A sharp interface version of the immersed boundary finite element method

Charles Puelz and Boyce Griffith

April 23, 2020

IB finite element method

$$U = \text{ reference configuration}$$
$$\chi(\cdot, t) = \text{ motion map}$$
$$\mathbf{X} \in U = \text{ Lagrangian coordinates}$$
$$\mathbf{x} \in \Omega = \text{ current coordinates}$$

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}^{\mathrm{f}}(\mathbf{x}, t) + \begin{cases} \boldsymbol{\sigma}^{\mathrm{e}}(\mathbf{x}, t) & \text{for } \mathbf{x} \in \chi(U, t) \\ \mathbf{0} & \text{otherwise} \end{cases}$$

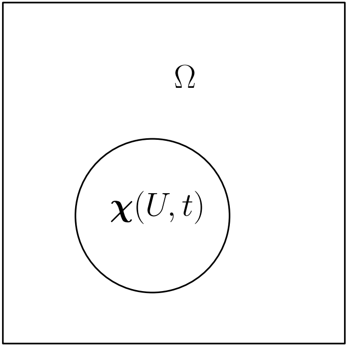


Figure: fluid–solid system at time t

first Piola–Kirchoff stress:

$$\mathbb{P}^{\mathrm{e}}(\mathbf{X}, t) = J(\mathbf{X}, t)\, \boldsymbol{\sigma}^{\mathrm{e}}(\chi(\mathbf{X}, t), t)\, \mathbb{F}^{-T}(\mathbf{X}, t), \quad \mathbf{X} \in U$$

Equations of motion

balance of momentum

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\right)=-\nabla p(\mathbf{x},t)+\mu\Delta\mathbf{u}(\mathbf{x},t)+\mathbf{f}(\mathbf{x},t)+\mathbf{f}_{\text{ext}}(\mathbf{x},t)$$

mass conservation

$$\nabla\cdot\mathbf{u}(\mathbf{x},t)=0$$

force density from the solid

$$\begin{aligned}\mathbf{f}(\mathbf{x},t)=&\int_U \nabla_{\mathbf{X}}\cdot\mathbb{P}^{\mathrm{e}}(\mathbf{X},t)\,\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{X}\\ &-\int_{\partial U}\mathbb{P}^{\mathrm{e}}(\mathbf{X},t)\mathbf{N}(\mathbf{X})\,\delta(\mathbf{x}-\chi(\mathbf{X},t))dA(\mathbf{X})\end{aligned}$$

no slip between solid and fluid

$$\frac{\partial\chi}{\partial t}(\mathbf{X},t)=\int_\Omega \mathbf{u}(\mathbf{x},t)\,\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{X}$$

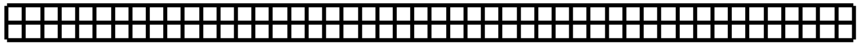
Boffi et al., *Comput Methods Appl Mech Eng*, 2008

First example: thick pre-stressed ring

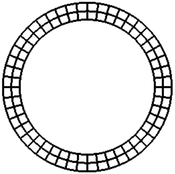
$$\mathbf{s} \in U = [0, 2\pi R] \times [0, w],$$

$$\boldsymbol{\chi}(\mathbf{s}, 0) = (\, \cos(s_1/R(R + s_2) + 0.5,\ \sin(s_1/R)(R + \gamma + s_2)\,)\,,$$

$$\mathbb{P}^e = \frac{\mu^e}{w}\mathbb{F}.$$



U



$\boldsymbol{\chi}(U, 0)$

Pressure field: thick pre-stressed ring

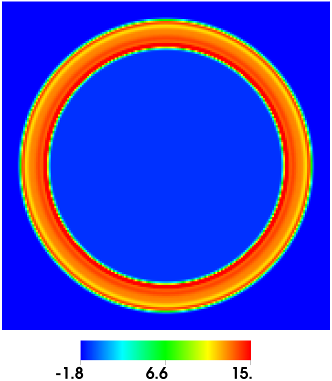
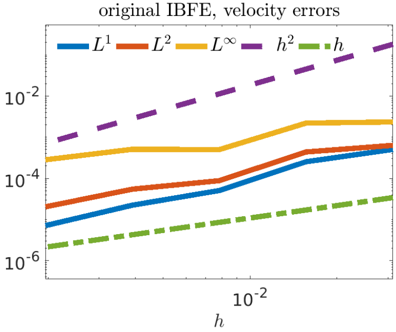
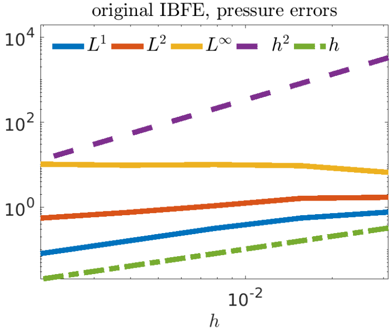


Figure: Pressure field.

