**Abstract**

Self-folding origami has emerged as a tool to make functional objects in material science. The common idea is to pattern a sheet with creases and activate them to have the object fold spontaneously into a desired configuration. This article shows that quadrilateral metasheets are able to fold into the predetermined configurations, if we only impose strain on part of their creases. In this study, we find the optimal pattern of strain on a collinear quadrilateral metasheet, such that the self-folding metasheet can fold into the desired configuration with minimum "functional" creases. We calculate the available configurations of a collinear quadrilateral metasheet under the optimal pattern of stains, and we compare the energy evolution along the folding pathway of each folded state. We conclude that the energy predominance of the desired pathway during the initial period of time accounts for the foldability under the optimal pattern of stain.

1 Introduction

One challenge in material science is to make functional objects at the nanoscale. These objects are far too small to build by hand, and self-folding origami has emerged as a tool for designing these three-dimensional structures from flat films [1, 2, 3]. Besides its great potential for the manufacture of complicated geometries and devices, self-folding origami opens up a number of research directions, such as origami design, the foldability and mechanics of origami, etc. Stern, Pinson and Murugan [7] discussed the complexity of refolding a previously folded sheet of paper and they provide fundamental limits on the programmability of energy landscapes in sheets. Chen and Santangelo [6] investigated the branches of a generic, triangulated origami crease pattern. Waitukaitis, Menaut, Chen, and Hecke [9] defined the origami multistability and proved that rigid, degree-four vertices are generically multistable. The criteria of choosing available Mountain and Valley assignments in the Depth-First-Search algorithm introduced in subsection 3.1 is based on their discussion on the foldability of a degree-four vertex.

An origami structure is a system of rigid flat plates jointed pairwise by hinges (or creases). A mathematical interpretation of origami structures is provided more precisely in subsection 2.1. The network formed by the creases and their junctions is called the crease pattern. Based on different assignments of Mountain and Valley options, an origami structure can take on a variety of configurations in 3D space.

Santangelo [8] discussed that there are basically two approaches to self-fold a crease. One way begins with a thin sheet of prestressed polymer glass. When heated locally, the release of the prestress causes the material to shrink locally and fold, if the origami structure is designed properly. A second approach starts with multilayer sheets of different materials. As the temperature changes, one surface expands to a greater extent than the other and the strip bends to the predetermined state. These methods all provide good control for producing folded structures. However, rather
than modifying the mechanisms of folding, can we simplify the imposed pattern of strain that forces a flat sheet to buckle into a desired configuration?

Based on the methods introduced in section 2, we simulate the folding process and find that quadrilateral metasheets are able to fold into the predetermined configurations, if we only impose strain on part of their creases. The widely-used model of folding a self-folding origami imposes strain on all the creases in its crease pattern. However, is there an optimal pattern of strain, such that an origami structure can fold into the desired configuration with minimum "functional" creases?

Here, we address the following question: what is the optimal pattern of strain for a quadrilateral metasheet to fold into its desired configuration and how can we authenticate its foldability? Miura-Ori is an origami structure that receives increasing attention for its negative Poisson’s ratio and its high degree of symmetry folding in its periodicity [4, 5]. The crease patterns of the Miura fold form a tessellation of the surface by parallelograms, that is a collinear quadrilateral metasheet. Here, we take Miura-ori as the desired configuration; yet, our work opens up more experimental directions on self-folding origami structures not necessarily reliant on Miura-Ori.

To answer these questions, in section 3, we first calculate the number of available configurations of a collinear quadrilateral metasheet under the optimal pattern of stains, and then we compare the energy evolution along each folding pathway. Finally, we analyze the results and explain why a collinear quadrilateral metasheet under the optimal pattern of stains is able to fold into the desired Miura-Ori. Future directions of this research project is discussed in section 4.

2 Math Modeling

The goal of this section is to introduce a computational model for self-folding origami structures, upon which the subsequent sections are built. We will first introduce the definition and the energy function of an origami structure. Then, we will explain how we model the folding process of an origami structure by finding the local minimum of its energy function. Finally, we test the robustness of our method and explain the modifications that make the numerical model more robust.

2.1 Definition of an Origami Structure and its Energy Function

An origami structure contains the information of its vertices, edges, faces, and the angles between adjacent faces. Now, we will explain the model of each part and how the corresponding coefficients represent the properties of the material.

