# Symmetry Breaking Phenomenon in Free Floating and Melting Ice

## Experiments and IBSE Method Simulation

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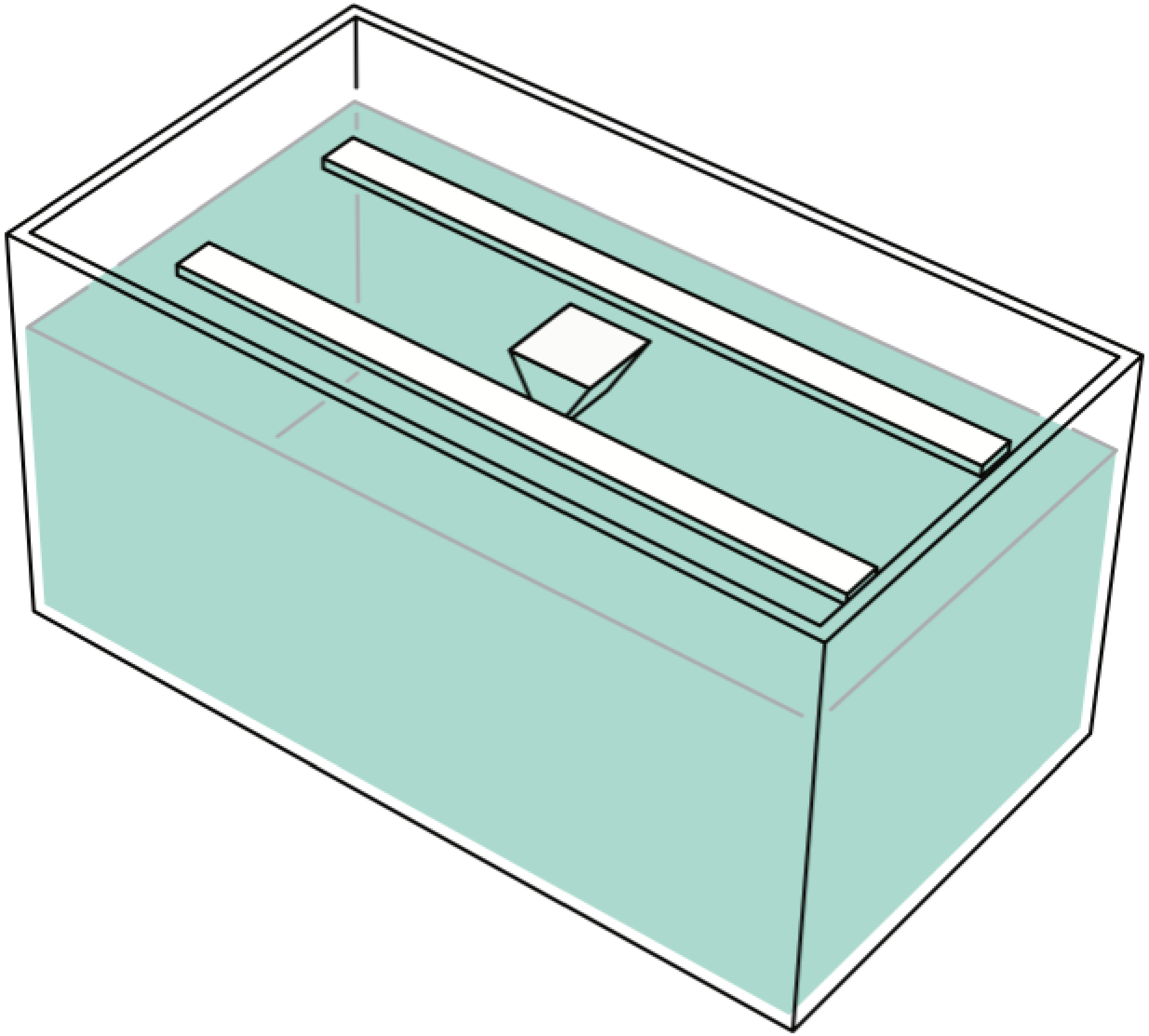
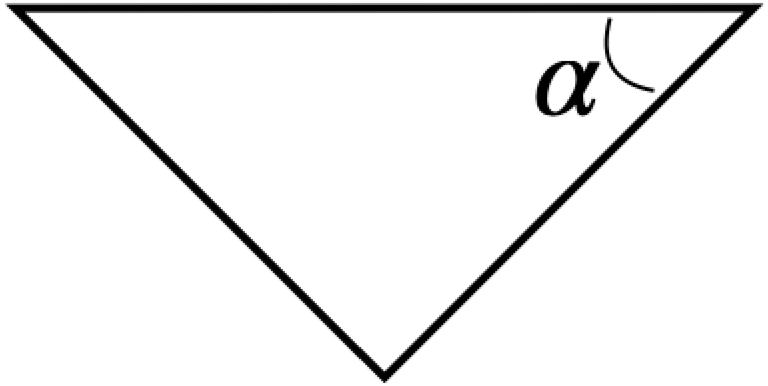
# Motivation

