Using Nonconservative Forces to Design Autonomous Robots

Winston Liang

Mentors: Robert V. Kohn and Raghavendra Venkatraman

Courant Institute of Mathematical Sciences

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Introduction

Model

Simulation

Conclusion

Supp Info

Background

- Brandenbourger (U. Amsterdam) and Vitelli (U. Chicago)’s group have built a polygonal robot that can climb uphill autonomously.

- Active matter: an external source of energy does work by means of nonconservative forces.

Our Approach

Focusing on this "low dimensional" problem with a specific geometry (the hexagon), our goal is to seek an understanding of the mechanism behind the wheel’s locomotion.

▶ Contact with the surface: difficult to model.

▶ Consider a bicycle wheel:

Applying external torque → Rotates in free space.
Putting it on the ground → Moves forward or backward.

Analogously, if the nonconservative forces can cause the freestanding robot to rotate, then contact with the surface should make it walk.
Label the vertices: $\alpha = 0, 1, \ldots, N - 1$.

Vertices: identical point masses.

Edges: identical massless springs.

Angles: identical massless angular springs.

Motors: apply active torsional forces $\tau^a_\alpha = \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1})$. 
Consider the conservative case:

- **Spring:** preferred length $a$, spring constant $k$.
- **Angular spring:** preferred angle $\theta^0$, spring constant $\kappa$.
- **Mass:** $m = 1$.

The potential energy:

$$
V = \frac{k}{2} \sum_{\alpha} (|x_{\alpha} - x_{\alpha+1}| - a)^2 + \frac{\kappa}{2} \sum_{\alpha} (\theta_{\alpha} - \theta^0)^2.
$$

- Hamiltonian mechanics with masses interacting by forces:

$$
\ddot{x}_\alpha = -\nabla_{x_\alpha} V.
$$
Equation when $\kappa^a = 0$

\[
\ddot{x}_\alpha = -\nabla_{x_\alpha} V
\]

\[
= k \left[ \frac{x_{\alpha+1} - x_\alpha}{|x_{\alpha+1} - x_\alpha|} (|x_{\alpha+1} - x_\alpha| - a) + \frac{x_{\alpha-1} - x_\alpha}{|x_{\alpha-1} - x_\alpha|} (|x_{\alpha-1} - x_\alpha| - a) \right]
\]

\[
+ \left[ \frac{(x_{\alpha+1} - x_\alpha)^\perp}{|x_{\alpha+1} - x_\alpha|^2} (\tau_\alpha(x) - \tau_{\alpha+1}(x)) - \frac{(x_{\alpha-1} - x_\alpha)^\perp}{|x_{\alpha-1} - x_\alpha|^2} (\tau_\alpha(x) - \tau_{\alpha-1}(x)) \right]
\]

where

\[
\tau_\alpha(x) = \kappa(\theta_\alpha - \theta^0), \text{ and } \xi^\perp = (-\xi_2, \xi_1).
\]
How to Include the Nonconservative Forces?

When $\kappa^a \neq 0$:

- The angular restoring forces are expressed in torques.
- The motors apply active torques: $\tau_\alpha^a = \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1})$.
- Should get the same equation with different $\tau_\alpha$:

$$\ddot{x}_\alpha = F(x)$$

$$= k \left[ \frac{x_{\alpha+1} - x_\alpha}{|x_{\alpha+1} - x_\alpha|} (|x_{\alpha+1} - x_\alpha| - a) + \frac{x_{\alpha-1} - x_\alpha}{|x_{\alpha-1} - x_\alpha|} (|x_{\alpha-1} - x_\alpha| - a) \right]$$

$$+ \left[ \frac{(x_{\alpha+1} - x_\alpha)^\perp}{|x_{\alpha+1} - x_\alpha|^2} (\tau_\alpha(x) - \tau_{\alpha+1}(x)) - \frac{(x_{\alpha-1} - x_\alpha)^\perp}{|x_{\alpha-1} - x_\alpha|^2} (\tau_\alpha(x) - \tau_{\alpha-1}(x)) \right]$$

where

$$\tau_\alpha = \kappa(\theta_\alpha - \theta^0) + \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1}), \text{ and } \xi^\perp = (-\xi_2, \xi_1).$$

Remark: The total force on the polygon is 0 in both cases.
For different initial conditions, we use the Runge-Kutta method to solve the ODE system **numerically** and observe:

- Without the nonconservative forces: **periodic orbits**.
- With the nonconservative forces: shearing and **rotation** (compared with the periodic orbit).

