[ \chi(s, 0) = \left( \cos\left(\frac{s_1}{R}(R + s_2) + 0.5\right), \sin\left(\frac{s_1}{R}(R + \gamma + s_2)\right) \right), \]

\[ P^e = \frac{\mu^e}{w} F. \]
Pressure field: thick pre-stressed ring

Figure: Pressure field.
Numerical errors: thick pre-stressed ring with sharp interface
Some ways for dealing with errors at the interface

- **immersed interface methods, cut-cell methods**: discrete operators like finite difference stencils are locally modified at the fluid/solid interface. (LeVeque and Li, *SIAM J Num Anal*, 1994).

- **immersed boundary smooth extension**: compute functions that are “smooth extensions” from the fluid to solid domain, and use them to modify the forcing on the fluid. (Stein et al., *J Comput Phys*, 2016).

**Our approach**: requires no modification of discrete operators, can deal with moving interfaces in 2D and 3D, and is not poorly conditioned.
Discontinuities at the fluid/solid interface

\[
[g(x)] = \lim_{\epsilon \downarrow 0} g(x + \epsilon n) - \lim_{\epsilon \downarrow 0} g(x - \epsilon n) := g^+(x) - g^-(x)
\]

continuity of the traction vector implies \([\sigma n] = 0\) on \(\partial \chi(U, t)\).

traction vector continuity is Newton’s third law:
“When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.”

Pressure discontinuity

Using traction vector continuity and $\sigma^{e,+} = 0$ on $\partial \chi(U, t)$:

$$- [p] n + \mu [\nabla u + (\nabla u)^T] n - \sigma^{e,-} n = 0.$$ 

Lemma: Let $t$ and $b$ form a basis for the tangent plane to the point at which we are considering the jump. Then $[(\nabla u) t] = [(\nabla u) b] = 0$.

Proof: Consider a parametrized curve $\beta = \beta(s)$ defined on $\partial \chi(U, t)$ that contains the point at which we consider the jump. This curve is constructed so its tangent vector $d\beta/ds$ is equal to $t$ at this point.

$$\frac{d}{ds} u_i(\beta(s)) = \frac{d\beta}{ds} \cdot \nabla u_i(\beta(s)) = t \cdot \nabla u_i(\beta(s)).$$
Pressure discontinuity

\[-[p] \mathbf{n} + \mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \mathbf{n} - \sigma^e, \n \mathbf{n} = 0.\]

**Lemma:** \([\mathbf{n} \cdot (\nabla \mathbf{u}) \mathbf{n}] = [\mathbf{n} \cdot (\nabla \mathbf{u})^T \mathbf{n}] = 0.\]

**Proof:** Use the incompressibility condition and \([\mathbf{(\nabla \mathbf{u})_t}] = [\mathbf{(\nabla \mathbf{u})_b}] = 0.\]

\([p] = -\mathbf{n} \cdot \sigma^e, \n \mathbf{n}\)
Shear stress discontinuity

For a unit tangent vector $t$ to $\partial \chi(U, t)$:

$$\mu t \cdot \left[ \nabla u + (\nabla u)^T \right] n = t \cdot \sigma^{e, -} n$$

This jump condition can be reformulated in terms of the normal derivative of $u$:

$$\mu \left[ (\nabla u) n \right] = (I - n n^T) \sigma^{e, -} n.$$
Let \( \mathbf{v} \) be the unit tangent vector to a curve \( \gamma(s) \), i.e.

\[
\mathbf{v} = \frac{d\gamma}{ds}.
\]

Also, let \( \tilde{\gamma}(s) = \chi(\gamma(s)) \).

\[
\frac{d\tilde{\gamma}}{ds} = \frac{d}{ds} \chi(\gamma(s)) = \mathbb{F} \frac{d\gamma}{ds} = \mathbb{F} \mathbf{v}.
\]
Transforming area elements