# Mathematician's Iceberg:

- Clear ice
- Fine cut prism shape
- 1d motion: 2 symmetric inclined surfaces

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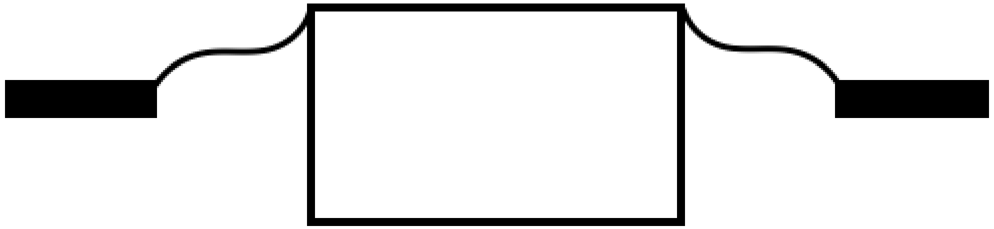
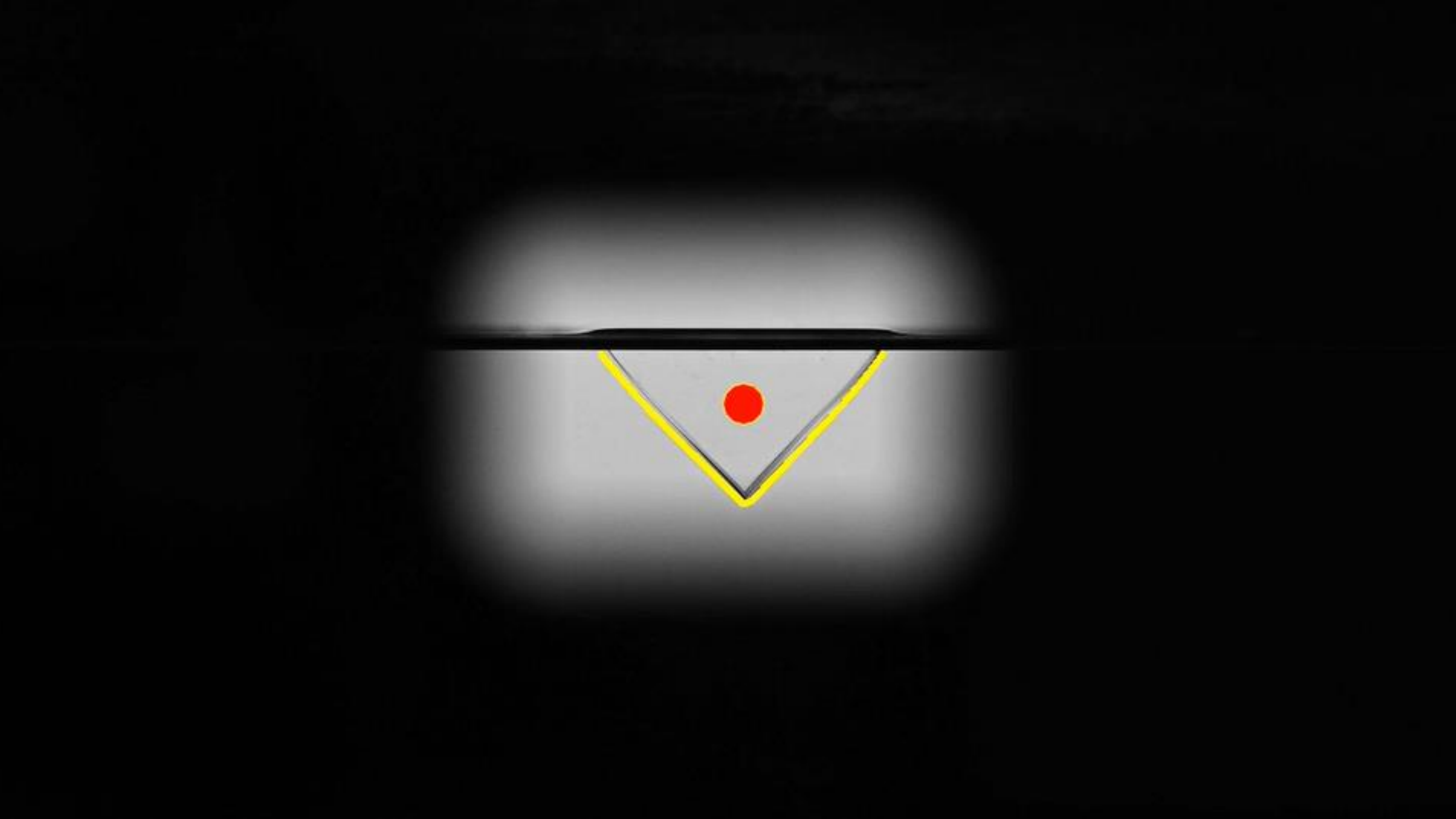
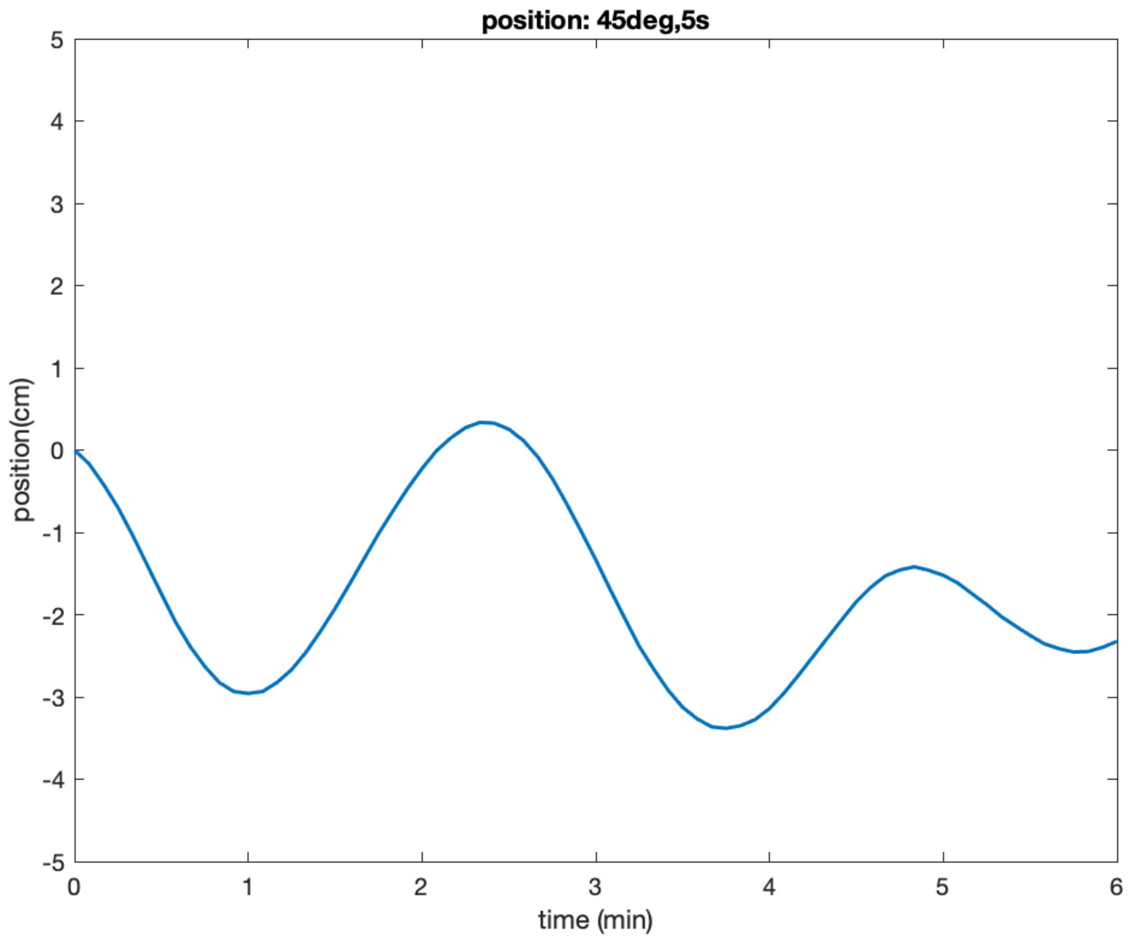
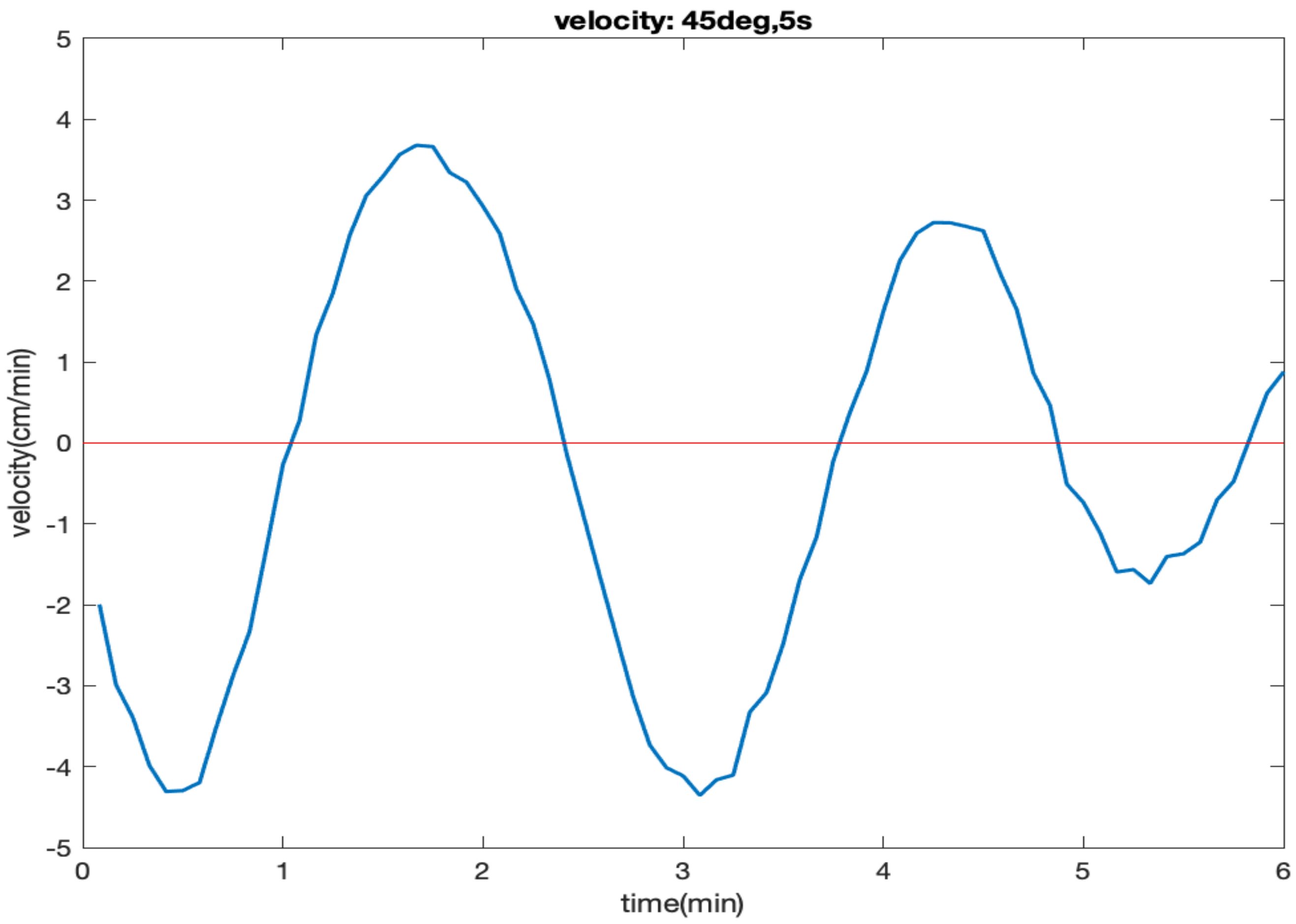






