Escape rates for rotor walk in $\mathbb{Z}^d$

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- Rotor walks - dependent on mechanism, initial configuration
- Escape rates - specific theorems
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- Open problems - analogous questions to random walks: recurrence/transience of arbitrary configurations, number of sites visited

Rotor walks

Attach arrows at each site pointing in any direction. At each step, rotate the arrow counter-clockwise and move the particle in that direction.

Similar questions as for random walks:

- hitting times of sets,
- number of visits to a site,
- number of sites visited,
- recurrent/transient configurations and mechanisms.

Some applications

- TCS: Diffusion model: task distribution/parallel processors
- TCS: load-balancing: efficient utilization of computational resources in parallel and distributed systems. Aim: reallocate the load such that at the end each node has approximately the same load.
- TCS: design principles for navigation problems and optimal transport in networks: what are the most efficient ways to visit all sites with the least amount of resources/knowledge
- TCS: broadcasting information in networks
- Physics: model of self-organized criticality
- Chemistry: motion of particle affected by its medium

Connection to random walk

How close is it?

Recurrence/transience

- ρ is *recurrent* if the rotor walk with initial configuration ρ returns to the origin infinitely often ($x_n = o$ for infinitely many n);
- otherwise, we say that ρ is *transient*.

$I(\rho, n)$: the number of walks that escape to infinity.

Schramm

Schramm:

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n} \leqslant \alpha_d \tag{1}$$

where α_d is the probability that simple random walk in $\mathbb{Z}^d$ does not return to the origin.

Schramm

$2dm$ particles at $x \in \mathbb{Z}^d \to m$ particles move to each of the $2d$ neighbors of x.

- $N = (2d)^r m$ particles at the origin,
- each particle takes a single rotor walk step,
- $r - 1$ times: each particle that is not at the origin take a single rotor walk step.

$\forall$ paths of length $\ell \leqslant r$ looping at o, exactly $(2d)^{-\ell} N$ particles traverse this path.

At origin:

$$N \sum_{\gamma : o \to o,\, |\gamma| \leqslant r} (2d)^{-|\gamma|} = Np$$

Schramm

Now letting each particle that is not at the origin continue performing rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$, the number of particles that stop in $\partial\mathcal{B}_r$ is at most $N(1-p)$, so

$$\frac{I_r(\rho, N)}{N} \leqslant 1 - p.$$

For general n, let N be the smallest multiple of $(2d)^r$ that is $\geqslant n$. Then

$$\frac{I_r(\rho, n)}{n} \leqslant \frac{I_r(\rho, N)}{N - (2d)^r}$$

The right side is at most $(1-p)(1 + 2(2d)^r/N)$, so

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n} \leqslant \limsup_{n\to\infty} \frac{I_r(\rho, n)}{n} \leqslant 1 - p = \mathbb{P}(T_o^+ > r).$$

$r \to \infty$ the right side converges to α_d.

Question...

Is the aligned configuration on $\mathbb{Z}^d, d \geqslant 3$ as transient as random walk?

Main results

Theorem

For the rotor walk on $\mathbb{Z}^d$ with $d \geqslant 3$ with all rotors initially aligned $\uparrow$, a positive fraction of particles escape to infinity; that is,

$$\liminf_{n\to\infty} \frac{I(\uparrow, n)}{n} > 0.$$

Question...

How about in 2 dimensions?

Main results

Theorem

For rotor walk in $\mathbb{Z}^2$ with any rotor configuration ρ, we have

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n/\log n} \leqslant \frac{\pi}{2}.$$

Moreover, if all rotors are initially aligned $\uparrow$, then

$$\liminf_{n\to\infty} \frac{I(\uparrow, n)}{n/\log n} > 0.$$

Main ideas

- in order to estimate the number of escapes to infinity, look at number of particles exiting a large ball before returning to origin
- asymptotics of green's function
- vertical coordinates hit by particles

Odometer - monotonicity and convergence

Notation: $I(u())$: the number of times the rotor walk goes to infinity before the nth return to the origin.

Similar notation in finite setting: $I_r(u())$: the number of times the rotor walk hits B_r before the n^{th} return to a.

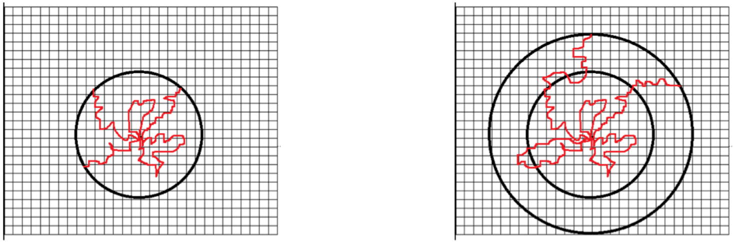


Figure : Monotonicity and convergence. Figure: thanks Shirshendu Ganguly.

$$I_r(\rho, n) > I_R(\rho, n) \to I(\rho, n)$$

Odometer

$u_n^r(x)$: total number of exits from x by first n particles stopped on hitting B_r.(odometer)

Lemma

For any $r > 0$ and $n \in \mathbb{N}$ and any initial rotor configuration ρ, we have

$$I_r(\rho, u_n^r(o)) = n.$$

Odometer proof

Starting with $N = u_n^r(o)$ particles at the origin, consider the following two experiments:

1. Let n of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r$.
2. Let N of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$.

By the definition of u_n^r, in the first experiment the total number of exits from the origin is exactly N. Therefore the two experiments have exactly the same outcome: n particles reach $\partial\mathcal{B}_r$ and $N - n$ remain at the origin.