Let \( \mathbf{N} = \mathbf{v} \times \mathbf{w} \) be a normal vector in the reference configuration. Define:

\[
\mathbf{n} = \frac{\mathbf{F} \mathbf{v} \times \mathbf{F} \mathbf{w}}{||\mathbf{F} \mathbf{v} \times \mathbf{F} \mathbf{w}||}
\]

to be the normal vector in the current configuration. The ratio of sizes of area elements is:

\[
\frac{dA}{da} = \frac{||\mathbf{v} \times \mathbf{w}||}{||\mathbf{F} \mathbf{v} \times \mathbf{F} \mathbf{w}||} = \frac{1}{||\mathbf{F} \mathbf{v} \times \mathbf{F} \mathbf{w}||}
\]

This, combined with the identity: \( \mathbf{F} \mathbf{v} \times \mathbf{F} \mathbf{w} = \text{det}(\mathbf{F}) \mathbf{F}^{-T} \mathbf{v} \times \mathbf{w} \) gives Nanson’s relation:

\[
\mathbf{n} \, da = \text{det}(\mathbf{F}) \mathbf{F}^{-T} \mathbf{N} \, dA.
\]
Removing the pressure jump

By definition of $P^e$ and Nanson’s relation:

$$\sigma^e n \, da = P^e \, N \, dA \quad \text{and} \quad n \, da = J F^{-T} \, N \, dA.$$ 

These equations imply:

$$n = \frac{F^{-T} N}{|F^{-T} N|} \quad \text{and} \quad \sigma^e n = J^{-1} \frac{P^e \, N}{|F^{-T} N|},$$

so the jump in the pressure is:

$$[p] = -J^{-1} \frac{n \cdot P^e \, N}{|F^{-T} N|}.$$ 

Idea to “remove” the jump: define a modified stress $\tilde{P}^e$ so that

$$n \cdot \tilde{P}^e N = 0.$$
Removing the pressure jump

Let \( \varphi \) be some scalar valued function defined on \( U \) and
\[
\tilde{P}^e = P^e - J \varphi F^{-T}.
\]

Then \( n \cdot \tilde{P}^e N = 0 \iff n \cdot P^e N = J \varphi n \cdot F^{-T} N, \)

implying we want \( \varphi \) to satisfy:
\[
\varphi = J^{-1} \frac{F^{-T} N}{||F^{-T} N||^2} \cdot P^e N := g \quad \text{on } \partial U.
\]

We can find such a function by solving:
\[
-\nabla^2 \varphi = 0 \quad \text{in } \quad U, \quad \varphi = g \quad \text{on } \quad \partial U.
\]
Obtaining the physical pressure

Recall that our modified Cauchy stress is given by:

\[ \tilde{\sigma}^e = J^{-1}\tilde{P}^e F^T = \sigma^e - \varphi I. \]

**Sharp interface algorithm:**

- Solve harmonic problem
  \[ -\nabla^2 \varphi = 0 \quad \text{in} \quad U, \quad \varphi = g \quad \text{on} \quad \partial U. \]
- Solve FSI problem with \( \tilde{P}^e \) for variables \( u \) and \( \pi \).
  By construction, \( [\pi] = 0 \) on \( \partial \chi(U, t) \).
- Reconstruct the physical pressure
  \[ p(x, t) = \pi(x, t) + \varphi(\chi^{-1}(x, t), t). \]
Penalty method

The penalty method is an approach for approximating:

\[-\nabla^2 u = 0 \text{ in } U\]
\[u = g \text{ on } \partial U\]

Multiply by a test function \( \varphi \) and integrate by parts:

\[\int_U \nabla u \cdot \nabla \varphi - \int_{\partial U} \mathbf{n} \cdot \nabla u \varphi = 0\]

The Dirichlet condition looks almost like a Robin condition:

\[\varepsilon \mathbf{n} \cdot \nabla u + u = g \text{ on } \partial U\]
Penalty method cont.

Plugging this in gives

$$\int_U \nabla u \cdot \nabla \varphi + \varepsilon^{-1} \int_{\partial U} u \varphi = \varepsilon^{-1} \int_{\partial U} g \varphi$$

A finite element scheme is then:

Find $u_h \in V_h$ so that

$$\int_U \nabla u_h \cdot \nabla \varphi_h + \varepsilon^{-1} \int_{\partial U} u_h \varphi_h = \varepsilon^{-1} \int_{\partial U} g \varphi_h$$

for all $\varphi_h \in V_h$.