Numerical errors: thick pre-stressed ring with sharp interface





Some ways for dealing with errors at the interface

- **immersed interface methods, cut-cell methods**: discrete operators like finite difference stencils are locally modified at the fluid/solid interface. (LeVeque and Li, *SIAM J Num Anal*, 1994).

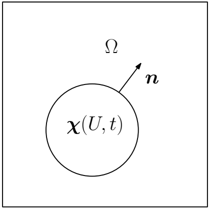
- **immersed boundary smooth extension**: compute functions that are "smooth extensions" from the fluid to solid domain, and use them to modify the forcing on the fluid.
(Stein et al., *J Comput Phys*, 2016).

Our approach: requires no modification of discrete operators, can deal with moving interfaces in 2D and 3D, and is not poorly conditioned.

Discontinuities at the fluid/solid interface

$$[\mathbf{g}(\mathbf{x})] = \lim_{\epsilon \downarrow 0} \mathbf{g}(\mathbf{x} + \epsilon\,\mathbf{n}) - \lim_{\epsilon \downarrow 0} \mathbf{g}(\mathbf{x} - \epsilon\,\mathbf{n}) := \mathbf{g}^{+}(\mathbf{x}) - \mathbf{g}^{-}(\mathbf{x})$$

continuity of the traction vector implies $[\boldsymbol{\sigma}\,\mathbf{n}] = 0 \quad \text{on} \quad \partial\boldsymbol{\chi}(U,t)$.



traction vector continuity is Newton's third law:

"When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body."

https://en.wikipedia.org/wiki/Newton%27s_laws_of_motion

Pressure discontinuity

Using traction vector continuity and $\boldsymbol{\sigma}^{\mathrm{e},+} = 0$ on $\partial\boldsymbol{\chi}(U, t)$:

$$-[p]\,\mathbf{n} + \mu\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\,\mathbf{n} - \boldsymbol{\sigma}^{\mathrm{e},-}\,\mathbf{n} = 0.$$

Lemma: Let $\mathbf{t}$ and $\mathbf{b}$ form a basis for the tangent plane to the point at which we are considering the jump. Then $[(\nabla\mathbf{u})\,\mathbf{t}] = [(\nabla\mathbf{u})\,\mathbf{b}] = 0$.

Proof: Consider a parametrized curve $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ defined on $\partial\boldsymbol{\chi}(U, t)$ that contains the point at which we consider the jump. This curve is constructed so its tangent vector $d\boldsymbol{\beta}/ds$ is equal to $\mathbf{t}$ at this point.

$$\frac{d}{ds}u_i(\boldsymbol{\beta}(s)) = \frac{d\boldsymbol{\beta}}{ds}\cdot\nabla u_i(\boldsymbol{\beta}(s)) = \mathbf{t}\cdot\nabla u_i(\boldsymbol{\beta}(s)).$$

Pressure discontinuity

$$-[p]\,\mathbf{n} + \mu\,[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\,\mathbf{n} - \boldsymbol{\sigma}^{\mathrm{e},-}\,\mathbf{n} = 0.$$

Lemma: $[\mathbf{n}\cdot(\nabla\mathbf{u})\,\mathbf{n}] = [\mathbf{n}\cdot(\nabla\mathbf{u})^T\mathbf{n}] = 0.$

Proof: Use the incompressibility condition and $[(\nabla\mathbf{u})\,\mathbf{t}] = [(\nabla\mathbf{u})\,\mathbf{b}] = 0.$

$$\boxed{[p] = -\mathbf{n}\cdot\boldsymbol{\sigma}^{e,-}\mathbf{n}}$$

Shear stress discontinuity

For a unit tangent vector $\mathbf{t}$ to $\partial\boldsymbol{\chi}(U,t)$:

$$\mu\,\mathbf{t}\cdot[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]\,\mathbf{n}=\mathbf{t}\cdot\boldsymbol{\sigma}^{e,-}\mathbf{n}$$

