

Overview for Introduction to Mathematical Modeling (MATH-UA 251)

Synopsis: Formulation and analysis of mathematical models. Mathematical tools include dimensional analysis, optimization, simulation, probability, and elementary differential equations. Applications to biology, sports, economics, and other areas of science. The necessary mathematical and scientific background will be developed as needed. Students will participate in formulating models as well as in analyzing them.

Grading: Grading will be based on a combination of weekly homework assignments, weekly quizzes and tests. There will also be a class project towards the end of the term. The assignments (including the project) will count for the majority of the course grade. The quizzes and the tests will make up the remaining portion of the course grade.

Syllabus: The topics are listed by week, although the timeline is only an approximation.

Mean-field population-dynamics models (chemical kinetics): Week 1: Ill begin by discussing mean-field population-dynamics models. These models, which typically take the form of a system of ordinary-differential-equations (ODEs), are often derived from something like a difference equation which attempts to characterize some underlying conservation law. Models like this are sometimes very well justified (e.g., chemical kinetics) but other times are less well justified (e.g., epidemiology and economics). Drawing from some recent work in epidemiology (Ivorra, Ngom and Ramos: Be-CoDiS: A mathematical model to predict the risk of human diseases spread between countries. Validation and application to the 2014 Ebola Virus Disease epidemic. arXiv:1410.6153v4) and (Chowell and Nishiura: Transmission dynamics and control of Ebola virus disease (EVD): a review. BMC Medicine 2014, 12:196), well introduce and derive a mean-field population dynamics model. Well also spend quite some time discussing the assumptions underlying this kind of model, and when these assumptions may not be valid (e.g., the assumption that events are independent and distributed with Poisson statistics).

Assumptions underlying mean-field models (statistical independence): Week 2: Continuing in the same vein, Ill introduce some of the mechanisms at work along the neuronal cell membrane, and slowly derive some of the equations governing the chemical kinetics describing ion-transport across this membrane (see, e.g., Dayan and Abbot: Theoretical neuroscience, or Koch: Biophysics of computation). These mean-field models are relatively well justified, and serve as the building blocks for the Hodgkin-Huxley equations for the dynamics of the neuronal membrane potential. These Hodgkin-Huxley equations, in turn, serve as the basis for much of theoretical neuroscience, and are used in components in many kinds of neural networks. Ill also point out that, in many cases, mean-field models lean heavily on the concept of regression: that is to say, using a simple functional relationship to describe the observed correlation between various measurements. Well discuss regression more later, but for now well talk a little bit about how these mean-field models actually behave.

Analyzing simple systems of ODEs (eigenvalues): Week 3: Using the previous examples as motivation, Ill discuss a few very simple ways to analyze systems of ordinary-differential-equations (i.e., ODEs). As many of the ODEs associated with mean-field-equations are time-independent (i.e., autonomous), well begin by looking at the induced flow across the phase-plane (using examples in 1- and 2-dimensions), and considering the behaviour around fixed-points (i.e., equilibria). Importantly, the behavior close to equilibria can be understood by considering the eigensystem borne from linearization. As an example we'll discuss a few simple predator-prey models (e.g., the lotke-volterra equations, see Canale, R.P. 'An analysis of Models describing Predator-Prey Interaction'. Biotechnology and Bioengineering, 12: 353-378, 1970).

Finding equilibria (Newton's method): Week 4: Finding the equilibria of a system of autonomous ODEs is equivalent to finding roots of the right-hand-side (i.e., fixed-points of the flow). At this point Ill mention bisection and newtons method. The former is of course very robust but only applicable in one dimension, whereas the latter is less robust, but generalizable to multiple dimensions. Ill also mention a few applications of root-finding that are not immediately related to ODEs (e.g., determining your position+time using GPS).

Summarizing data (regression): Week 5+6: At this point, going back to the mean-field models we discussed previously, we discuss the use of regression. We begin with the simplest case of linear regression, and discuss how linear-regression (and more generally polynomial-regression) has been used to determine many of the constants that show up in the chemical kinetics equations. Of course, linear-regression is not always applicable; sometimes systems are better described using discrete states which arent well described via continuous variables. In these latter situations logistic-regression is usually a better choice. Nevertheless, even logistic-regression isnt perfect: logistic-regression is rather sensitive to outliers, and carries with it many assumptions which may not always be valid. At this point we can attempt to apply our intuition to some studies carried out in the social sciences, e.g., (Sjollund, Hemmingsson and Allebeck: IQ and Level of Alcohol Consumption Findings from a National Survey of Swedish Conscripts. Alcohol Clin Exp Res, Vol **, No *, 2015: pp 18.) and (<http://arxiv.org/pdf/1503.03021v1.pdf>).

