An analogue of Krasnoselskii's theorem for Riesz spaces

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April 2020

Abstract

The goal of this Independent Study, is to develop some new results combining two well known fixed point theorems, in the context of Riesz spaces (ordered Banach spaces with certain special properties). The result proved, will be complemented with certain examples and counterexamples, along with suggested directions for future research.

1 Introduction

Krasnoseslkii's theorem is a very powerful result, which provides a bridge between classical fixed point theory and non-linear analysis. This is by providing a generalized version of Banach's contraction mapping theorem and Schauder's fixed point principle, for Banach spaces. It has numerous applications to Economics, as well as to the study of existence of solutions of non-linear PDE's. This work was motivated by the goal of trying to find a similar bridge, between Classcial/Metric Fixed- Point theory and Order- Theoretic Fixed- Point Theory. Any such result, could potentially have applications to the study of economic equilibria, choice-based markets, decision theory etc.

In this study, the two theorems of importance are Banach's contraction mapping theorem and Tarski's fixed point theorem. These theorems are quite different in flavor, given that Banach's theorem is a theorem about metric spaces, while Tarski's theorem is about certain posets. Naturally, a common landscape for their study would be that of ordered Banach spaces.

The result of interest will be developed analogously to Krasnoselskii's theorem, which proves the existence of a fixed point, for a contraction map perturbed by a Schauder map in the context of a Banach space. In particular, we will study the conditions under which the sum of an increasing map and a contraction have a fixed point for an ordered Banach space.

In order to develop our results, we must first review the existing results, the objects of study, and various related definitions.

2 Review of Metric-Fixed point-theorems

Krasnoselskii's theorem, is an extremely important result with applications to Economics and the theory of differential equations. The idea behind this theorem, is to combine Banach and Schauder's theorems in the context of a Banach space. It does so by considering a map which is the sum of a contraction and a Schauder map (defined shortly) defined on some closed subset of a Banach space. It turns out, that provided the map is a self map, it has a fixed point. First, we state these component theorems, and then provide a proof of Krasnoselskii's theorem.

2.1 Theorem

Banach's contraction mapping theorem: Let (X, d) be a complete metric space, and let $T : X \to X$ be a contraction i.e. $\forall x, y \in X$, $d(Tx, Ty) \leq cd(x, y)$, where $c \in (0, 1)$. Then, $\exists!\ x_0 \in X$ such that $T(x_0) = x_0$

Remark 1: Banach's contraction mapping theorem was first formulated by the mathematician Stefan Banach in 1922. It's first proof was given by the mathematician Renato Caccioppoli in 1931.[1]. A concise proof of the theorem can be found in [2].

2.2 Theorem

Schauder's fixed point theorem: Let $(X, \| \|)$ be a normed linear space, and let S be a closed and convex subset of X. If f is a continuous self map on S such that $f(S) \subseteq S$ is relatively compact i.e. $cl(f(S))$ is a compact subset of S, then f has a fixed point. A map with this property is defined as a relatively compact map.

Remark 2: Schauder's fixed point theorem, in the case of Banach spaces was first proved by Schauder in 1930. Further generalizations were achieved by the Russian mathematician Tychonoff, with the most general case (assuming that the subset is not compact but just relatively compact), being proved by B.V Singbal. A proof can be found in [4].

We will now state and prove Krasnoselskii's theorem, and highlight certain parts of the proof, as they will be important for later parts of the paper.

2.3 Krasnoselskii's Fixed point theorem

Krasnoselskii's theorem combines Banach and Schauder's fixed point theorems for Banach spaces by considering a map formed from the addition of a contraction and a relatively compact map.

Krasnoselsii's Fixed point theorem: Let $(X, \| \ \|)$ be a Banach space (complete, normed linear space) and let S be a closed, convex and nonempty subset of X. Let $g : S \to X$ be a contraction and $h : S \to X$ a relatively compact map such that $(g + h)(S) \subseteq S$. Then, $g + h$ has a fixed point in S.