This jump condition can be reformulated in terms of the normal derivative of $\mathbf{u}$:

$$\boxed{\mu\,[(\nabla\mathbf{u})\,\mathbf{n}]=(\mathbf{I}-\mathbf{n}\,\mathbf{n}^T)\,\boldsymbol{\sigma}^{e,-}\,\mathbf{n}.}$$

Transforming vectors

Let $\mathbf{v}$ be the unit tangent vector to a curve $\gamma(s)$, i.e.

$$\mathbf{v} = \frac{d\gamma}{ds}.$$

Also, let $\tilde{\gamma}(s) = \chi(\gamma(s))$.

$$\frac{d\tilde{\gamma}}{ds} = \frac{d}{ds}\chi(\gamma(s)) = \mathbb{F}\,\frac{d\gamma}{ds} = \mathbb{F}\,\mathbf{v}.$$

Transforming area elements

Let $\mathbf{N} = \mathbf{v} \times \mathbf{w}$ be a normal vector in the reference configuration. Define:

$$\mathbf{n} = \frac{\mathbb{F}\,\mathbf{v} \times \mathbb{F}\,\mathbf{w}}{\|\mathbb{F}\,\mathbf{v} \times \mathbb{F}\,\mathbf{w}\|}$$

to be the normal vector in the current configuration. The ratio of sizes of area elements is:

$$\frac{dA}{da} = \frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbb{F}\,\mathbf{v} \times \mathbb{F}\,\mathbf{w}\|} = \frac{1}{\|\mathbb{F}\,\mathbf{v} \times \mathbb{F}\,\mathbf{w}\|}$$

This, combined with the identity: $\mathbb{F}\,\mathbf{v} \times \mathbb{F}\,\mathbf{w} = \det(\mathbb{F})\,\mathbb{F}^{-T}\mathbf{v} \times \mathbf{w}$ gives Nanson's relation:

$$\mathbf{n}\,da = \det(\mathbb{F})\,\mathbb{F}^{-T}\mathbf{N}\,dA.$$

Removing the pressure jump

By definition of $\mathbb{P}^e$ and Nanson's relation:

$$\boldsymbol{\sigma}^e\,\mathbf{n}\,da = \mathbb{P}^e\,\mathbf{N}\,dA \quad \text{and} \quad \mathbf{n}\,da = J\,\mathbb{F}^{-T}\,\mathbf{N}\,dA.$$

These equations imply:

$$\mathbf{n} = \frac{\mathbb{F}^{-T}\mathbf{N}}{\|\mathbb{F}^{-T}\mathbf{N}\|} \quad \text{and} \quad \boldsymbol{\sigma}^e\,\mathbf{n} = J^{-1}\frac{\mathbb{P}^e\,\mathbf{N}}{\|\mathbb{F}^{-T}\mathbf{N}\|},$$

so the jump in the pressure is:

$$[p] = -J^{-1}\frac{\mathbf{n}\cdot\mathbb{P}^e\,\mathbf{N}}{\|\mathbb{F}^{-T}\mathbf{N}\|}.$$

Idea to "remove" the jump: define a **modified stress** $\tilde{\mathbb{P}}^e$ so that

$$\mathbf{n}\cdot\tilde{\mathbb{P}}^e\mathbf{N} = 0.$$

Removing the pressure jump

Let φ be some scalar valued function defined on U and

$$\tilde{\mathbb{P}}^{\mathrm{e}} = \mathbb{P}^{\mathrm{e}} - J\,\varphi\,\mathbb{F}^{-T}.$$

Then $\mathbf{n} \cdot \tilde{\mathbb{P}}^{\mathrm{e}}\,\mathbf{N} = 0 \iff \mathbf{n} \cdot \mathbb{P}^{\mathrm{e}}\mathbf{N} = J\,\varphi\,\mathbf{n} \cdot \mathbb{F}^{-T}\mathbf{N}$,

implying we want φ to satisfy:

$$\varphi = J^{-1}\frac{\mathbb{F}^{-T}\mathbf{N}}{\|\mathbb{F}^{-T}\mathbf{N}\|^2} \cdot \mathbb{P}^{\mathrm{e}}\mathbf{N} := g \quad \text{on } \partial U.$$

We can find such a function by solving:

$$-\nabla^2\varphi = 0 \quad \text{in} \quad U, \quad \varphi = g \quad \text{on} \quad \partial U.$$

