Motion of a Particle under Asymmetric Forces

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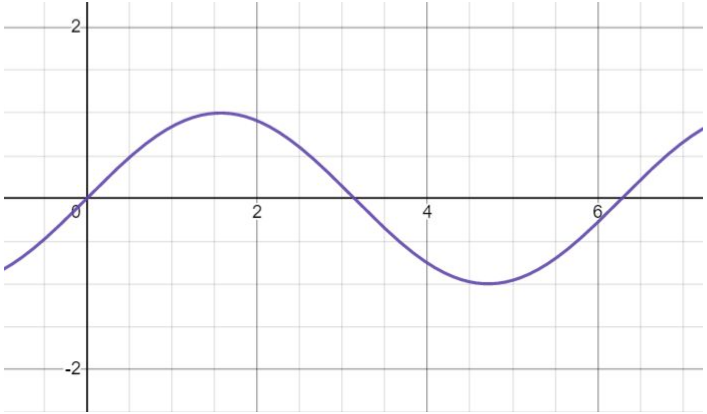
NYU COURANT

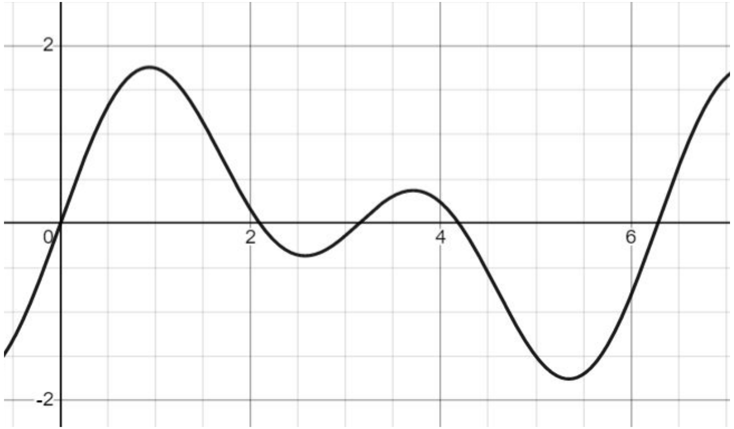
Definition

Let f(t) be a periodic function. We say f is **symmetric** if the second half of its period is the negative of the first half. Otherwise, f is **asymmetric**.

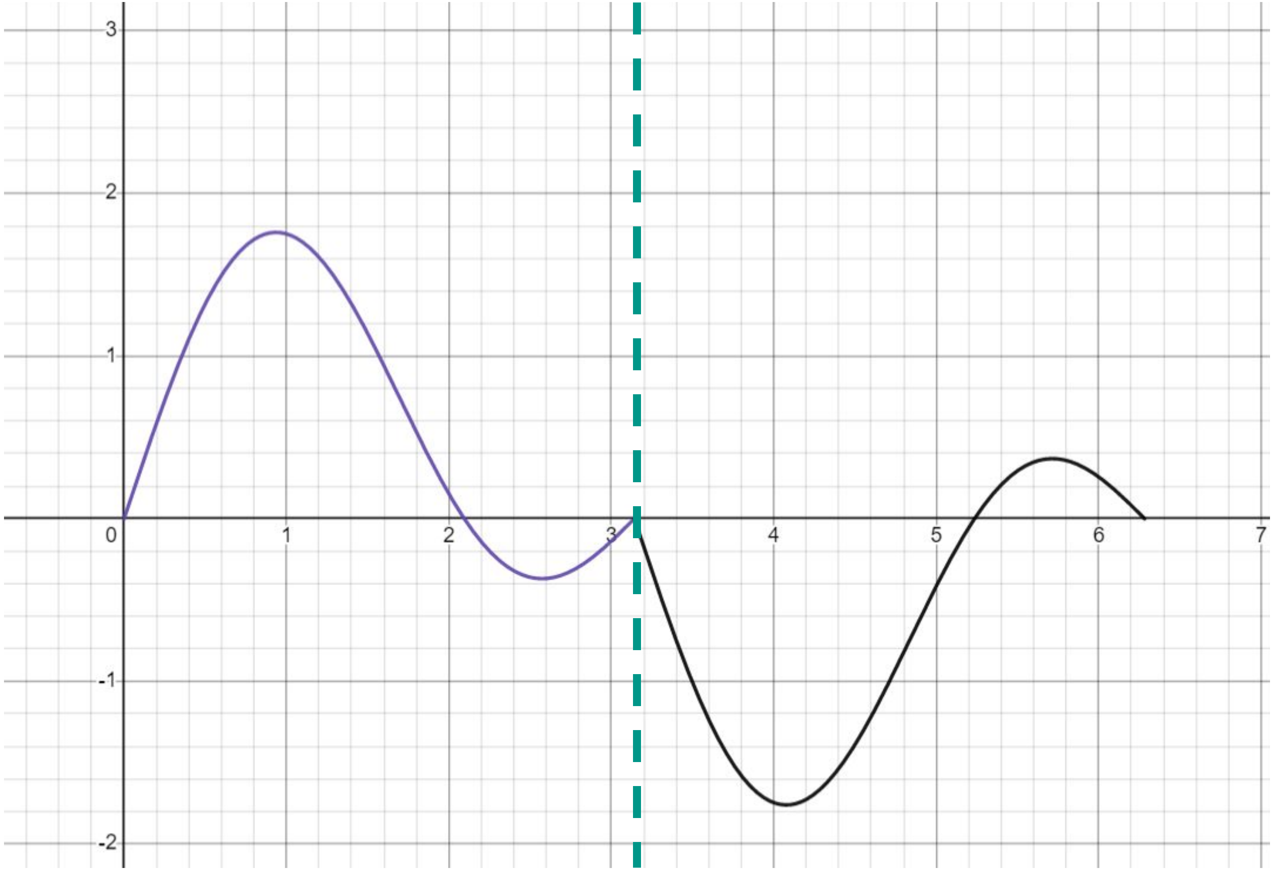
Ex:



f(x) = sin(x) (symmetric)



f(x) = sin(x) + sin(2x) (asymmetric)

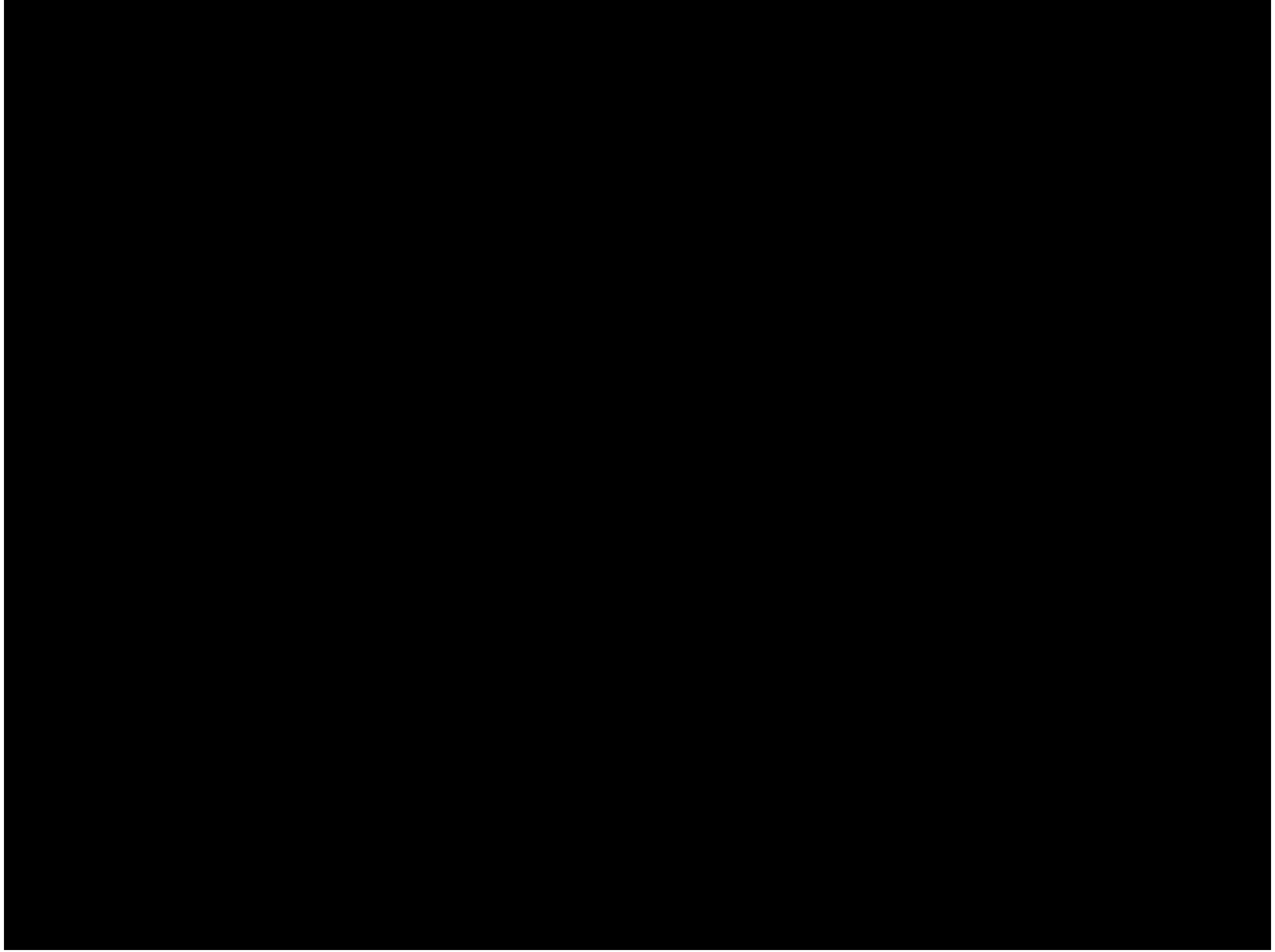


sin(t) + sin(2t) if it *were* symmetric

Particle under an Asymmetric Force with Nonlinear Drag

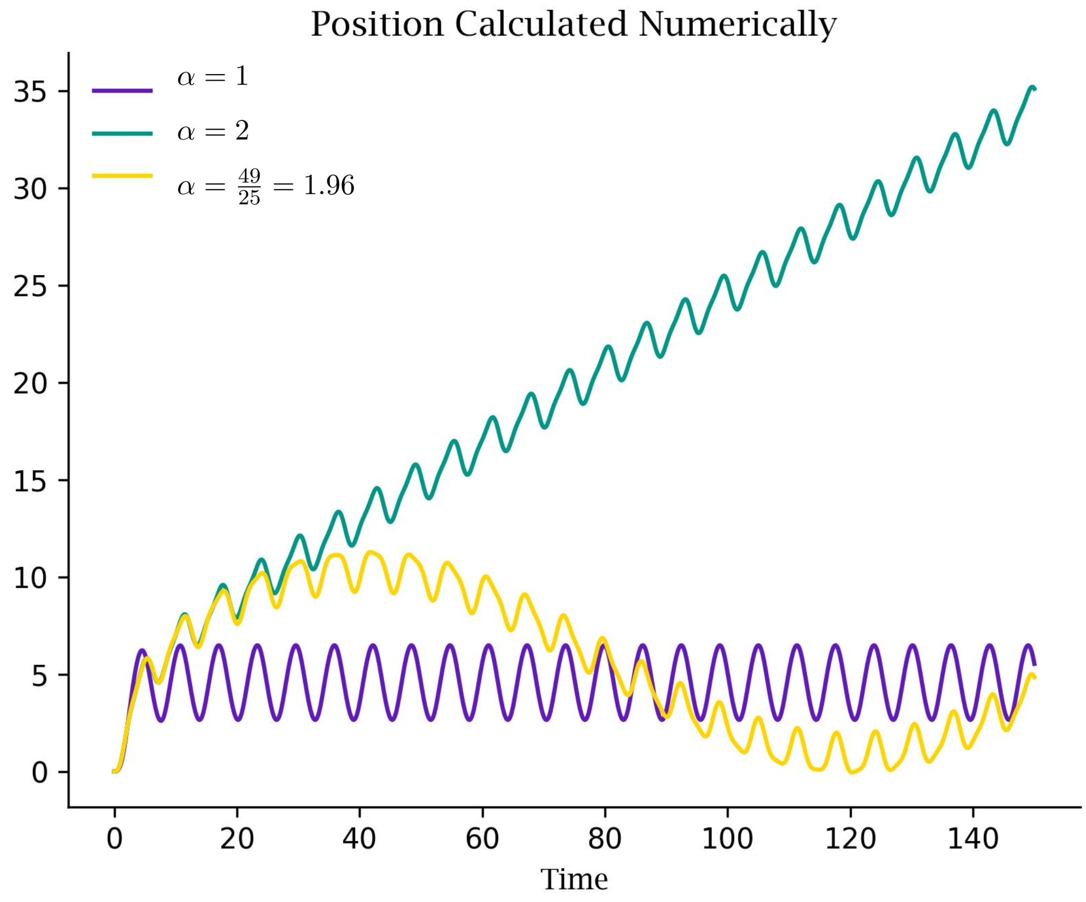
- Particle under an asymmetric force, say
  $$F(t) = \sin(t) + \sin(\alpha t), \alpha > 1$$
- Time-average value of $F(t)$= 0
- Newton's second law then gives (after nondimensionalization of units): $\frac{dv}{dt} = \sin(t) + \sin(\alpha t) - \epsilon v^3$ for $0 < \epsilon \ll 1$.