Therefore, the nonconservative forces can break some periodic orbits and induce overall rotations.
Initial condition: deform the hexagon into a rectangle and assume the initial velocities are 0.

We observe the following evolution:

**Motors off:**

![Images of motors off evolution]

**Motors on:**

![Images of motors on evolution]
Capturing Rotation

We observe rotation empirically, but how do we quantify it?

- Since the total force is 0, the center of mass stays fixed.
- Go to polar coordinates:

  \[ x_\alpha = r_\alpha (\cos \varphi_\alpha, \sin \varphi_\alpha). \]

- Introduce the average of the angles:

  \[ \varphi_{\text{avg}} = \frac{1}{N} \sum_{\alpha} \varphi_\alpha. \]
Capturing Rotation (Cont’d)

▶ After calculation, we get:

\[ \dot{\varphi}_\alpha = \frac{1}{r_\alpha^2} \langle \dot{\mathbf{x}}_\alpha, \mathbf{x}_\perp \rangle. \]

▶ Singular when \( r \) near 0 → Need to suppose that the vertices always stay away from \( 0 \).

▶ Then,

\[ \dot{\varphi}_{\text{avg}} = \frac{1}{N} \sum_\alpha \dot{\varphi}_\alpha = \frac{1}{N} \sum_\alpha \frac{1}{r_\alpha^2} \langle \dot{\mathbf{x}}_\alpha, \mathbf{x}_\perp \rangle. \]

Observation: Our periodic orbits exhibit no rotation.

Elaboration: Periodic orbits with reflection-symmetry about one axis ensures that \( \sum_\alpha \varphi_\alpha \) is constant in time.
Solving the ODE for $\varphi_{avg}$ in the two mentioned settings,

- Motors off $\Rightarrow$ no overall rotation (blue).
- Motors on $\Rightarrow$ overall rotation (orange).
We introduce damping:

\[ \ddot{x}_\alpha = \ddot{F}(x) = F(x) - \Gamma \dot{x}_\alpha \]

where \( \Gamma > 0 \) is the damping coefficient.

Then we observe:

**Motors on, damping on:**

![Diagram of motors on, damping on](image1)

**Motors up, damping on:**

![Diagram of motors up, damping on](image2)
Solving the ODE for $\varphi_{avg}$ in the four mentioned settings,

- conservative case $\Rightarrow$ **no** overall rotation (blue).
- motors on $\Rightarrow$ exhibit overall **rotation** (orange).
- damping makes the rotation **steadier** (green & red).
Conclusions and Puzzles

Conclusions:

▶ Nonconservative forces convert the periodic oscillations into slowly rotating ones.

▶ Rotation of the freestanding robot should make it walk when it comes into contact with the surface.

Puzzles:

▶ Why does the conservative system have periodic orbits?

▶ Do motors also induce rotation of non-periodic conservative orbits?
Reference:

Martin Brandenbourger, Colin Scheibner, Jonas Veenstra, Vincenzo Vitelli and Corentin Coulais.
Let $\delta \theta_1, \delta \theta_2$ denote the deviation from the preferred angle $\theta^0$.

In the conservative case:

$$
\begin{pmatrix}
\tau_1 \\
\tau_2
\end{pmatrix} =
\begin{pmatrix}
-\kappa & 0 \\
0 & -\kappa
\end{pmatrix}
\begin{pmatrix}
\delta \theta_1 \\
\delta \theta_2
\end{pmatrix}
$$

This means each vertex only feels itself.

The nonconservative forces replace $\tau_\alpha$ with $\tau_\alpha + \tau^a_\alpha$, where

$$
\begin{pmatrix}
\tau^a_1 \\
\tau^a_2
\end{pmatrix} =
\begin{pmatrix}
0 & -\kappa^a \\
\kappa^a & 0
\end{pmatrix}
\begin{pmatrix}
\delta \theta_1 \\
\delta \theta_2
\end{pmatrix}
$$

The matrix associated with $\tau^a$ is skew symmetric.

Hence, the system cannot be written in potential form.