- $X := \{x_i\}_{i=1}^n$ is the coordinates of the $n \in \mathbb{N}^+$ vertices. Notice that each configuration is uniquely represented by vertex coordinates.

- $G := \{(x_i, x_j)\}$ is the set of edges with predetermined fixed length. Each pair of vertices $(x_i, x_j)$ in the set is connected and the order of vertices has no influence. For example, a flat origami structure that is able to fold into a tetrahedron shown in Figure 1 has an edge set \{$(x_1, x_2), (x_2, x_3), (x_1, x_4), (x_2, x_4), (x_3, x_5), (x_4, x_5), (x_1, x_6), (x_5, x_6)$\}. Each edge $e_{ij}$ is modeled by a spring with a resting length $l_{ij}$ and a stiffness coefficient $0 < k_{ij} < 1$.

Figure 1: A flat origami structure of a tetrahedron.
When \( k_{ij} \to 1, l_{ij} \), the actual length of \( e_{ij} \), is more reluctant to change; and when \( k_{ij} \to 0 \), \( e_{ij} \) is easy to be enlarged or squeezed.

- \( A := \{(x_k, x_i, x_j, x_\ell)\} \) is the set of angles, whose pivots are \( e_{ij} \in E \). Each angle \( (x_k, x_i, x_j, x_\ell) \) is modeled by a spring with a resting angle \( \pi/2 \) and a stiffness coefficient \( 0 < g_{ij} < 1 \). When \( g_{ij} \to 1 \), the angle has greater tendency to bend towards \( \pi/2 \); when \( g_{ij} \to 0 \), the angle is less likely to do so.

\[ A = \{ (x_k, x_i, x_j, x_\ell) \} \] is the set of angles, whose pivots are \( e_{ij} \in G \). Each angle \( (x_k, x_i, x_j, x_\ell) \) is modeled by a spring with a resting angle \( \pi/2 \) and a stiffness coefficient \( 0 < g_{ij} < 1 \). When \( g_{ij} \to 1 \), the angle has greater tendency to bend towards \( \pi/2 \); when \( g_{ij} \to 0 \), the angle is less likely to do so.

\[ A = \{ (x_k, x_i, x_j, x_\ell) \} \] is the set of angles, whose pivots are \( e_{ij} \in G \). Each angle \( (x_k, x_i, x_j, x_\ell) \) is modeled by a spring with a resting angle \( \pi/2 \) and a stiffness coefficient \( 0 < g_{ij} < 1 \). When \( g_{ij} \to 1 \), the angle has greater tendency to bend towards \( \pi/2 \); when \( g_{ij} \to 0 \), the angle is less likely to do so.

\[ \tilde{A} := \{ (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \} \] and \( \hat{A} := \{ (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \} \) are the sets of assistant edges and angles respectively, which are used to model the stiffness of each face. \( \tilde{e}_{ij} \) is the selected diagonal of its corresponding face, and \( \hat{e}_{ij} \) is the pivot of the face bending angle \( (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \). A stiffness coefficient \( 0 < \tilde{k}_{ij} < 1 \) is assigned to each \( \tilde{e}_{ij} \), and \( 0 < \hat{g}_{ij} < 1 \) to each angle \( (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \).

\[ \tilde{G} := \{ (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \} \] and \( \hat{A} := \{ (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \} \) are the sets of assistant edges and angles respectively, which are used to model the stiffness of each face. \( \tilde{e}_{ij} \) is the selected diagonal of its corresponding face, and \( \hat{e}_{ij} \) is the pivot of the face bending angle \( (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \). A stiffness coefficient \( 0 < \tilde{k}_{ij} < 1 \) is assigned to each \( \tilde{e}_{ij} \), and \( 0 < \hat{g}_{ij} < 1 \) to each angle \( (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \).