Nonlinear Drag Term



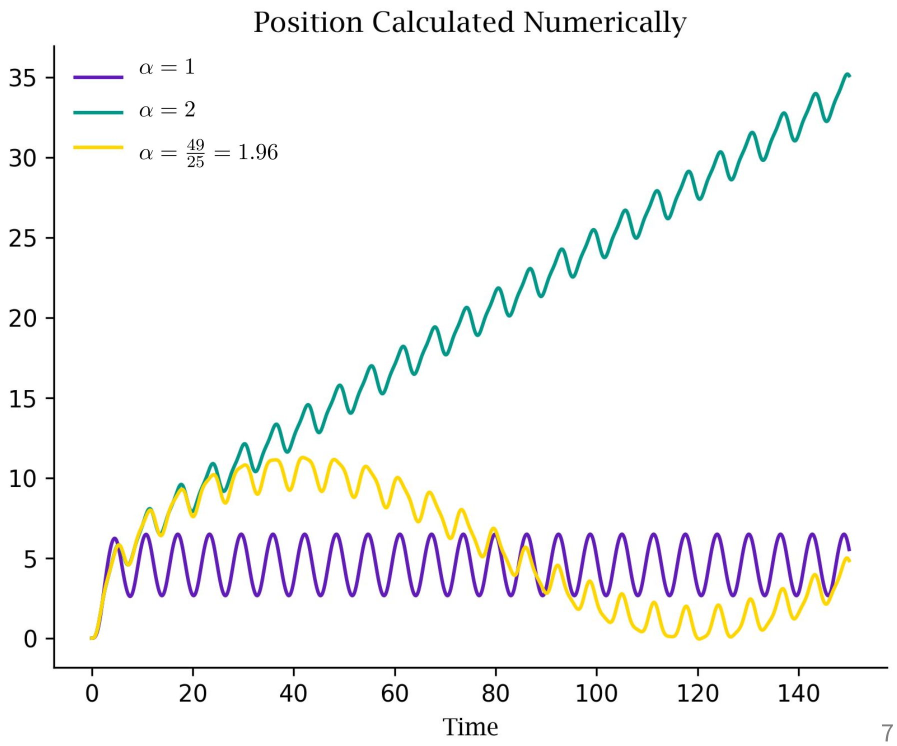
Hashemi et al. Phys. Rev. E, 2022

$$\frac{dv}{dt} = \sin(t) + \sin(\alpha t) - \epsilon v^3$$



Hallmarks of the Long-Term Motion (From Experimental/Numerical Observation)

- Time-average always constant
- Varies wildly with different choices of **α**
  - **α** = 2 vs. **α** = 49/25 = 1.96
  - ‘Simpler’ ratios more likely to induce motion
    - 4/3 vs. 25/24
    - odd/odd



Goals

- Find a good analytical approximation for the solution to the ODE:

$$\frac{dv}{dt} = \sin(t) + \sin(\alpha t) - \epsilon v^3$$

  - Accurate way to find the motion of a particle under such forces
  - Could provide intuition into why the particle moves
- Understand how different choices of α affect the motion

Methods

- Analytical approximations using perturbation and two-timing
  - Power series in ϵ
  - Multiple time scales