position: 45deg,5s
position(cm)
time (min)

velocity: 45deg,5s
velocity(cm/min)
time(min)

Melting

Conservation of momentum

Movement

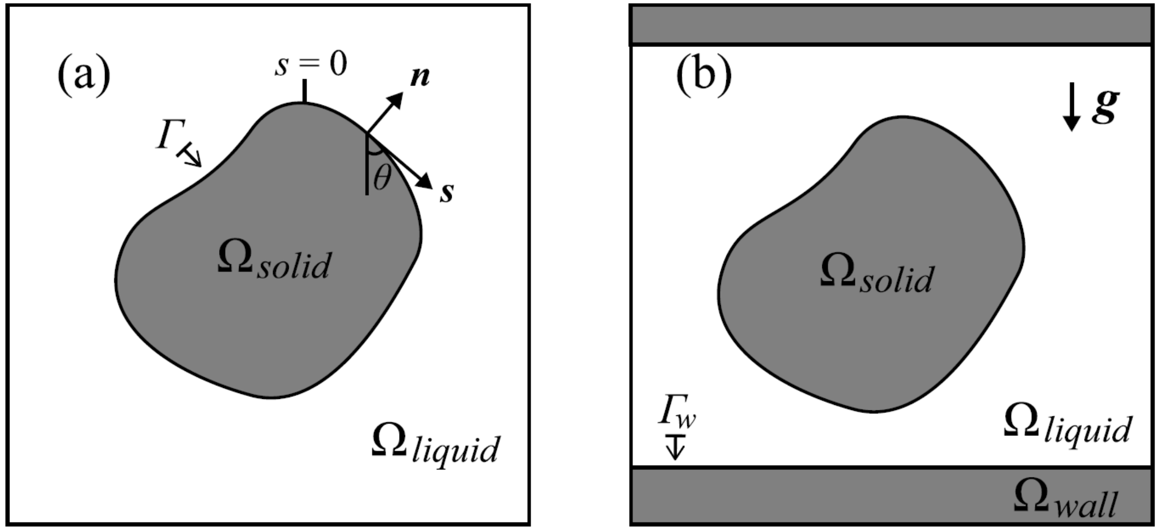
Melting

?