Obtaining the physical pressure

Recall that our modified Cauchy stress is given by:

$$\tilde{\boldsymbol{\sigma}}^e = J^{-1}\tilde{\mathbb{P}}^e\mathbb{F}^T = \boldsymbol{\sigma}^e - \varphi\mathbf{I}.$$

Sharp interface algorithm:

- Solve harmonic problem
  $-\nabla^2\varphi = 0 \quad \text{in} \quad U, \quad \varphi = g \quad \text{on} \quad \partial U.$
- Solve FSI problem with $\tilde{\mathbb{P}}^e$ for variables $\mathbf{u}$ and π.
  By construction, $[\pi] = 0$ on $\partial\boldsymbol{\chi}(U, t)$.
- Reconstruct the physical pressure
  $p(\mathbf{x}, t) = \pi(\mathbf{x}, t) + \varphi(\boldsymbol{\chi}^{-1}(x, t), t).$

Penalty method

The penalty method is an approach for approximating:

$$-\nabla^2 u = 0 \text{ in } U$$
$$u = g \text{ on } \partial U$$

Multiply by a test function φ and integrate by parts:

$$\int_U \nabla u \cdot \nabla \varphi - \int_{\partial U} \mathbf{n} \cdot \nabla u \, \varphi = 0$$

The Dirichlet condition looks almost like a Robin condition:

$$\varepsilon \, \mathbf{n} \cdot \nabla u + u = g \text{ on } \partial U$$

Penalty method cont.

Plugging this in gives

$$\int_U \nabla u \cdot \nabla \varphi + \varepsilon^{-1} \int_{\partial U} u\, \varphi = \varepsilon^{-1} \int_{\partial U} g\, \varphi$$

A finite element scheme is then:

Find $u_h \in V_h$ so that

$$\int_U \nabla u_h \cdot \nabla \varphi_h + \varepsilon^{-1} \int_{\partial U} u_h\, \varphi_h = \varepsilon^{-1} \int_{\partial U} g\, \varphi_h$$

for all $\varphi_h \in V_h$.

This approach is also called **Nitsche's method**.

Pressure field: thick pre-stressed ring

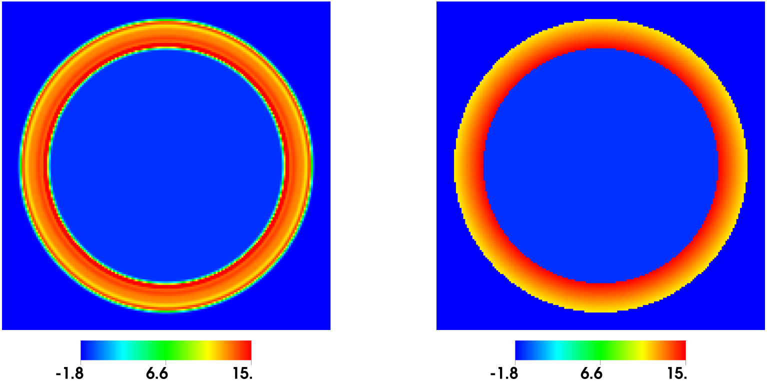


Figure: Sharp interface on the right.

π and φ fields: thick pre-stressed ring

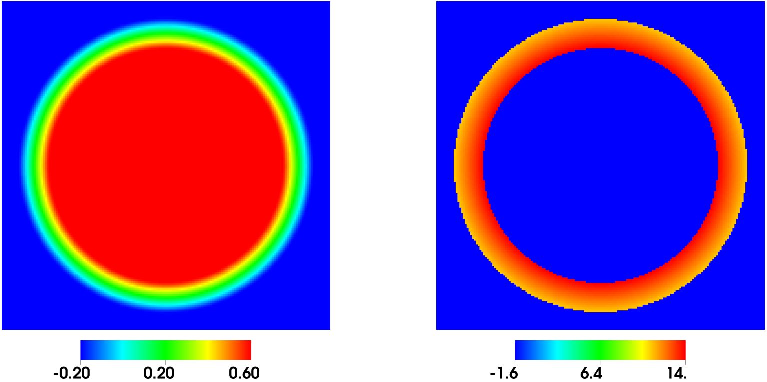
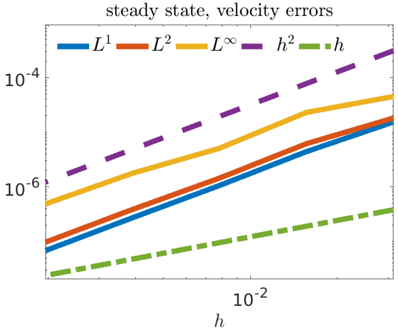
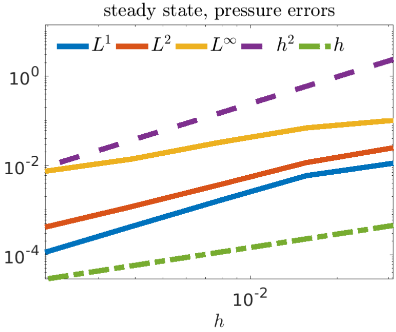