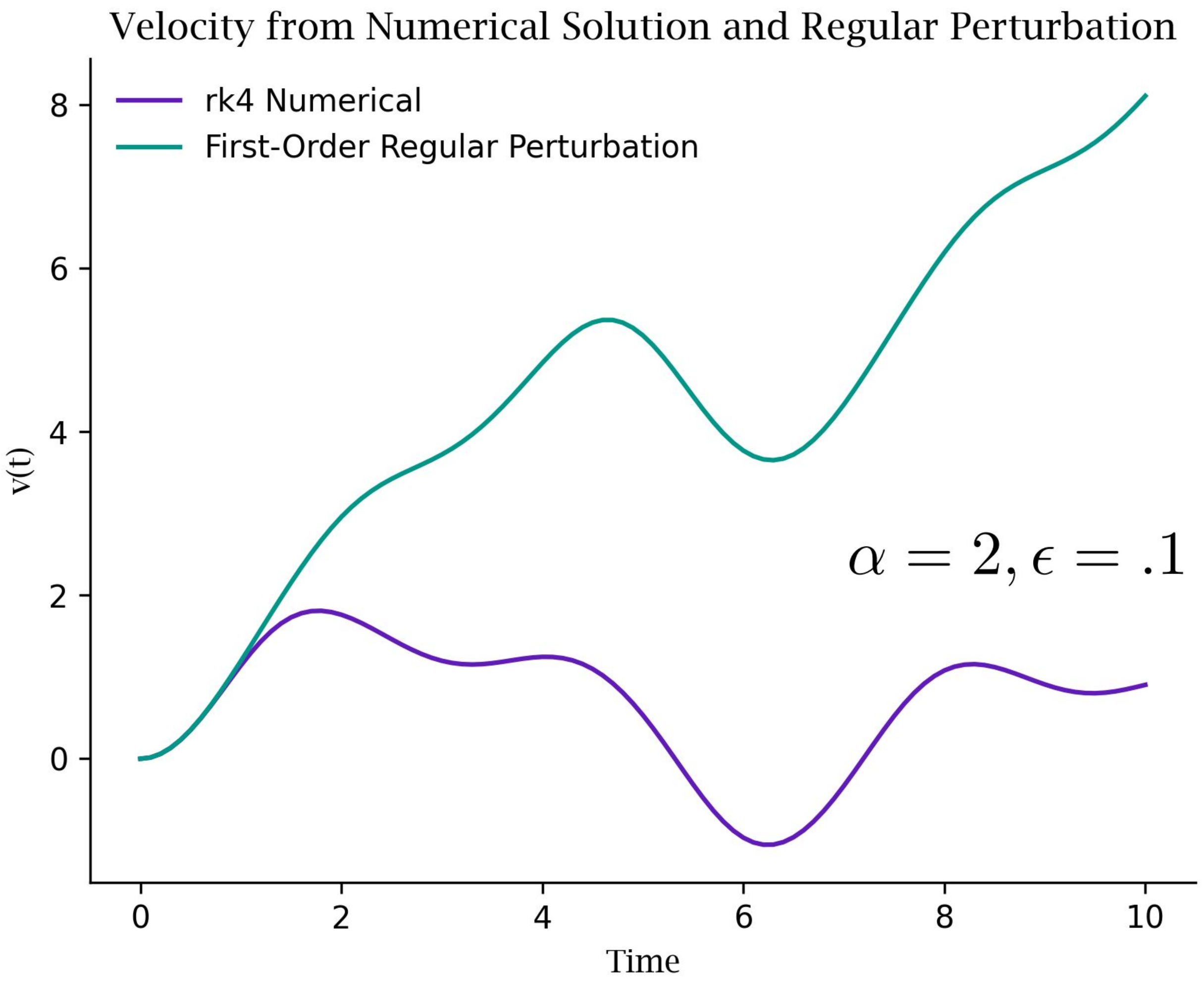
Regular Perturbation Attempt

$$\frac{dv}{dt} = \sin(t) + \sin(\alpha t) - \epsilon v^3$$

- For small ϵ , approximate: $v(t, \epsilon) = v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \ldots$
- Differentiate: $\sin(t) + \sin(\alpha t) - \epsilon v^3 = v_0'(t) + \epsilon v_1'(t) + O(\epsilon^2)$
- Expand: $\sin(t) + \sin(\alpha t) - \epsilon v_0^3 + O(\epsilon^2) = v_0'(t) + \epsilon v_1'(t) + O(\epsilon^2)$
- Solve: $v_0(t) = \int \sin(t) + \sin(\alpha t) dt$

$$v_0(0) = 0 \text{ and } v_1(0) = 0.$$

$$-v_1(t) = \int (v_0)^3 \, dt$$



Issues caused by **secular terms:** t, tsin(t)

Solution with Two-Timing

- Two **independent** time scales: τ = t (small time), T = **ε**t (big time)
- Same power series and process as before (w/ chain rule):

$$\sin(\tau) + \sin(\alpha\tau) - \epsilon\,(v_0(\tau, T))^3 + O(\epsilon^2) = \frac{\partial v_0}{\partial \tau} + \epsilon\left(\frac{\partial v_0}{\partial T} + \frac{\partial v_1}{\partial \tau}\right) + O(\epsilon^2)$$

- Solve:

$$\sin(\tau) + \sin(\alpha\tau) = \frac{\partial v_0}{\partial \tau}$$

$$-(v_0)^3 = \frac{\partial v_0}{\partial T} + \frac{\partial v_1}{\partial \tau}$$

$$\Longrightarrow$$

$$v_0(\tau, T) = -\cos(\tau) - \frac{\cos(\alpha\tau)}{\alpha} + A(T)$$

$$-\frac{\partial v_1}{\partial \tau} = \left(-\cos(\tau) - \frac{\cos(\alpha\tau)}{\alpha} + A(T)\right)^3 + A'(T)$$