Notice that the above formula only works for \( 0 \leq a_{ij} \leq \pi \). Therefore, we should preassign a Mountain and Valley option to each angle, where a Mountain crease means \( \pi \leq a_{ij} \leq 2\pi \) and a Valley crease means \( 0 \leq a_{ij} \leq \pi \). For a Mountain crease, if \( \overrightarrow{x_i x_j} \) is in the same direction as \( \overrightarrow{x_i \tilde{x}_j} \) and \( \overrightarrow{x_\ell \hat{x}_k} \), we hold the result \( a_{ij} \) calculated by equation 1; otherwise, the actual angle should be \( 2\pi - a_{ij} \). The conditions are the opposite for a Valley crease. Since the vertex order of each angle matters, we should be careful when we input the angles in practice. For example, one possible angle set for the origami shown in Figure 1 could be \( \{(x_1, x_2, x_4, x_5), (x_3, x_5, x_2, x_4), (x_6, x_1, x_5, x_2)\} \), where each angle has an M/V option.

\[ \tilde{G} := \{ (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \} \] and \( \hat{A} := \{ (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \} \) are the sets of assistant edges and angles respectively, which are used to model the stiffness of each face. \( \tilde{e}_{ij} \) is the selected diagonal of its corresponding face, and \( \hat{e}_{ij} \) is the pivot of the face bending angle \( (\tilde{x}_k, \tilde{x}_i, \tilde{x}_j, \tilde{x}_\ell) \). A stiffness coefficient \( 0 < \tilde{k}_{ij} < 1 \) is assigned to each \( \tilde{e}_{ij} \), and \( 0 < \hat{g}_{ij} < 1 \) to each angle \( (\hat{x}_k, \hat{x}_i, \hat{x}_j, \hat{x}_\ell) \).

Notice that an \( n \)-vertex face needs \( 2n - 6 \) assistant edges and angles, and the combination of assistant edges and angles is not unique. However, the test shows that on a quadrilateral face, the combination of one assistant edge and one corresponding assistant angle yields the most stable simulation, that is the face is less likely to bend itself under the same coefficients. Therefore, this combination is chosen for all experiments with quadrilateral faces in this report. More tests for choosing assistant edges and angles on the faces with vertices more than four is useful, but not necessary in this report.

Now, we are able to define an origami structure as a set of \( \{ G, A, \tilde{G}, \hat{A} \} \) together with all the coefficients \( \{ k, a, \tilde{k}, \hat{a} \} \), the resting lengths and angles. Notice that all the angles and edge lengths can be translated into the coordinates of vertices. Therefore, we define the energy of an origami structure as a function of vertices, which is

\[ E(x) = E_{\text{edge}}(x) + E_{\text{angle}}(x) + E_{\text{assistant edge}}(x) + E_{\text{assistant angle}}(x). \] 

(2)

As we model each edge and angle by a spring, the energy of each origami structure can be written as a sum of spring energy functions. For example, the energy of edges is

\[ E_{\text{edge}}(x) = \frac{1}{2} \sum_{(x_i, x_j) \in G} k_{ij} \left( ||x_i - x_j|| - l_{ij} \right)^2. \]
2.2 Dynamic Origami Folding Model

In subsection 2.1, we have defined an origami structure and its energy function. Now, to fold an origami structure in a way such that the energy function decreases, the following ordinary differential equation with respect to time should be solved.

\[
\frac{dx}{dt} = -\frac{\nabla E(x)}{\gamma} + \delta \eta + F_{\text{external}},
\]

where \(\gamma\) is the parameter of friction. Here, we add a stochastic term \(\eta\), which is the white noise \(dW_t/dt\), to model the random forcing it feels if it is in a fluid, or subject to random forcing in some other way (random turbulent heat flows, jiggling or shaking, etc). The coefficient \(\delta\) is defined as \(\sqrt{2\beta^{-1}}\gamma^{-1}\). \(\beta\), the parameter of inverse temperature, is \(1/(k_B \cdot T)\), where \(T\) is temperature and \(k_B\) is Boltzmann’s constant. Besides the physical meaning of the stochastic term, \(\eta\) also helps the numerical method to traverse more possible energy states before getting stuck in some local minimum.

While solving the ODE system shown in equation 3, we discretize the time space into \(N_t = T/\Delta t\) steps and update the vertices in each time step by

\[
x_{j+1} = x_j - \nabla E(x_j) \Delta t + \xi + F_{\text{external}} \Delta t,
\]

where \(\xi = \int_t^{t+1} \delta dW_t = N(0, \delta^2 \Delta t)\). In each simulation, we define a fault-tolerance parameter \(\epsilon\), and when \(\nabla E(x_j) < \epsilon\), we believe that the origami structure has reached its final state. Calculating the energy gradients of the length functions is fairly straightforward; however, the gradients of the angle energy is quite involved. More details about calculations are illustrated in appendix A at the end of the report.

