Using Nonconservative Forces to Design Autonomous Robots

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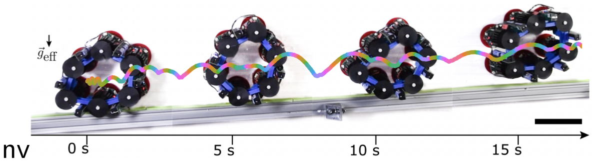
Background

- Brandenbourger (U. Amsterdam) and Vitelli (U. Chicago)'s group have built a polygonal robot that can climb uphill **autonomously**.
- Active matter: an external source of energy does work by means of **nonconservative** forces.



Martin Brandenbourger, Colin Scheibner, Jonas Veenstra, Vincenzo Vitelli and Corentin Coulais.
Limit cycles turn active matter into robots, 2021, arXiv:2108.08837.

Our Approach

Focusing on this "low dimensional" problem with a specific geometry (the hexagon), our goal is to seek an understanding of the mechanism behind the wheel's locomotion.

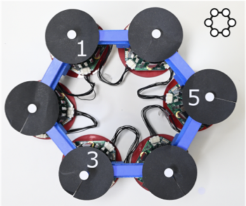
- Contact with the surface: **difficult** to model.
- Consider a bicycle wheel:

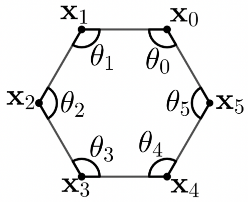
  Applying external torque → **Rotates** in free space.

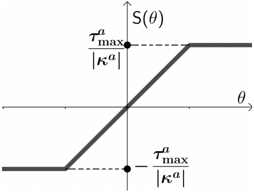
  Putting it on the ground → **Moves** forward or backward.

Analogously, if the nonconservative forces can cause the freestanding robot to **rotate**, then contact with the surface should make it **walk**.

Spring-Mass Model







- Label the vertices: $\alpha = 0, 1, \ldots, N-1$.
- Vertices: identical point masses.
- Edges: identical massless springs.
- Angles: identical massless angular springs.
- Motors: apply active torsional forces $\tau^a_\alpha = \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1})$.

Potential Energy (with Only Conservative Forces)

Consider the conservative case:

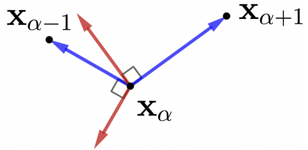
- Spring: preferred length a, spring constant k.
- Angular spring: preferred angle θ^0, spring constant κ.
- Mass: $m = 1$.
- The potential energy:

$$V = \frac{k}{2} \sum_{\alpha} (|\mathbf{x}_\alpha - \mathbf{x}_{\alpha+1}| - a)^2 + \frac{\kappa}{2} \sum_{\alpha} (\theta_\alpha - \theta^0)^2.$$

- Hamiltonian mechanics with masses interacting by forces:

$$\ddot{\mathbf{x}}_\alpha = -\nabla_{\mathbf{x}_\alpha} V.$$

Equation when $\kappa^a = 0$



$$
\begin{aligned}
\ddot{\mathbf{x}}_\alpha &= -\nabla_{\mathbf{x}_\alpha} V \\
&= k\left[\frac{\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha}{|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha|}(|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha|-a) + \frac{\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha}{|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha|}(|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha|-a)\right] \\
&+ \left[\frac{(\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha)^\perp}{|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha|^2}(\tau_\alpha(\mathbf{x})-\tau_{\alpha+1}(\mathbf{x})) - \frac{(\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha)^\perp}{|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha|^2}(\tau_\alpha(\mathbf{x})-\tau_{\alpha-1}(\mathbf{x}))\right]
\end{aligned}
$$

where

$$\tau_\alpha(\mathbf{x}) = \kappa(\theta_\alpha - \theta^0), \text{ and } \xi^\perp = (-\xi_2, \xi_1).$$

How to Include the Nonconservative Forces?

When $\kappa^a \neq 0$:

- The angular restoring forces are expressed in **torques**.
- The motors apply active **torques**: $\tau_\alpha^a = \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1})$.
- Should get the **same** equation with **different** τ_α:

$$\begin{aligned}\ddot{\mathbf{x}}_\alpha &= \mathbf{F}(\mathbf{x}) \\ &= k\left[\frac{\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha}{|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha|}(|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha| - a) + \frac{\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha}{|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha|}(|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha| - a)\right] \\ &+ \left[\frac{(\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha)^\perp}{|\mathbf{x}_{\alpha+1}-\mathbf{x}_\alpha|^2}(\tau_\alpha(\mathbf{x}) - \tau_{\alpha+1}(\mathbf{x})) - \frac{(\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha)^\perp}{|\mathbf{x}_{\alpha-1}-\mathbf{x}_\alpha|^2}(\tau_\alpha(\mathbf{x}) - \tau_{\alpha-1}(\mathbf{x}))\right]\end{aligned}$$

where

$$\tau_\alpha = \kappa(\theta_\alpha - \theta^0) + \kappa^a S(\theta_{\alpha+1} - \theta_{\alpha-1}), \text{ and } \xi^\perp = (-\xi_2, \xi_1).$$

Remark: The total force on the polygon is **0** in both cases.