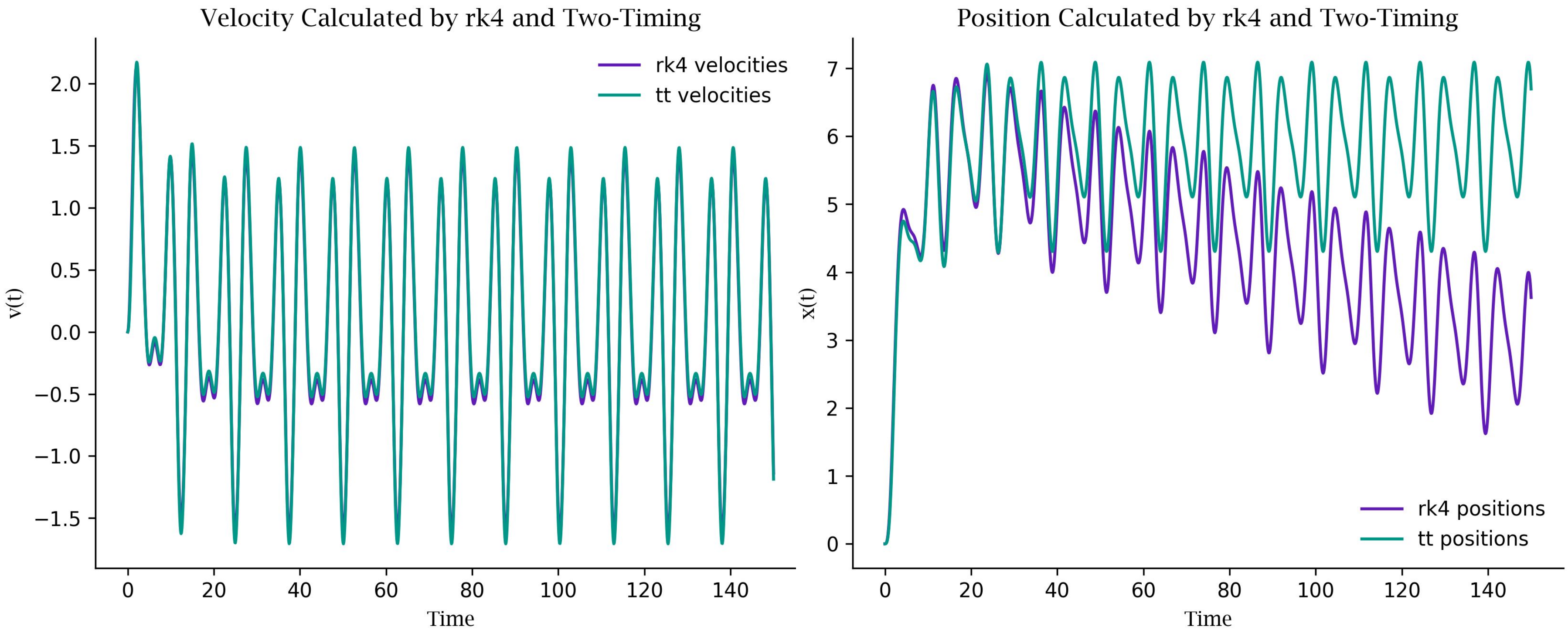
Choosing A(T) Correctly

$$-\frac{\partial v_1}{\partial \tau} = \left(-\cos(\tau) - \frac{\cos(\alpha\tau)}{\alpha} + A(T)\right)^3 + A'(T)$$

- Eliminate **secular terms** to ensure constant velocity
- Find non-zero time average terms like $\cos^2(\tau),\ \cos^2(\alpha\tau)$; set up a differential equation for A(T) to get rid of them

$$A'(T) = -\frac{3\alpha^2+3}{2\alpha^2}A(T) - (A(T))^3 \iff A(T) = \sqrt{\frac{\frac{3\alpha^2+3}{2\alpha^2}}{\left(\frac{\frac{3\alpha^2+3}{2}}{(1+\alpha)^2}+1\right)e^{\frac{3\alpha^2+3}{\alpha^2}T} - 1}}$$