2.3 Robustness Test and Modifications of the Model

So far, we have determined the model of folding a self-folding origami. Before analyzing the results, we show that the discrete numerical method, illustrated by equation 4, of solving ODE system in equation 3 is robust. Notice that the folding method should be an ergodic dynamical system, which means that the mechanism has the same behavior averaged over time as averaged over all of the space states. Therefore, we compare the energy ensemble \(E_{\text{ensemble}}\) at time \(T\), which is the ensemble average over many realizations, and \(\frac{1}{T} \int_0^T E(t) dt\), which is the average over long time for one realization.

![Figure 3: \(E_{\text{ensemble}}\) and \(\frac{1}{T} \int_0^T E(t) dt\) graphs for 50 realizations, \(T = 100, dt = 0.2\). The unfolded origami is shown in Figure 1, with constant resting edge length 1 and constant target angle \(\arccos\frac{1}{3}\).](image-url)
Figure 3 compares the ensemble energy with \( \frac{1}{T} \int_0^T E(t) dt \) at time \( T = 100 \), when folding a tetrahedron under different \( \delta \) conditions, that is applying different coefficients on the stochastic term. The plot shows that the ensemble energy and \( \frac{1}{T} \int_0^T E(t) dt \) are close to each other in an allowable error scope, which gives us confidence on the ergodicity of the algorithm and it also makes sense that the ensemble energy is increasing with \( \delta \). Further tests might be useful, but might not be necessary.

During the experiments of folding an origami structure, one of the crucial challenges is that the folding mechanism cannot prevent face crossing. We have tried two coping approaches: adding a backtracking line-search method to the algorithm, and modifying the angle function to be continuous.

### 2.3.1 Backtracking Line-search Method

Based on the definition of angles, we notice that the face crossing leads to an energy jump, because the angle function is not continuous at zero degree: \( 0^- = 2\pi \) in the angle function. Taking the advantage of this property, the backtracking line-search modifies the step length of the steepest gradient descent, and tells the algorithm to stop folding when two adjacent faces touch each other. At each time point \( t_k \), the method will

- Start with large step length \( \alpha_k^0 = 1 \).
- If \( E(x_k + \alpha_k \Delta x_k \Delta t) < E(x_k) \), the algorithm accepts the step length \( \alpha_k \).
- Otherwise, compute \( \alpha_k^{i+1} = \rho \alpha_k^i \) with \( \rho = \frac{1}{2} \) and go back to the previous step. The algorithm stops folding when \( \alpha_k^{i+1} < \epsilon \).

Therefore, the backtracking line-search method helps us to reach a state on the pathway towards a local minimum of the energy function before stepping beyond the constraint of no face crossing.

### 2.3.2 Continuous Angle Function

Besides the energy local minimums with no face crossing, we are also interested in the energy states without constraints, although the folding process might not be realizable in practice. This goal motivates us to make the angle function continuous.

Firstly, we calculate \( \hat{a}_{ij}^k \) according to the vertices coordinates at the current time point \( t_k \). Then we choose \( a_{ij}^k \) such that

\[
\hat{a}_{ij}^k = \text{arg min}_{a_{ij}^k \in \{\hat{a}_{ij}^k + 2n\pi | n \in \mathbb{Z}\}} \|a_{ij}^k - a_{ij}^{k-1}\|.
\]

Note that the resting angle \( \hat{a}_{ij} \) should be modified in the same way, that is

\[
\hat{a}_{ij} = \text{arg min}_{a_{ij} \in \{\hat{a}_{ij} + 2n\pi | n \in \mathbb{Z}\}} \|a_{ij} - a_{ij}^k\|.
\]

Additionally, the Mountain and Valley option of each angle \( a_{ij} \) is checked at each time point, because the modification of the angle function could change the sign of \( \nabla E_{a_{ij}}(x) \). Since the sign of \( \nabla E_{a_{ij}}(x) \) decides whether two faces linked by \( e_{ij} \) tend to attack or repel each other, we modify the energy gradient such that whenever two adjacent faces go across each other, the energy gradient of the angle changes its sign, and therefore the two faces tend to go back in the next time step.

Based on the above modifications on the model, the algorithm can search for local minimums of the energy function with no constraints. All the results presented in section 3 are provided by the algorithm with a continuous angle function.