Proof. The proof is divided into two parts:
Part 1: Let $K \in (0, 1)$ be the Lipschitz constant for g. Define $f(x) = x - g(x)$ $\forall x \in S$. We claim, f is an embedding of S into X. To this end, observe if $f(x) = f(y)$, then $x - y = g(x) - g(y)$. But, we know $\|g(x) - g(y)\| \leq K\|x - y\| < \|x - y\|$. Thus, this can only occur when $\|x - y\| = 0$ i.e. $x = y$. Thus, $f : S \to X$ is injective. Continuity of f follows since $\|f(x) - f(y)\| = \|(x-y) - (g(x) - g(y))\| \leq (1 + K)\|x - y\|$ (in particular, f is a Lipschitz map). Finally, we must show that $f^{-1} : f(S) \to S$ is continuous. This too follows because f^{-1} is Lipschitz. To see this, we pick $u, v \in f(S)$. Then, $\exists!\ x, y \in S$ such that $f(x) = u, f(y) = v$. Then, $\|x - y\| - \|g(x) - g(y)\| \leq \|f(x) - f(y)\|$ (By reverse triangle inequality). So, $(1 - K)\|x - y\| \leq \|f(x) - f(y)\|$.
Therefore, f is an embedding.

Part 2: Having shown that f is an embedding, we will now use this to establish the theorem. In particular, we observe that showing $g + h$ has a fixed point, is equivalent to showing that $f^{-1} \circ h$ has a fixed point. This is because, if $f^{-1} \circ h$ has a fixed point $x \in S$, then, $f(f^{-1} \circ h)(x) = f(x) \implies h(x) = f(x) \implies g(x) + h(x) = (g + h)(x) = x$.
To establish this, we must first show that $h(S) \subseteq f(S)$ so that the map $f^{-1} \circ h$ is well defined. We can in fact show that $cl(h(S)) \subseteq f(S)$. To this end, let $y \in cl(h(S))$. Define $f_y(x) = y + g(x)$, $\forall x \in S$. Then, $y + g(x) \in cl(h(S)) + g(S) \subseteq cl(h(S) + g(S)) \subseteq S$ (The last inclusion holds because S is closed in X). This shows, that $\forall y \in cl(h(S))$ and $\forall x \in S$, $f_y(x) \in S$. Also, observe that given $y \in cl(h(S))$, $f_y : S \to S$ is a contraction mapping since it is nothing but the translation of a contraction map. Then, since X is a Banach space, and S is closed in X, S is also a Banach space. Fix y and thus f_y. Then, by Banach's contraction mapping theorem applied to f_y, $\exists! x \in S$ such that $f_y(x) = x$. This implies, $y = f(x)$ i .e. $y \in f(S)$. Since y was arbitrary, we conclude that $cl(h(S)) \subseteq f(S)$.
Now, consider $f^{-1} \circ h : S \to S$ (this is well defined because $h(S) \subseteq cl(h(S)) \subseteq f(S)$). We already know that S is closed and convex. Further, we claim $f^{-1} \circ h$ is a Schauder map i.e. it is a relatively compact self map on S. This follows because, given that f is a homeomorphism $cl((f^{-1} \circ h)(S)) = f^{-1} \circ cl(h(S))$ and a homeomorphism sends compact sets to compact sets. Thus, by Schauder's theorem, $f^{-1} \circ h : S \to S$ must have a fixed point.

Therefore, as shown before we conclude that $g + h$ has a fixed point proving the theorem.[5]

□

Remark 3: Krasnoselskii's fixed point theorem was discovered by Mark Krasnoselskii in the 60's. His main goal was to use the result for the study of the solvability of certain non-linear PDE's. Since then, the result has had several generalizations. For a result achieved more recently, which implies the previous generalizations, see [6].

Krasnoselskii embedding: The first part of the proof was paramount to setting up the rest of the argument and allowed us to define an equivalent problem, by the use of Banach's contraction mapping theorem. As a result, it will be quite an important component of our proof of the main result in this paper, which seeks to combine the contraction mapping theorem with Tarski's order theoretic fixed point theorem for ordered Banach spaces. For this reason, we call this embedding constructed through a contraction map on a closed subset of a Banach space, a Krasnoselskii embedding, and we will use it through the rest of the paper without directly mentioning its construction.