This approach is also called **Nitsche’s method**.
Pressure field: thick pre-stressed ring

Figure: Sharp interface on the right.
π and ϕ fields: thick pre-stressed ring

Figure: The π field is on the left and the ϕ field is on the right.
Numerical errors: thick pre-stressed ring with the sharp interface method

**steady state, velocity errors**

- $L^1$
- $L^2$
- $L^\infty$
- $h^2$
- $h$

**steady state, pressure errors**

- $L^1$
- $L^2$
- $L^\infty$
- $h^2$
- $h$
Pressure field: thick inflating ring

Figure: pressure field... sharp interface on the right.
Deformations: thick inflating ring

Figure: The original IBFE method is on the left and the sharp interface method is on the right. The coloring in the final configuration corresponds to the determinant $J = \det(F)$. 
Actively contracting torus

\[ P^e(X, t) = \mu_e (F - F^{-T}) + T(G^{-1}(X), t) F f_0 \otimes f_0. \]

**Figure:** The mesh for the torus, with the fiber vector field $f_0$ superimposed.
Actively contracting torus

McQueen and Peskin, *J Comput Phys*, 1989

**Figure:** A visualization of contraction. Time increases from left–to–right and top–to–bottom. The color indicates the value of the tension function.
Velocity field: actively contracting torus

Figure: A slice of the velocity field and active tension function
Pressure field: actively contracting torus

Figure: A slice of the pressure field... sharp interface on the right.
Unified weak formulation

\[
\rho \left( \frac{\partial u}{\partial t}(x, t) + u(x, t) \cdot \nabla u(x, t) \right) = -\nabla p(x, t) + \mu \Delta u(x, t) + g(x, t)
\]
\[
\nabla \cdot u(x, t) = 0
\]
\[
g(x, t) = \int_{U} G(X, t) \delta(x - \chi(X, t)) dX
\]
\[
\frac{\partial \chi}{\partial t}(X, t) = \int_{\Omega} u(x, t) \delta(x - \chi(X, t)) dx
\]

where \( G(X, t) \) satisfies:

\[
\int_{U} G(X, t) \cdot V_h(X) dX = -\int_{U} p^e(X, t) : \nabla_x V_h(X) dX
\]
\[
+ \int_{U} F_{\text{bdy}}(X, t) \cdot V_h(X) dX + \int_{\partial U} F_{\text{surf}}(X, t) \cdot V_h(X) dA(X)
\]

for all \( V \) in some finite element space.
Unified weak formulation

\[
\rho \left( \frac{\partial u}{\partial t}(x, t) + u(x, t) \cdot \nabla u(x, t) \right) = -\nabla p(x, t) + \mu \Delta u(x, t) + g(x, t)
\]

\[
\nabla \cdot u(x, t) = 0
\]

\[
g(x, t) = \int_U G(X, t) \delta(x - \chi(X, t)) dX
\]

\[
\frac{\partial \chi}{\partial t}(X, t) = \int_\Omega u(x, t) \delta(x - \chi(X, t)) dx
\]

where \( G(X, t) \) satisfies:

\[
\int_U G(X, t) \cdot V_h(X) dX = -\int_U P^e(X, t) : \nabla x V_h(X) dX
\]

\[
+ \int_U F_{\text{bdy}}(X, t) \cdot V_h(X) dX + \int_{\partial U} F_{\text{surf}}(X, t) \cdot V_h(X) dA(X)
\]

for all \( V \) in some finite element space.
Isovolumetric/dilational split of the stress

given a strain energy density $W$, we define the PK1 stress in one of two ways:

through a deviatoric projection:

$$P^e = \text{DEV}\left[\frac{\partial W}{\partial F}\right] + \frac{\partial U}{\partial F}, \quad \text{DEV}[\bullet] = (\bullet) - \frac{1}{3}(\bullet : F) \mathbb{I}^{-T}$$

or by using modified invariants: $\tilde{W} = W(\bar{l}_1, \bar{l}_2, \ldots)$

$$P^e = \frac{\partial \tilde{W}}{\partial \bar{F}} + \frac{\partial U}{\partial \bar{F}}, \quad \bar{F} = J^{-1/3}F$$

$$l_1(F) = F^TF, \quad l_2(F) = \frac{1}{2}\left[(\text{tr} F^TF)^2 - \text{tr} (F^TF)^2\right]$$