$$\alpha = \frac{3}{2}, \epsilon = .1$$



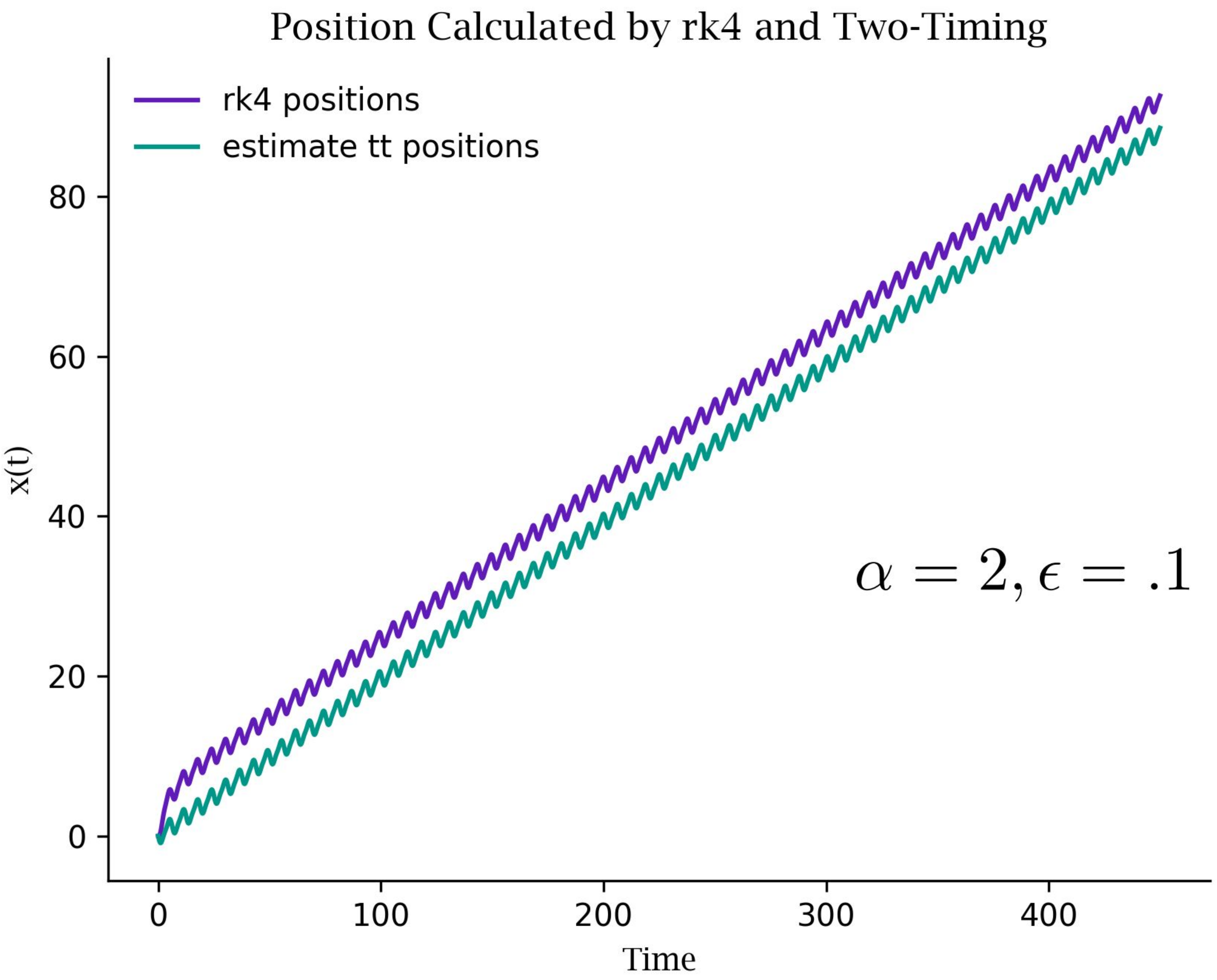
Special case: α = 2

$$-\frac{\partial v_1}{\partial \tau} = \left(-\cos(\tau) - \frac{\cos(\alpha\tau)}{\alpha} + A(T)\right)^3 + A'(T)$$

- $-\frac{1}{\alpha}\cos^2(\tau)\cos(\alpha\tau)$ term has nonzero time average *only if* **α** = 2 (Recall sin and cos power reduction)
- Changes the differential equation for A(T)