Oscillation

Negative feedback mechanism?

# Numerical Simulation: PDE system




$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \frac{1}{\mathrm{Pr}\sqrt{\mathrm{Gr}}} \Delta T \text{ in } \Omega_{\text{liquid}}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{\sqrt{\mathrm{Gr}}} \Delta \boldsymbol{u} - \nabla p + T\hat{\mathbf{y}} \text{ in } \Omega_{\text{liquid}}$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega_{\text{liquid}}$$

$$V_n = -\frac{\mathrm{St}}{\mathrm{Pr}\sqrt{\mathrm{Gr}}} \frac{\partial T}{\partial n} \text{ on } \Gamma$$

St: Stefan number
Pr: Prandtl number
Gr: Grashof number.

Take convection-diffusion eq as example: $\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pe} \Delta T$
Time discretization: $\frac{\partial T}{\partial t} \approx \frac{T^{t+\Delta t} - c^t}{\Delta t} \Rightarrow$

$$\left( \mathbb{I} - \frac{\Delta t}{Pe} \Delta \right) T^{t+\Delta t} = T^t - \Delta t \left( \mathbf{u}^t \cdot \nabla T^t \right)$$

General form(Helmholtz eq):

$$\left( \mathbb{I} - \sigma_T \Delta \right) T^{t+\Delta t} = f_T \left( T^t, \mathbf{u}^t \right)$$

If we write Helmholtz operator as $\mathcal{L}_T = (\mathbb{I} - \sigma_T \Delta)$, then this becomes

$$\mathcal{L}_T T = f_T$$

History: Peskin, 1972 $\rightarrow$ Stein et al, 2016, $\rightarrow$ Mac et al, 2021.

Idea: Find an $\eta$ smoothly extending the unknown $T$ from $\Omega_{liquid}$ to $C = \Omega_{liquid} \cup \Omega_{solid}$, such that $\eta \in C^k(C)$ and first k directives of T and $\eta$ match on the boundary $\Gamma = \partial\Omega_{liquid}$. To impose boundary conditions on $\Gamma$, we need to

- Interpolate the fluid velocity (as well as mass) to the structure (Lagrangian)
- spread structural forces to the fluid grids (Eulerian)

Interpolation operators $S^*_{(j)}$ :

$$\left(S^*_{(j)}\xi\right)(\alpha) = (-1)^j \int_C \xi(\chi) \frac{\partial^j \delta(\chi - X(\alpha))}{\partial n^j} d\chi$$

Spread operators $S(j)$:

$$\left(S_{(j)}F\right)(\chi) = (-1)^j \int_\Gamma F_j(\alpha) \frac{\partial^j \delta(\chi - X(\alpha))}{\partial n^j} dX(\alpha)$$

We compute a smooth enough $\eta$ by solving a high-order PDE of the form $\mathcal{H}^k\eta = 0$ in $C$, where $\mathcal{H}^k$ is e.g. polyharmonic operator $\Delta^{k+1}$. This $\eta$ defines an extension to the original RHS $f_T$ to be $\chi_{\Omega_{solid}}\mathcal{L}_T\eta + \chi_{\Omega_{liquid}}f_T$, defined all over $C$.