3 Definitions and Tarski's fixed point theorem

Below, we provide the relevant definitions from order theory.

1. **Poset:** A poset is a set equipped with a relation $(X, \succcurlyeq)$ such that this relation is reflexive, anti-symmetric and transitive. The relation is known as a partial order.

2. **Chain:** Let $(X, \succcurlyeq)$ be a poset. Then, $S \subseteq X$ is said to be a $\succcurlyeq$- chain in X, if $\forall\, x, y \in S$, either $x \succcurlyeq y$ or $y \succcurlyeq x$.

3. **Dedekind complete poset:** A poset $(X, \succcurlyeq)$, is said to be Dedekind complete, if each $S \subseteq X$ bounded from above has a supremum i.e. $\exists x_* \in X$ such that $x_* \succcurlyeq y\ \forall y \in S$ and $x \succcurlyeq x_*\ \forall x \in \{x \in X : x \succcurlyeq S\}$ (by $x \succcurlyeq S$, we mean $x \succcurlyeq y,\ \forall\, y \in S$).

4. **Complete Lattice:** A poset is said to be a complete lattice if it is Dedekind complete and has a global maximum and minimum.

5.**Riesz Space:** A Real Banach space $(X, \|\|, \succcurlyeq)$ equipped with a partial order, is said to be Riesz if it satisfies the following compatibility laws:
i) If $x \succcurlyeq 0$, then $\lambda x \succcurlyeq 0\ \forall \lambda \in \mathbb{R}_+$.
ii) If $x \succcurlyeq y$, then $x + z \succcurlyeq y + z, \quad \forall z \in X$
iii)Given $x \succcurlyeq y \succcurlyeq 0$, we must have $\|x\| \geq \|y\|$.

6. **Increasing map:** A self map f on a poset, X is said to be increasing, provided $f(x) \succcurlyeq f(y)$ whenever $x \succcurlyeq y$.

7. **Set of fixed points:** From here on, $Fix(f)$ will represent the set of fixed

points of a self map f.

3.1 Tarski's fixed point Theorem

Tarski's fixed point theorem: Let $(X, \succcurlyeq)$ be a complete lattice, and $f : X \to X$ be an increasing self-map on S. Then, $Fix(f)$ is nonempty and is closed under maximum and minimum. (In particular, the set of fixed points of f has a maximum and minimum within it.)

Remark 4: Tarski's fixed point theorem was stated by him in a paper published in 1955. The mathematician Knaster also contributed, to the result, by establishing the result for certain special lattices. For this reason, it is often known as the Knaster-Tarski theorem. To see the result as orginally published, see [7]

Note: A map of this nature will be known as a Tarski map.

4 Main Result

Now, we will state and prove the central result of this paper.

4.1 Theorem

Theorem 1. *Let $(X, \|\|, \succcurlyeq)$ be a Riesz space and let $S \subseteq X$ such that:*
i) S is a $\succcurlyeq$ chain in X
ii) S is a complete lattice
iii) S is closed in X.
Let $g : S \to X$ be a contraction on S with Lipschitz coefficient $K \in (0, 1)$ and let $h : S \to X$ be an increasing map on S. Then, if $g(S)+h(S) \subseteq S$, $Fix(h+g)$ is a nonempty subset of S and has a $\succcurlyeq$ maximum and $\succcurlyeq$ minimum.

Proof. Let $f : S \to X$ be the Krasnoselskii embedding constructed using the contraction map. Then, we can define $f^{-1} : f(S) \to S$ (as before, given that f is an embedding). Now, $f^{-1} \circ h$ is well defined, since $h(S) \subseteq f(S)$ (The proof is identical to what was shown in the section on Krasnoselskii's theorem).
As before, in order to show that $h + g$ has a fixed point, it suffices to show that $f^{-1} \circ h : S \to S$ has a fixed point.
Claim: $f^{-1} \circ h$ is an increasing self map on S. To demonstrate this claim, suppose $x \succcurlyeq y$. Then, since h is an increasing map, $h(x) \succcurlyeq h(y)$. Let $(f^{-1} \circ h)(x) = \alpha, (f^{-1} \circ h)(y) = \beta$. Then, $h(x) = f(\alpha) = \alpha - g(\alpha), h(y) = \beta - g(\beta)$. So, we have $(\alpha - g(\alpha)) \succcurlyeq (\beta - g(\beta))$. This gives, $\alpha - \beta \succcurlyeq g(\alpha) - g(\beta)$.