### 3 Results

In section 2, we have introduced the definition of an origami structure and the model of the folding process. Remember that we have introduced the folding mechanisms in section 1. There are different methods to install a dynamic "motor" on a crease, such that the crease is able to self-fold into the target state. The imposed strain on a crease implies that the angle coefficient of this crease is positive. The widely-used model of folding a self-folding origami imposes strain on all
the creases in its crease pattern, that is the imposed pattern of strain assignments is exactly the crease pattern of this origami structure. Now, we want to answer the question: can we simplify the imposed pattern of strain that forces a flat sheet to buckle into a desired configuration?

Let us first clarify some terminologies so that we could discuss the issue more precisely. We define an optimal pattern of strain, which records the minimum "functional" creases that enable the origami structure to self-fold into the desired configuration. This optimal pattern may not be unique, if there are more than one arrangement of those "functional" creases that could do the job. Here, we address the following question: what is the optimal pattern of strain for a collinear quadrilateral metasheet to fold into its desired configuration and how can we authenticate its foldability?

Figure 4: The crease pattern and Mountain and Valley assignment of a 5 by 6 Miura-Ori.

In this study, we answer the above question in the case of Miura-Ori, which is a method of folding a flat surface into a smaller area. The crease patterns of the Miura-Ori origami form a tessellation of the surface by parallelograms, which is a collinear quadrilateral metasheet shown in Figure 4. It is the M/V assignment that makes a Miura-Ori: each of the zigzag paths of creases consists solely of mountain folds or of valley folds, with mountains alternating with valleys from one zigzag path to the next. Each of the straight paths of creases alternates between mountain and valley folds. [11] Notice that each desired configuration is determined by a unique M/V assignment. Here, we explore the optimal pattern of strain for a collinear quadrilateral metasheet, whose desired configuration is determined by Miura-Ori folds.

In subsection 3.1, we introduce how we work out the M/V assignments determined by a set of crease-pinchings. Therefore, we know the available ending configurations (or branches), if we only pinch the creases in the optimal strain pattern. Then, in subsection 3.2, we design an optimal pattern of strain on a 2 by 3 metasheet and confirm its stability to fold into a Miura-Ori configuration. Finally, in subsection 3.3, we analyze the energy evolution along the pathway of each folded state and argue that the energy predominance of a certain period of time along the pathway of Miura-Ori explains the feasibility of the optimal strain pattern.

3.1 Available Folded States Determined by a Set of Crease-Pinchings

Figure 5: A degree-four vertex.

Notice that the folded states (or branches) determined by a set of crease-pinchings, is the same as the available M/V assignments determined by a set of M/V options. First, we look at the constraints of M/V assignments on the essential component of a metasheet, a degree-four vertex, which is four rigid faces attached by four edges that meet a vertex. In Figure 5, we specify the flat-state geometry by the set of sector angles \( \{\alpha_i\} \), where \( 0 < \alpha_i < \pi \) and \( \sum \alpha_i = 2\pi \). Huffman [10] noted that one folding angle must have the opposite Mountain and Valley option from the rest. Additionally, Waitukaitis [9] showed that edge \( e_{ij} \) can be the "unique" fold with the M/V option
opposite from the rest, only if the sum of sector angles of the two plates connected by edge $e_{ij}$ is less than or equal to $\pi$. For example, the case shown in Figure 5 is an available M/V assignment, since $\alpha_2 + \alpha_3 \leq \pi$.

![Figure 6: A directed graph of degree-four vertices on a 5 by 6 metasheet.](image)

Based on the above two constraints on the M/V assignments at a degree-four vertex, we design a Depth First Search (DFS) algorithm to traverse a directed graph, which is made of degree-four vertices on a metasheet. At each vertex, the algorithm works out the available M/V assignments that satisfy the constraints, and then calls next vertex if there are any.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>18</td>
<td>54</td>
<td>162</td>
<td>486</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>82</td>
<td>374</td>
<td>1706</td>
<td>7782</td>
</tr>
<tr>
<td>4</td>
<td>54</td>
<td>374</td>
<td>2604</td>
<td>18150</td>
<td>126534</td>
</tr>
<tr>
<td>5</td>
<td>162</td>
<td>1706</td>
<td>18150</td>
<td>193662</td>
<td>2068146</td>
</tr>
<tr>
<td>6</td>
<td>486</td>
<td>7782</td>
<td>126534</td>
<td>2068146</td>
<td>33865632</td>
</tr>
</tbody>
</table>

Table 1: The number of available M/V assignments of a collinear quadrilateral metasheet of different sizes. The entry in row $i$ and column $j$ is the number of available M/V assignments of a $i$ by $j$ collinear quadrilateral metasheet.