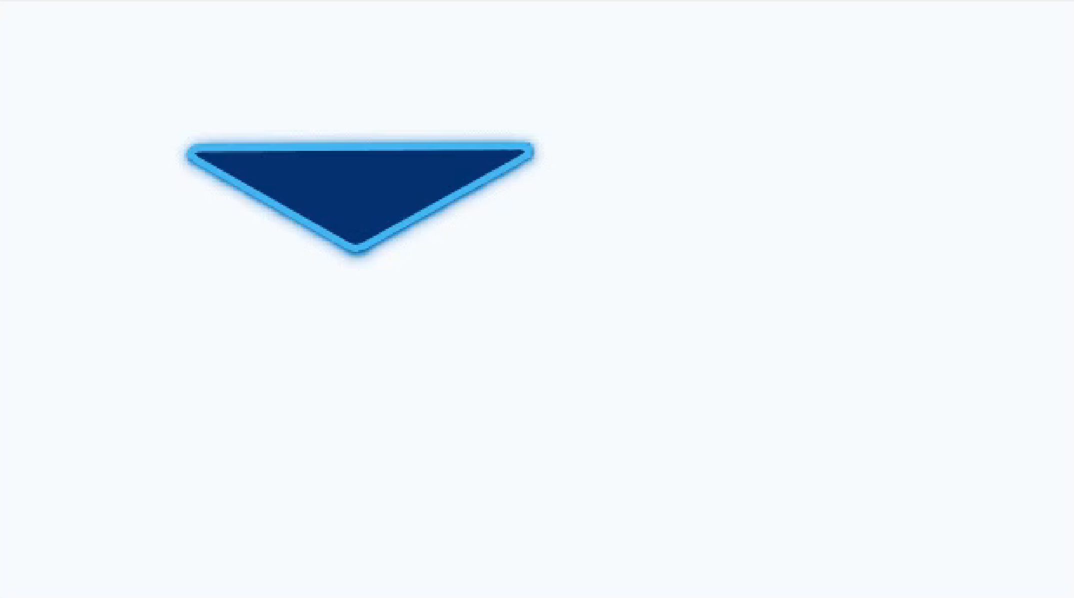
$$
\begin{aligned}
\mathcal{L}_T T &= \chi_{\Omega_{solid}}\mathcal{L}_T\eta + \chi_{\Omega_{liquid}}f_T s && \text{in } C,\\
\mathcal{H}^k\eta + T_k F_T &= 0 && \text{in } \Omega_{solid},\\
R_k^*(\eta - T) &= 0 && \text{at } \Gamma,\\
S_{(0)}^* T &= g && \text{at } \Gamma,
\end{aligned}
$$

where $R_k^* = \begin{pmatrix} S_{(1)}^* & \cdots & S_{(k)}^* \end{pmatrix}^\top$ provides an interpolation of the first k normal derivatives to the boundary, excluding the function value itself.

Above eq can be written in block form as

$$\left(\begin{array}{cc|c} \mathcal{L}_T & -\chi\Omega_{solid}\mathcal{L}_T & \mathbf{0} \\ \mathbf{0} & \mathcal{H}^k & T_k \\ \hline R_k^* & -R_k^* & \mathbf{0} \\ S_{(0)}^* & \mathbf{0} & \mathbf{0} \end{array}\right) \left(\begin{array}{c} T \\ \eta \\ F_T \end{array}\right) = \left(\begin{array}{c} \chi\Omega_{liquid} f_c \\ \mathbf{0} \\ \hline \mathbf{0} \\ g \end{array}\right).$$

To reduce computation, we compute the Schur complement (SC) current time: 0.9980, $f_x$= 0.0002, $L/L_0$= 99.92 %




# Next Steps

- Stably reproduce experiment
- Track & analyze simulation data

David B. Stein, Robert D. Guy, Becca Thomases, Immersed boundary smooth extension: A high-order method for solving PDE on arbitrary smooth domains using Fourier spectral methods, Journal of Computational Physics, Volume 304, 2016, Pages 252-274, ISSN 0021-9991, https://doi.org/10.1016/j.jcp.2015.10.023.

Jinzi Mac Huang, Michael J. Shelley, David B. Stein, A stable and accurate scheme for solving the Stefan problem coupled with natural convection using the Immersed Boundary Smooth Extension method, Journal of Computational Physics, Volume 432, 2021, 110162, ISSN 0021-9991, https://doi.org/10.1016/j.jcp.2021.110162.