Now, given that S is a chain, we must have that either $\alpha \succcurlyeq \beta$ or $\beta \succcurlyeq \alpha$.
Case 1: $\beta \succcurlyeq \alpha$. This implies $g(\beta) - g(\alpha) \succcurlyeq \beta - \alpha \succcurlyeq 0$. Since X is a Riesz space, this would imply that $\|g(\beta) - g(\alpha)\| \geq \|\beta - \alpha\|$. However, given that g is a contraction, this is a contradiction. Thus, this case cannot occur.
Therefore, we must have $\alpha \succcurlyeq \beta$. This shows that $f^{-1} \circ h$ is an increasing self map on S. Thus, by Tarski's theorem, $f^{-1} \circ h$ has a nonempty set of fixed points which is closed under maximum and minimum. However, since $Fix(f^{-1} \circ h) = Fix(h + g)$, the result holds for $h + g$. This completes the proof. □

5 Discussion and Counterexample

The proof of this theorem, would not be valid if we dropped the chain assumption as it would not even be necessary for α and β to be $\succcurlyeq$ comparable. This prompts the question - Would this result hold in general, if one dropped the chain assumption, and simply assumed the subset to be closed and a complete lattice?
It turns out that the result fails to hold for a general non-chain subset of a Riesz space. To show this we provide a counterexample below.

5.1 Counterexample

Let $X = \mathbb{R}^2$ be the Riesz space under consideration. We will construct a special subset S of X, which is a closed subset and a complete lattice, but is not a chain in X. It will be shown, that the theorem does not hold for this particular subset. First, consider the anti-chain A $= \{(x, y) \in \mathbb{R}^2 : y = -x\} \subset \mathbb{R}^2$. Clearly, A is closed, but it is not a complete lattice as no two elements are comparable with respect to the standard component-wise partial order, and it's components are unbounded. So, fix $x_* \in \mathbb{R}_+$ and define S $= \{(x, y) \in \ A : -x_* \leq x \leq x_*\} \cup \{(x_*, x_*), (-x_*, -x_*)\}$. In particular, S is constructed by taking the union of a finite segment of A with two additional points. These points serve as a maximum and minimum for the set, and since A is an anti-chain, S is a complete lattice. Also, S is closed in X since it is a union of two closed sets.However, it is clearly not a chain in X.

Now, we construct a contraction $g : S \to X$ and an increasing map $h : S \to X$ such that $h(S) + g(S) \subset S$ (which are the conditions of the theorem), but with the property that $Fix(h + g) = \phi$.
First, we define $C = \{(x, y) \in \ A : -x_* \leq x \leq x_*\}$ and $D = \{(x_*, x_*), (-x_*, -x_*)\}$ (Effectively, we consider the two closed sets which separate S causing its disconnectedness). Now, define $g(-x, x) = (\frac{x}{3}, \frac{x}{3})\ \forall (-x, x) \in \ C$ (Since C is the bounded anti-chain $y = -x$, all points are of the form $(-x, x)$ for some unique $x \in (-x_*, x_*)$). Geometrically, this is nothing but a rotational isometry, followed by a scaling by $\frac{1}{3}$. Extend g to the entire S by defining $g(x_*, x_*) = (\frac{-2x_*}{3}, \frac{-2x_*}{3})$