Table 1 shows the number of available M/V assignments on a collinear quadrilateral metasheet of different sizes. It makes sense that the number of available assignments grows exponentially with the size of metasheets. Finding available folded states under the optimal pattern of strain could be solved using the same method, besides that the M/V options on the creases in the pattern should be fixed during the DFS traverse.

### 3.2 Stability of the Optimal Pattern of Strain

![Figure 7: The crease pattern and Mountain and Valley assignment of a 2 by 3 Miura-Ori.](image)

For simplicity, we now look at a 2 by 3 collinear quadrilateral metasheet, whose crease pattern is shown in Figure 7. The M/V assignment on the metasheet defines the target folding pathway of a 2 by 3 Miura-Ori.
To choose the optimal pattern of strain for a 2 by 3 collinear quadrilateral metasheet to fold in a Miura-Ori, we first determine the minimum number of creases-pinching. Our test shows that pinching only one crease is not enough to fold this metasheet into a Miura-Ori configuration; however, any combination of one horizontal crease and one vertical crease could do the job. Therefore, we believe that the minimum number of creases-pinching is two. Then, pinching any two creases, we calculate the probability of ending up in Miura-Ori configuration over 200 realizations, starting from a flat 2 by 3 collinear quadrilateral metasheet with noise coefficient $\delta = 0.1$. The optimal pattern of strain is the selection of creases, pinching which the metasheet has the highest probability to fold in a Miura-Ori pathway. In the case of a 2 by 3 collinear quadrilateral metasheet, the red creases in Figure 7 is the optimal pattern of strain and we fix it for further tests.

Figure 8: Four available folded configurations of a 2 by 3 collinear quadrilateral metasheet with Mountain and Valley assignment consistent to the optimal pattern of strain.

Calculated by the algorithm introduced in subsection 3.1, there are a total of four folded states that are consistent with the optimal pattern of strain, which is shown in Figure 8. The state 4 is the Miura-Ori configuration, which is the desired folded state. Table 2 shows the probability of ending up in each folded state over 200 realizations, starting from a flat 2 by 3 collinear quadrilateral metasheet under the optimal pattern of strain with noise coefficient $\delta = 0.1$. Even if we add a large noise during the folding process, most of the time the origami folds into a Miura-Ori configuration, which gives us confidence on the stability of the optimal pattern of strain.

<table>
<thead>
<tr>
<th>States</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob</td>
<td>1%</td>
<td>.5%</td>
<td>0%</td>
<td>98.5%</td>
</tr>
</tbody>
</table>

Table 2: The probability of ending up in each folded state over 200 realizations when a flat 2 by 3 collinear quadrilateral metasheet folds under the optimal pattern of strain, with constant resting angles $\arccos(0.95)$, and constant angle coefficients $0.05$, $0 < t < 80$, $dt = 0.1$, and $\delta = 0.1$.

3.3 Energy Evolution on Folding Pathways

After designing a convincing optimal pattern of strain, we explain its feasibility by comparing the energy evolution on the folding pathway of each available folded state. We first define two energy functions: $E_{\text{optimal}}$ and $E_{\text{state}}^{(i)}$ for $i = 1 \cdots 4$. $E_{\text{optimal}}$ is the energy function in which only the angles (creases) in the optimal pattern of strain have nonzero coefficients. That is, the angles (creases), which have not been preassigned a Mountain or Valley option, are omitted in $E_{\text{optimal}}$. On the opposite, $E_{\text{state}}^{(i)}$ for $i = 1 \cdots 4$ is the energy function that takes account of each angle (crease).
Figure 9 and 10 plot the average $E_{\text{optimal}}$ and the average gradient of $E_{\text{optimal}}$, which is $\frac{\nabla E_{\text{desired}} \cdot \nabla E_{(i)}}{\| \nabla E_{\text{temporary}} \|}$, over 100 realizations on the pathway of each folded state $i$. When $t$ is less than 15 or so, $E_{\text{optimal}}$ on the pathway of Miura-ori, which is state 4, is the lowest among the four pathways. It makes sense that the energy predominance of the initial period of time may account for the feasibility of the optimal pattern of strain, since after $t = 15$, the metasheet has already been stuck in its Miura-Ori pathway.