Figure: The π field is on the left and the φ field is on the right.

Numerical errors: thick pre-stressed ring with the sharp interface method





Pressure field: thick inflating ring

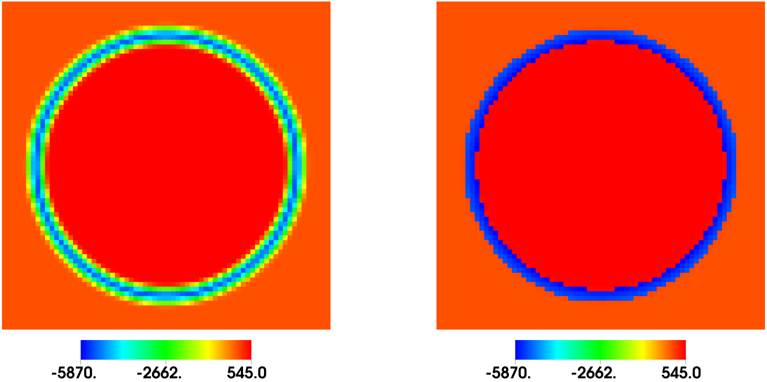
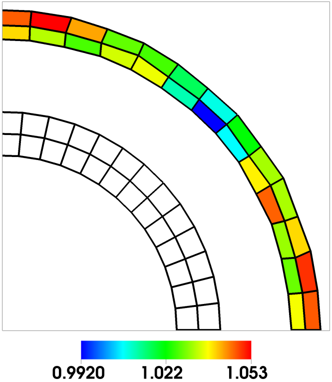


Figure: pressure field... sharp interface on the right.

Deformations: thick inflating ring



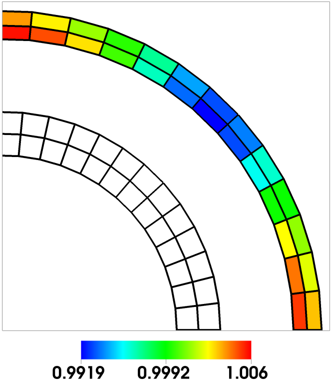


Figure: The original IBFE method is on the left and the sharp interface method is on the right. The coloring in the final configuration corresponds to the deteminant $J = \det(\mathbb{F})$.

Actively contracting torus

$$\mathbb{P}^{e}(\mathbf{X}, t) = \mu_e \left(\mathbb{F} - \mathbb{F}^{-T} \right) + T(\mathcal{G}^{-1}(\mathbf{X}), t)\, \mathbb{F}\, \mathbf{f}_0 \otimes \mathbf{f}_0.$$

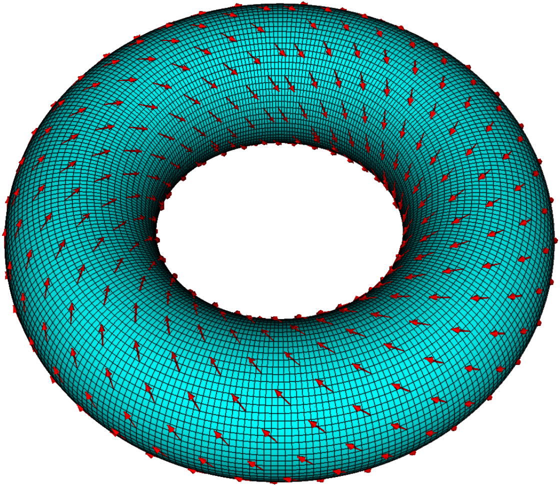


Figure: The mesh for the torus, with the fiber vector field $\mathbf{f}_0$ superimposed.