Green function

$$G_r(x,y) = \mathbb{E}_x \#\{j < T | X_j = y\}$$

Lawler:

$$G_r(x,o) = \begin{cases} a_d(|x|^{2-d} - r^{2-d}) + O(|x|^{1-d}), & d \geqslant 3 \\ \frac{2}{\pi}(\log r - \log |x|) + O(|x|^{-1}), & d = 2. \end{cases} \tag{2}$$

$$G_r(o,o) = \frac{2}{\pi}\log r + O(1). \tag{3}$$

Odometer differential conditions

Levine, Peres:

Lemma (Gradient)

For a directed edge (x, y) *in* $\mathbb{Z}^d$*, denote by* $\kappa(x, y)$ *the net number of crossings from* x *to* y *by* n *rotor walks started at the origin and stopped on exiting* $\mathcal{B}_r$*. Then*

$$\nabla u_n^r(x, y) = -2d\,\kappa(x, y) + R(x, y)$$

for some edge function R *satisfying* $|R(x, y)| \leqslant 4d - 2$ *for all edges* (x, y)*.*

Lemma (Laplacian)

$$\Delta u(x) = \operatorname{div} R(x), \;\; x \neq o$$

$$\Delta u(o) = -n + \operatorname{div} R(o)$$

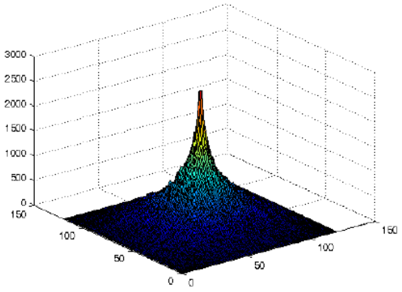
Approximating odometer function by the Green function

Let $f(x) = nG_r(x, o)$.

- $f(x)$ vanishes on ∂B_r
- $\Delta f(x) = 0$ for $x \in B_r - \{o\}$
- $\Delta f(o) = -n$

Natural to compare odometer with f.

Greens function and odometer with $n = 1000$ and $r = 60$



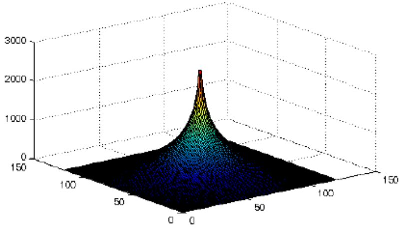


Figure : Figure: thanks Shirshendu Ganguly.

Approximating odometer function by the Green function

Levine, Peres: Let $f(x) = nG_r(x, o)$.

Lemma

In $\mathbb{Z}^d$, let $x \in \mathcal{B}_r$ and $\rho = r + 1 - |x|$. Then,

$$|u_n^r(x) - f(x)| \leqslant C\rho \log \frac{r}{\rho} + 8d^2.$$

where u_n^r is the odometer function for n particles performing rotor walk stopped on exiting $\mathcal{B}_r$.

Positive odometer

We need to show that near the boundary the error term is smaller than $f(x)$ which ensures positivity of the odometer.

Lemma

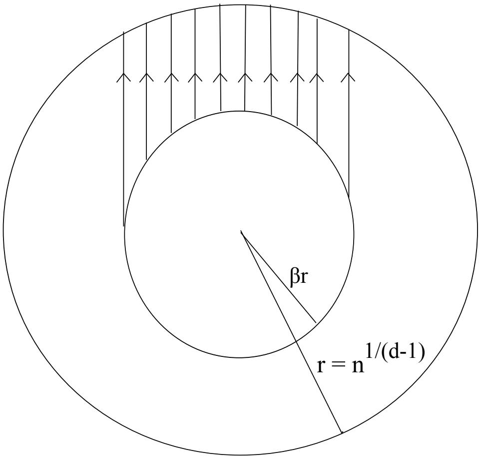
There exists a constant $\beta > 0$ *depending only on* d, *such that for any initial rotor configuration and* $r = n^{1/(d-1)}$ *we have* $u_n^r(x) > 0$ *for all* $x \in \partial\mathcal{B}_{\beta r}$.

Idea of proof: for $x \in \partial B_{\beta r}$ we have

$$|u(x) - f(x)| \leqslant C(1-\beta) r \log \frac{1}{1-\beta}$$

Need to look at columns

where initial configurations matters:



Sites at top surface contribute at least 1 to the number of escapes.

One more odometer

Lemma

For any initial rotor configuration in $\mathbb{Z}^2$ we have

$$u_n^n(o) = \frac{2}{\pi} n \log n + O(n).$$

Proof.

We have $f(o) = nG_n(o,o) = n(\frac{2}{\pi}\log n + O(1))$, and $|u_n^n(o) - f(o)| = O(n)$. $\square$

Almost done

Thus, the total number of escapes is $\Theta(n)$ among $u(o)$ particles.

Lemma ($\mathbb{Z}^d$, $d \geqslant 3$)

$u(o) = nG_r(o,o) + O(r) = \Theta(n)$

Lemma ($\mathbb{Z}^2$)

$u(o) = \frac{2}{\pi} n \log n + O(n)$

Now look at proportions:

$$\limsup_{n\to\infty} \frac{I(\rho, n)}{n/\log n} \leqslant \frac{\pi}{2}.$$

$$\liminf_{n\to\infty} \frac{I(\uparrow, n)}{n} > 0.$$

Open problems

- transience/recurrence of iid rotors configuration
- number of sites visited $\approx n^{2/3}$: constant sequence of rotors, length of excursions.
- Consider rotor walk in $\mathbb{Z}^2$ with a drift to the north: each rotor mechanism is period 5 with successive exits cycling through North, North, East, South, West. Is this walk transient for all initial rotor configurations?