YOUR SIGNATURE: ______________________________

(1) (a) (5 Points) Give an example of a surface S, oriented appropriately to use Stoke's Theorem, which has as its boundary the circle C of radius 1 centered at the origin, lying in the xy-plane, and oriented counterclockwise, when viewed from above.

(b) (5 Points) Is the following statement true or false? There exists a scalar function f and a vector field $\vec{F}$ satisfying $\text{div}(\text{grad}(f)) = \text{grad}(\text{div}(\vec{F}))$. Explain your reasoning.

(c) (5 Points) Give an example of a nonconstant vector field $\vec{F}$ and an oriented surface S such that $\int_S \vec{F} \cdot d\vec{A} = 1$.

(d) (5 Points) If possible, give an example of a vector field $\vec{H}(x, y, z)$ such that the curl $\vec{H} = \vec{k}$. If not possible, explain why.

(2) (a) (10 Points) If $\vec{F}(x,y,z) = \sin(y+z)\vec{i} + x\cos(y+z)\vec{j} + x\cos(y+z)\vec{k}$ and C is the upper half-circle in the xy-plane, of radius 3, and centered $(2,1,0)$, traversed counterclockwise when viewed from above, then

$$\int_C \vec{F}\cdot d\vec{r} = \underline{\hspace{3cm}}.$$

(b) (10 Points) A smooth vector field G has $\operatorname{curl}\vec{G}(x,y,z) = -3\vec{i} - 2\vec{j} + 7\vec{k}$. The circulation of $\vec{G}$ around a square of side $\sqrt{17}$ in the plane $x - y - z = 0$, oriented clockwise when viewed from the positive z-axis, is equal to ____________.

(3) (a) (10 Points) A smooth vector field $\vec{F}$ has div $\vec{F}(-1, 0, -2) = 13$. Estimate the flux of $\vec{F}$ out of a small sphere of radius 0.03 centered at the point $(-1, 0, -2)$.

(b) (10 Points) A smooth vector field $\vec{G}$ has curl $\vec{G}(0, 0, 0) = 11\vec{i} - 13\vec{j} + 7\vec{k}$. Estimate the circulation around a circle of radius 0.03 centered at the origin in the yz-plane and oriented counterclockwise when viewed from the positive x-axis.

(4) (a) (10 Points) Suppose $\vec{G}$ is a smooth vector field with div $\vec{G} = ||\vec{r}||$. Find the flux of $\vec{G}$ through the sphere of radius 3 centered at the origin and oriented outward.

(b) (10 Points) Let $\vec{F}$ be a smooth vector field, everywhere except at the origin, with div $\vec{F} = 0$ and flux 4π through any sphere oriented outward. Find the flux of $\vec{F}$ through the ellipsoid $2x^2 + y^2 + 3y^2 = 25$ oriented inward. Show your reasoning.

YOUR SIGNATURE:

(5) (a) (10 Points) Find a vector potential for the vector field $\vec{F} = 6x\vec{i} + (7y - z^2)\vec{j} + (x - 13z)\vec{k}$.

(b) (10 Points) Use Stoke's theorem to compute the flux of $\vec{F}$ through the hemisphere $z = \sqrt{1 - x^2 - y^2}$ oriented outward.

Useful formulas

- The volume and surface area of sphere of radius R are $\frac{4}{3}\pi R^3$ and $4\pi R$, respectively.

- $\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

- The divergence, or flux density, or a smooth vector field $\vec{F}$ is
$$\text{div } \vec{F}(x,y,z) = \lim_{\text{Volume}\to 0} \frac{\int_S \vec{F}\cdot d\vec{A}}{\text{Volume of } S},$$
where S is a sphere centered at (x,y,z), oriented outward, that contracts down to (x,y,z) in the limit.

- $\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$

- The circulation density of a smooth vector field $\vec{F}$ at (x,y,z) around the direction of the unit vector $\vec{n}$ is defined, provided the limit exists, to be
$$\text{circ}_{\vec{n}}\vec{F}(x,y,z) = \text{curl}\vec{F}\cdot\vec{n} = \lim_{\text{Area}\to 0}\frac{\text{Circulation around } C}{\text{Area inside } C} = \lim_{\text{Area}\to 0}\frac{\int_C \vec{F}\cdot d\vec{r}}{\text{Area inside } C}.$$
The circle C is in the plane perpendicular to $\vec{n}$ and oriented by the right-hand rule.

- The flux of a smooth vector field $\vec{F}$ through a smooth oriented surface S parameterized by $\vec{r} = \vec{r}(s,t)$, where (s,t) varies in a parameter region R, is given by
$$\int_S \vec{F}\cdot d\vec{A} = \int_R \vec{F}(\vec{r}(s,t))\cdot\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right) dsdt.$$

- The area of a surface S parameterized by $\vec{r} = \vec{r}(s,t)$, where (s,t) varies in a parameter region R, is given by
$$\int_S dA = \int_R \left|\left|\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right|\right| dsdt.$$

- **The Divergence Theorem (when applicable):**
$$\int_S \vec{F}\cdot d\vec{A} = \int_W \text{div } \vec{F}\, dV.$$

- **Stokes' Theorem (when applicable):**
$$\int_C \vec{F}\cdot d\vec{r} = \int_S \text{curl } \vec{F}\cdot d\vec{A}.$$