$$A'(T) = \frac{3}{8} - \frac{15}{8}A(T) - (A(T))^3$$

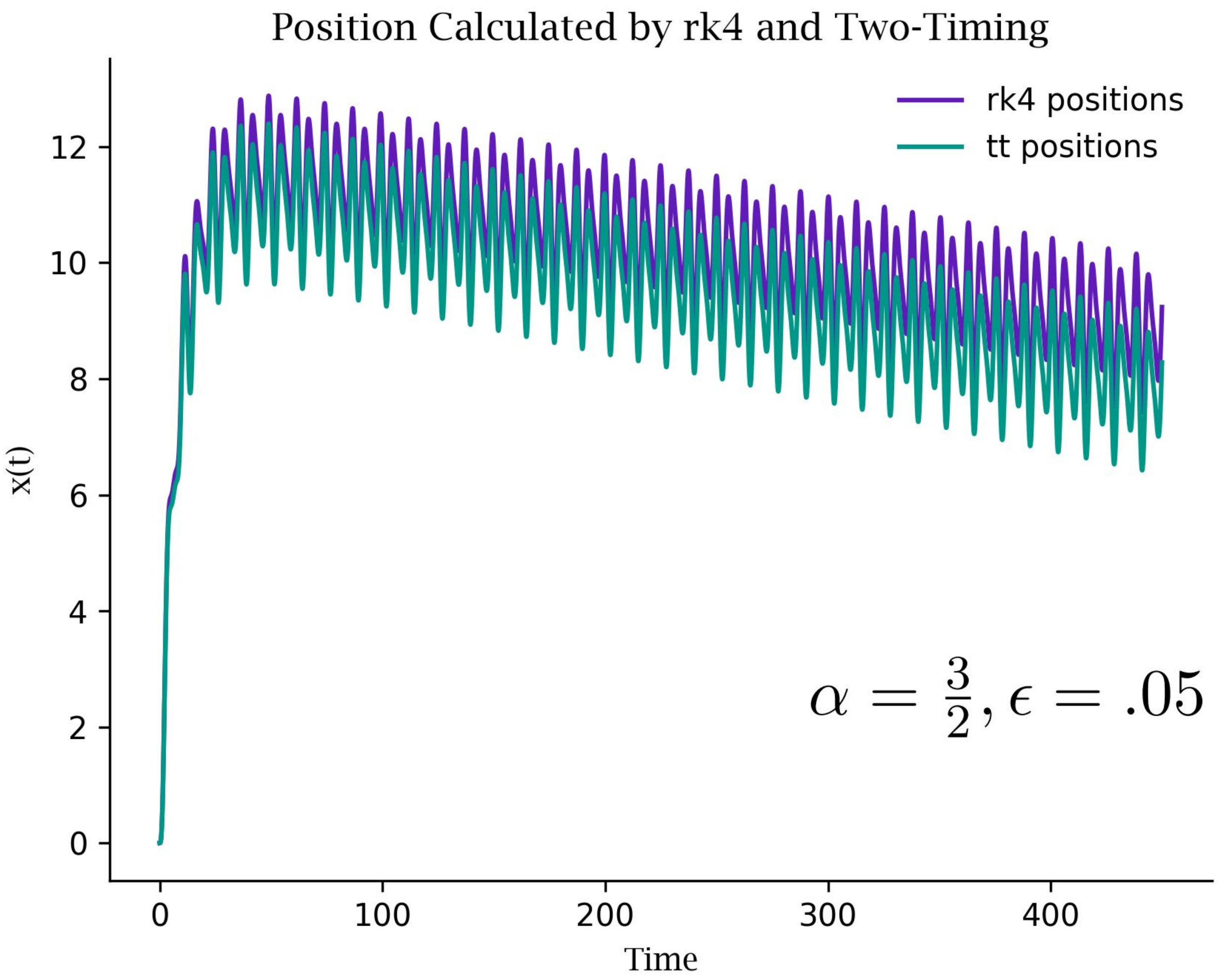
- Not analytically solvable, but solving for an equilibrium point successfully explains the motion in the **α** = 2 case



$\alpha = 2, \epsilon = .1$

Continuing in this Way

- Higher order expansion reveals similar terms for different values of α
  - Experimentally, $\alpha = 3/2$ and $\alpha = 4/3$ give the next strongest motion after $\alpha = 2$, and extra secular terms arise at the level of ϵ^2 in those cases

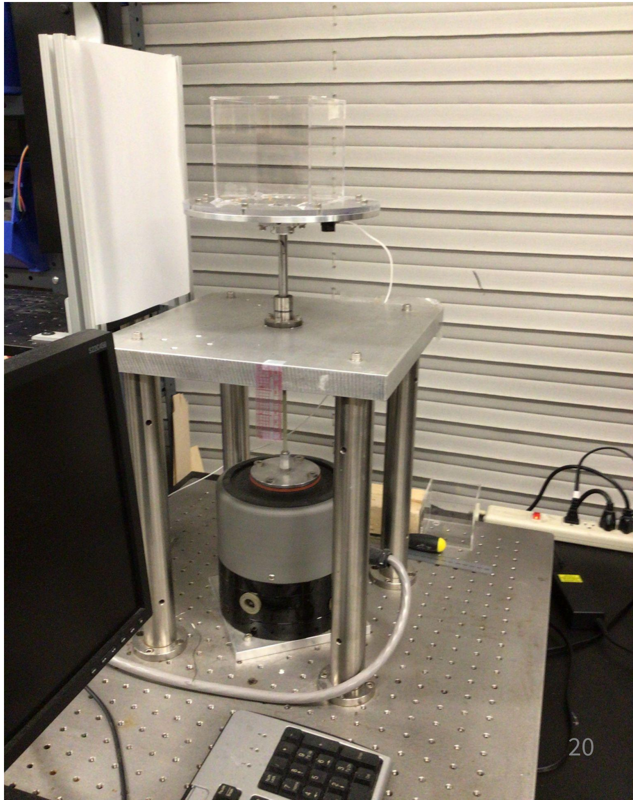

Position Calculated by rk4 and Two-Timing
rk4 positions
tt positions
x(t)
Time
0
2
4
6
8
10
12
0
100
200
300
400
$\alpha = \frac{3}{2}, \epsilon = .05$

Miscellaneous Remaining Thoughts/Observations

- For $\alpha = 3/2$ and $\alpha = 4/3$, smaller values of epsilon are required for accuracy
  - Possibility: even higher order terms
- Only ever see motion induced at even powers of epsilon
  - Terms depend on the parity of the level of expansion
- Explains why ‘simpler’ ratios induce motion ‘earlier’
- Explains why $\alpha = 2$ is qualitatively different from $\alpha = 49/25$

Next Steps

- Design and conduct an experiment
  - Vibration machine in the lab
- Four-timing



Questions?