Actively contracting torus

McQueen and Peskin, *J Comput Phys*, 1989

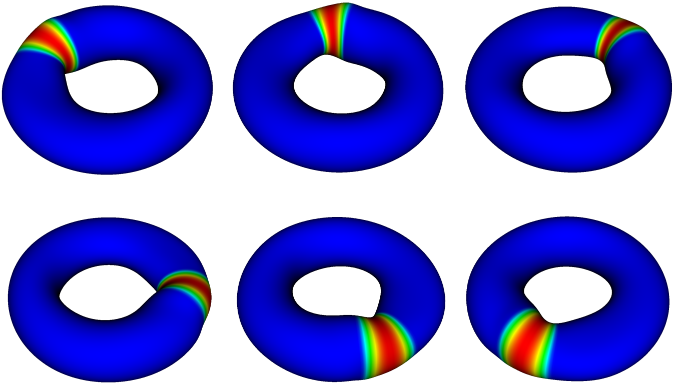


Figure: A visualization of contraction. Time increases from left–to–right and top–to–bottom. The color indicates the value of the tension function.

Velocity field: actively contracting torus

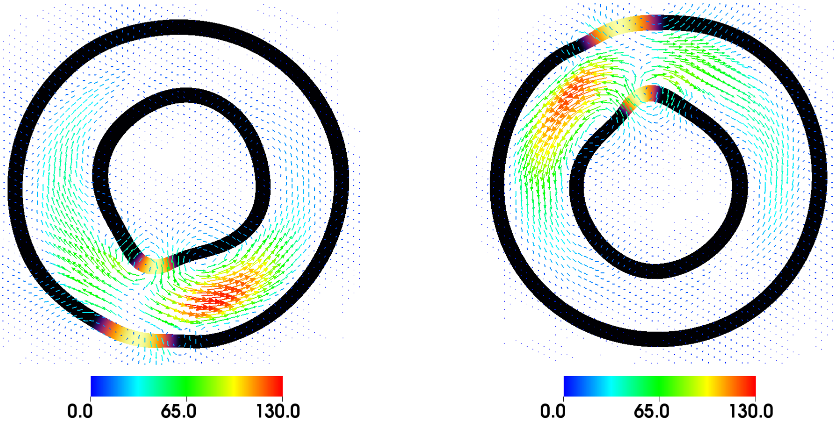


Figure: A slice of the velocity field and active tension function

Pressure field: actively contracting torus

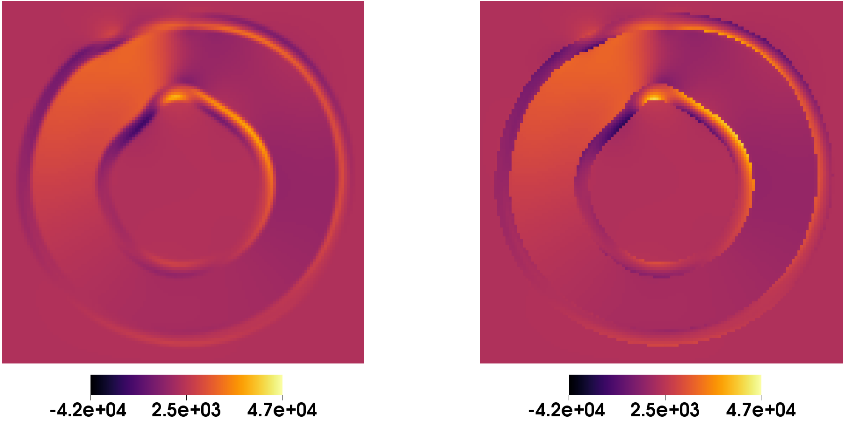
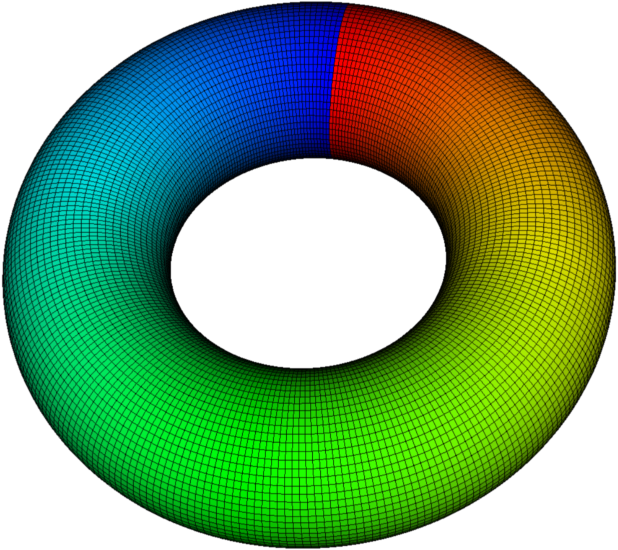


Figure: A slice of the pressure field... sharp interface on the right.