and $g(-x_*, -x_*) = (\frac{2x_*}{3}, \frac{2x_*}{3})$. So, g is defined piecewise on C and D. Clearly, g restricted on C is a Lipschitz map with coefficent $\frac{1}{3}$. Therefore, to show g is a contraction on S it suffices to prove that there is some $c \in (0, 1)$ such that $d(g(x_1, y_1), g(x_2, y_2)) \leq cd((x_1, y_1), (x_2, y_2))\ \forall (x_1, y_1) \in D$ and $\forall (x_2, y_2) \in C$, as then $max(c, \frac{1}{3})$ would be a global Lipschitz coefficient for the function g, proving that it is a contraction. By symmetry, we consider $(x_1, y_1) = (x_*, x_*)$. Let $(x_2, y_2) = (x, -x)$. Then, $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_* - x)^2 + (x_* + x)^2} = \sqrt{2}\sqrt{(x_*)^2 + (x)^2}$. On the other hand, $d(g(x_1, y_1), g(x_2, y_2)) = \sqrt{2}\sqrt{(\frac{x}{3} + \frac{2x_*}{3})^2}$. It is readily seen using the fact that $x^2 \leq x_*^2$, that the quantity is bounded above by $\sqrt{2}\sqrt{\frac{8}{9}}\sqrt{(x_*)^2 + (x)^2}$. In particular, $\sqrt{\frac{8}{9}}$ serves as a good enough value c for our purposes. Thus, $L = max(\sqrt{\frac{8}{9}}, \frac{1}{3}) = \sqrt{\frac{8}{9}}$ is a Lipshcitz coefficient for the map g, proving g is a contraction.

Now, we construct the increasing map h on S as follows- Define $h(x, -x) = (\frac{2x}{3}, 0)\ \forall (x, -x) \in C \setminus 0$. At 0, define $h(0, 0) = (-\frac{x_*}{3}, \frac{x_*}{3})$. Clearly, h is an increasing map when restricted to C because C is an anti-chain and every map on it is an increasing map. Extend, h to S by defining $h(x_*, x_*) = (\frac{2x_*}{3}, \frac{2x_*}{3})$ and $h(-x_*, -x_*) = (-\frac{2x_*}{3}, -\frac{2x_*}{3})$. It is apparent that h is an increasing map on S. Observe, $g(S) + h(S) \subset S$ since all non-zero points on the anti-chain, map to their own rescaling by $\frac{1}{3}$ (as, $(g + h)(x, -x) = (\frac{x}{3}, -\frac{x}{3})$, 0 maps to the point $h(0) \in S$ given that the contraction maps it to itself. Also, $(g + h)(D) = \{(0, 0)\}$. However, even though all the conditions of our theorem were satisfied for the closed, complete lattice S, we observe $g + h$ has no fixed point. Therefore, this method provides a counterexample albeit a pathological one, to a more general version of our theorem with the chain assumption removed.

6 Conclusion and further discussion

From the past two sections, we have seen that while there is a partial result on the combination of Banach and Tarski's theorems in the context of special subsets of a Riesz space, there is no generalization to arbitrary closed, complete lattice subsets of a Riesz space. The former was shown through our theorem and the latter was illustrated via a counterexample. But, chains are far too special objects, and for this theorem to be useful, we would hope for a better result. This prompts the questions: Are there certain special kinds of sub-lattices of a poset for which this result holds, which are more general than chains? If so, what are the potential applications of such a result? An example, could be studying Boolean lattices and their sub-lattice structure and the validity of such results.

7 Acknowledgements

I am greatly indebted to Professor Efe Ok, whose guidance and support enabled me to complete this Independent Study in time, as well as learn a lot in the process. I am also grateful to the Mathematics Department for providing me with this opportunity.

References

[1] Stefan Banach *"Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales"* Fundamenta Mathematicae, 3: 133–181. doi:10.4064/fm-3-1-133-181, 1922

[2] Walter Rudin, *Principles of Mathematical Analysis*, McGraw Hill Education; Third Edition (1 July 2017)

[3] Juliusz Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math.2 (1930), 171-180

[4] F. F. Bonsall *Lectures on some fixed point theorems of functional analysis*, Bombay 1962

[5] Efe Ok *Fixed point Theory-2*, Lecture notes, Graduate Economics, New York University, 2019

[6] Sehie Park *Generalizations of the Krasnoselskii fixed point theorem*, The National Academy of Sciences, Republic of Korea, School of Mathematical Sciences, Seoul National University, ScienceDirect 2006.

[7] Alfred Tarski (1955) *"A lattice-theoretical fixpoint theorem and its applications"*., Pacific Journal of Mathematics. 5:2: 285–309.