Idea

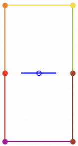
For different initial conditions, we use the Runge-Kutta method to solve the ODE system **numerically** and observe:

- Without the nonconservative forces: **periodic orbits**.
- With the nonconservative forces: shearing and **rotation** (compared with the periodic orbit).

Therefore, the nonconservative forces can break some periodic orbits and induce overall rotations.

Examples

Initial condition: deform the hexagon into a rectangle and assume the initial velocities are **0**.



We observe the following evolution:

Motors **off**:

Motors **on**:

Capturing Rotation

We observe rotation empirically, but how do we quantify it?

- Since the total force is **0**, the center of mass stays **fixed**.
- Go to polar coordinates:

$$\mathbf{x}_\alpha = r_\alpha(\cos\varphi_\alpha, \sin\varphi_\alpha).$$

- Introduce the average of the angles:

$$\varphi_{avg} = \frac{1}{N}\sum_\alpha \varphi_\alpha.$$

Capturing Rotation (Cont'd)

- After calculation, we get:

$$\dot{\varphi}_\alpha = \frac{1}{r_\alpha^2}\langle \dot{\mathbf{x}}_\alpha, \mathbf{x}_\alpha^\perp \rangle.$$

- Singular when r near $0 \rightarrow$ Need to suppose that the vertices always stay away from $\mathbf{0}$.
- Then,

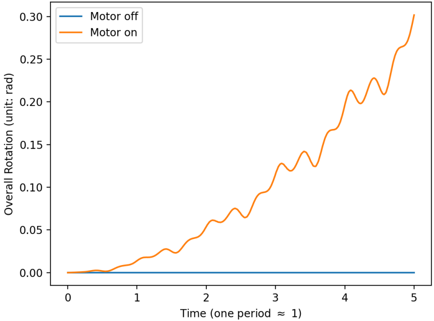
$$\dot{\varphi}_{avg} = \frac{1}{N}\sum_\alpha \dot{\varphi}_\alpha = \frac{1}{N}\sum_\alpha \frac{1}{r_\alpha^2}\langle \dot{\mathbf{x}}_\alpha, \mathbf{x}_\alpha^\perp \rangle.$$

Observation: Our periodic orbits exhibit no rotation.

Elaboration: Periodic orbits with reflection-symmetry about one axis ensures that $\sum_\alpha \varphi_\alpha$ is constant in time.

Results

Solving the ODE for φ_{avg} in the two mentioned settings,



- Motors off ⇒ **no** overall rotation (blue).
- Motors on ⇒ overall **rotation** (orange).

Can We Do Better?

We introduce damping:

$$\ddot{\mathbf{x}}_\alpha = \tilde{\mathbf{F}}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) - \Gamma \dot{\mathbf{x}}_\alpha$$

where $\Gamma > 0$ is the damping coefficient.

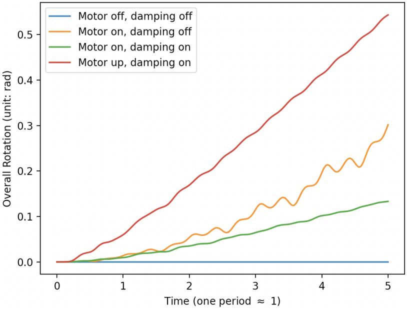
Then we observe:

Motors **on**, damping **on**:

Motors **up**, damping **on**:

Results

Solving the ODE for φ_{avg} in the four mentioned settings,



- conservative case ⇒ **no** overall rotation (blue).
- motors on ⇒ exhibit overall **rotation** (orange).
- damping makes the rotation **steadier** (green & red).

Conclusions and Puzzles

Conclusions:

- Nonconservative forces convert the **periodic oscillations** into slowly **rotating** ones.
- Rotation of the freestanding robot should make it walk when it comes into **contact with the surface**.

Puzzles:

- Why does the conservative system have **periodic** orbits?
- Do motors also induce rotation of **non-periodic** conservative orbits?

Reference

Reference:

Martin Brandenbourger, Colin Scheibner, Jonas Veenstra, Vincenzo Vitelli and Corentin Coulais.
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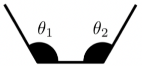
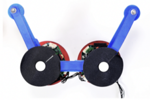
The Logic behind Their Choice of the Nonconservative Forces





- Let $\delta\theta_1$, $\delta\theta_2$ denote the deviation from the preferred angle θ^0.
- In the conservative case:

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -\kappa & 0 \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

  This means each vertex only feels itself.
- The nonconservative forces replace τ_α with $\tau_\alpha + \tau_\alpha^a$, where

$$\begin{pmatrix} \tau_1^a \\ \tau_2^a \end{pmatrix} = \begin{pmatrix} 0 & -\kappa^a \\ \kappa^a & 0 \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

  The matrix associated with τ^a is skew symmetric.
  Hence, the system cannot be written in potential form.