Unified weak formulation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t)+\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\right)=-\nabla p(\mathbf{x},t)+\mu\Delta\mathbf{u}(\mathbf{x},t)+\mathbf{g}(\mathbf{x},t)$$

$$\nabla\cdot\mathbf{u}(\mathbf{x},t)=0$$

$$\mathbf{g}(\mathbf{x},t)=\int_U \mathbf{G}(\mathbf{X},t)\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{X}$$

$$\frac{\partial \chi}{\partial t}(\mathbf{X},t)=\int_\Omega \mathbf{u}(\mathbf{x},t)\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{x}$$

where $\mathbf{G}(\mathbf{X},t)$ satisfies:

$$\int_U \mathbf{G}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,d\mathbf{X}=-\int_U \mathbb{P}^{\mathrm{e}}(\mathbf{X},t):\nabla_{\mathbf{X}}\mathbf{V}_h(\mathbf{X})\,d\mathbf{X}$$
$$+\int_U \mathbf{F}_{\mathrm{bdy}}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,d\mathbf{X}+\int_{\partial U}\mathbf{F}_{\mathrm{surf}}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,dA(\mathbf{X})$$

for all $\mathbf{V}$ in some finite element space.

Unified weak formulation

$$
\begin{aligned}
\rho\left(\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\right) &= -\nabla p(\mathbf{x},t) + \mu\Delta\mathbf{u}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t) \\
\nabla\cdot\mathbf{u}(\mathbf{x},t) &= 0 \\
\mathbf{g}(\mathbf{x},t) &= \int_U \mathbf{G}(\mathbf{X},t)\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{X} \\
\frac{\partial \chi}{\partial t}(\mathbf{X},t) &= \int_\Omega \mathbf{u}(\mathbf{x},t)\delta(\mathbf{x}-\chi(\mathbf{X},t))d\mathbf{x}
\end{aligned}
$$

where $\mathbf{G}(\mathbf{X},t)$ satisfies:

$$
\begin{aligned}
\int_U \mathbf{G}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,d\mathbf{X} = &-\int_U \mathbb{P}^{\mathrm{e}}(\mathbf{X},t) : \nabla_{\mathbf{X}}\mathbf{V}_h(\mathbf{X})\,d\mathbf{X} \\
&+ \int_U \mathbf{F}_{\mathrm{bdy}}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,d\mathbf{X} + \int_{\partial U}\mathbf{F}_{\mathrm{surf}}(\mathbf{X},t)\cdot\mathbf{V}_h(\mathbf{X})\,dA(\mathbf{X})
\end{aligned}
$$

for all $\mathbf{V}$ in some finite element space.

Isovolumetric/dilational split of the stress

given a strain energy density W, we define the PK1 stress in one of two ways:

through a **deviatoric projection**:

$$\mathbb{P}^{\mathsf{e}} = \mathsf{DEV}\Big[\frac{\partial W}{\partial \mathbb{F}}\Big] + \frac{\partial \mathcal{U}}{\partial \mathbb{F}}, \quad \mathsf{DEV}[\bullet] = (\bullet) - \frac{1}{3}(\bullet : \mathbb{F})\,\mathbb{F}^{-T}$$

or by using **modified invariants**: $\bar{W} = W(\bar{I}_1, \bar{I}_2, \ldots)$

$$\mathbb{P}^{\mathsf{e}} = \frac{\partial \bar{W}}{\partial \mathbb{F}} + \frac{\partial \mathcal{U}}{\partial \mathbb{F}}, \quad \bar{\mathbb{F}} = J^{-1/3}\mathbb{F}$$

$$I_1(\mathbb{F}) = \mathbb{F}^T\mathbb{F}, \quad I_2(\mathbb{F}) = \frac{1}{2}\Big[(\mathrm{tr}\,\mathbb{F}^T\mathbb{F})^2 - \mathrm{tr}\,(\mathbb{F}^T\mathbb{F})^2\Big]$$