Figure 9: $E_{\text{optimal}}$ along pathway $i$ for $i = 1, \cdots, 4$ vs time $t$ graphs for constant resting angles $\arccos(0.95)$, and constant angle coefficients 0.05 with $0 < t < 80$, $dt = 0.1$, $\delta = 0.1$.

Furthermore, the energy evolution of the pathway to state 3 is remarkable. The energy jump in the initial period of time represents the order of the folding process. That is we need to first fold horizontal creases and then vertical creases to end up with state 3. The low average gradient of $E_{\text{optimal}}$ demonstrates that the origami has to fold in a pathway violating the direction of $E_{\text{optimal}}$ during the initial period of time. The above evidence explains why state 3 is the least favorable folded state.
Figure 10: The energy gradient \( \frac{\nabla E_{\text{desired}} \cdot \nabla E_{i}}{\|\nabla E_{i}\|} \) along pathway \( i \) for \( i = 1, \cdots, 4 \) vs time \( t \) graphs for constant resting angles \( \arccos(0.95) \), and constant angle coefficients 0.05 with \( 0 < t < 80 \), \( dt = 0.1, \delta = 0.1 \).

4 Conclusion

In this study, we introduced the model of folding an origami structure. The obstacles of dealing with face crossing is identified and alleviated. We argued that there is an optimal pattern of strain, which promises a collinear quadrilateral metasheet to fold into a Miura-Ori with fewest creases pinched. We showed that the authenticity of this optimal pattern of strain can be confirmed by comparing the energy evolution along the pathway of each possible folded state.

Our work suggests many interesting possible directions in expansion of this project. Firstly, the minimum number of functional creases that enable a generalized origami structure to fold into the desired folded state should be determined. We suppose the answer relates to the geometry properties, the degree of freedom, etc. Additionally, the relationship between the energy evolution and the ending states opens up a new direction to design the folding pathways. For example, we can control the stability and feasibility of each folded state by adding external forces or updating angle coefficients over time. Furthermore, although we have verified a self-folding metasheet could fold into a Miura-Ori under the optimal pattern of strain, the answer is still unknown to the question that why the energy function favors Miura-Ori more than the other states. It is interesting that even though there are several possible local minima the system could go to, it overwhelmingly evolves to just one of these. Why is that? What property of the energy landscape makes this so? And how could we determine this, using only local information near the flat state? These are all interesting questions to answer in expansion of this project.

Acknowledgements

I thank Miranda Holmes-Cerfon, Pejman Sanaei, and Jason Kaye for insightful advice and discussions. I acknowledge National Science Foundation (DMS-1646339) for funding and the Courant Institute for computing resources.

References

Appendix A Calculation of the Gradient of Angle Energy

Let \( y_1(x) \) be a normal vector of triangle \((x_h, x_i, x_j)\), \( y_2(x) \) be a normal vector of triangle \((x_k, x_i, x_j)\), written as

\[
    y_1(x) = \overrightarrow{x_h x_i} \times \overrightarrow{x_h x_j}, \quad \text{and} \quad y_2(x) = \overrightarrow{x_k x_i} \times \overrightarrow{x_k x_j}.
\]

Then the actual angle \( a_{ij} \) is

\[
    a_{ij}(x) = \arccos\left(\frac{y_1(x) \cdot y_2(x)}{\|y_1(x)\| \cdot \|y_2(x)\|}\right).
\]
The energy of angle \((x_h, x_i, x_j, x_k)\) is

\[
E_{a_{ij}}(x) = \frac{1}{2} g_{ij} (a_{ij} - \overline{a_{ij}})^2
\]

The energy gradient of angle \((x_h, x_i, x_j, x_k)\) is

\[
\nabla E_{a_{ij}}(x) = -\frac{a_{ij} - \overline{a_{ij}}}{\sqrt{1 - \left(\frac{y_1 \cdot y_2}{\|y_1\| \|y_2\|}\right)^2}} \left[ y_2 \nabla \frac{\bar{\nabla} y_1}{\|y_1\| \|y_2\|} + y_1 \nabla \frac{\bar{\nabla} y_2}{\|y_1\| \|y_2\|} - y_1 y_2 \cdot \left( \frac{\bar{\nabla} y_1}{\|y_1\|^2 \|y_2\|} + \frac{\bar{\nabla} y_2}{\|y_1\| \|y_2\|^2} \right) \right].
\]