Application (Fitzhugh-Nagumo equations): Week 7: At this point we try and apply what we have learned so far to a specific set of interactions within the neuronal cell membrane. Given a collection of observations regarding membrane potentials, membrane currents, and ion concentrations, we first use regression to try and delineate a few simple functional relationships between these observations. Then we take these functional relationships, along with the conserved quantities within this system, and write a difference equation. This difference equation leads to a differential equation, which is similar to the single-compartment Fitzhugh-Nagumo equations. This system of ODEs has equilibria that depend on the input current; for sufficiently low input-current the single equilibrium is stable, but for sufficiently high input-current this equilibrium becomes unstable, leading to the generation of periodic action potentials. Obviously the dynamics of this system of ODEs is quite complicated; analytical solutions are out of reach and so we turn to numerical solutions.

Numerical solution of ODEs (Euler’s method); Week 8: Using these Fitzhugh-Nagumo equations as a case-study, we introduce explicit and implicit Eulers method. Also, referencing back to polynomial regression, we mention some of the more complicated numerical methods such as BDF-schemes and ABM-schemes. We perform a simple local- and global-error analysis for explicit and implicit Eulers method (as well as for the trapezoidal rule). We also discuss linear stability analysis (and regions of stability), relating back to the near-equilibrium behavior of an ODE (which depends on the eigensystem associated with the linearization).

Simulating a system of ODEs: Week 9: Now we have to tools to simulate a Fitzhugh-Nagumo-like system of ODEs, or the simpler integrate-and-fire (I&F) equations. These simulations serve as an introduction to stiffness and highlight the importance of implicit schemes whenever the linearized system is poorly conditioned. These simulations model something like a single neuron in the brain, assuming that that neuron receives unstructured (Poisson) input from other neurons around it. Consequently, this case-study gives us an opportunity to delve more deeply into the nature of Poisson processes, and how to simulate a Poisson process efficiently.

Complicated systems of ODEs (Chaos): Week 10: Using the single-neuron equations as a stepping-stone, we simulate networks of I&F-neurons, illustrating that even when the components of a network are simple and very well understood the network itself can exhibit emergent behavior that is substantially more complex. Ill use this case-study to argue that when dealing with complicated networks of interacting components mean-field models may not always be based on valid assumptions, and can often give misleading results. Given time, we’ll discuss the convolution theorem and the fast-fourier-transform, together which allow for the efficiently integration of stereotyped interactions across space (or time).

Markov Models (Discrete): Week 11: After reviewing and summarizing our work on ODEs so far, I’ll introduce simple discrete-state Markov Models. We’ll use this opportunity to discuss stochastic matrices, as well as the equilibrium distribution (and, by, extension, the power method for finding eigenvectors). We’ll describe several examples along these lines, including some simple markov models for gene-expression and the ‘page-rank’ algorithm.

Application (Pokemon-Go): Week 12: A recurring theme in our presentation will be the similarities between the evolution of a distribution of states (within a Markov Model) and the evolution of an autonomous system of ODEs. Importantly, both systems can be understood by considering the eigenvectors (respectively, eigenfunctions) of the associated evolution operator. To further highlight this relationship, we’ll discuss a simple population-dynamics model of the player ‘ecosystem’ associated with the mobile game ‘Pokemon-Go’. This population-dynamics model can be formulated in terms of either a Markov Model or an ODE; students will analyze both as a class project.

Markov Models (Heat-Equation): Week 13-14: As the term draws to a close we’ll use a simple infinite-dimensional discrete Markov-process (on a line) to derive the heat-equation (a partial-differential-equation) in 1-dimension. We’ll spend some time discussing this equation, drawing a connection between the statistical features of the underlying markov-process (i.e., microscopic diffusion) and the dynamical features of the green’s function for the heat-equation (i.e., the impulse-response). We’ll also use this opportunity to discuss the limiting behavior of the binomial-distribution. We will also review the properties of the fourier-transform (e.g., using the fourier-transform to solve the heat-equation).

More Markov Models (Brownian Motion): Week 15+: At this point it is unlikely that we’ll have too much time left. Typically one or two of the topics listed above will take up an extra lecture or so, and at this point we’ll start reviewing for the final. However, if we do have some time left, I’ll use the green’s function solution to the heat-equation to motivate the gaussian-increments used to define brownian motion. This should serve as a stepping stone towards the first few topics in stochastic calculus.