An analysis of the numerical stability of the Immersed Boundary Method

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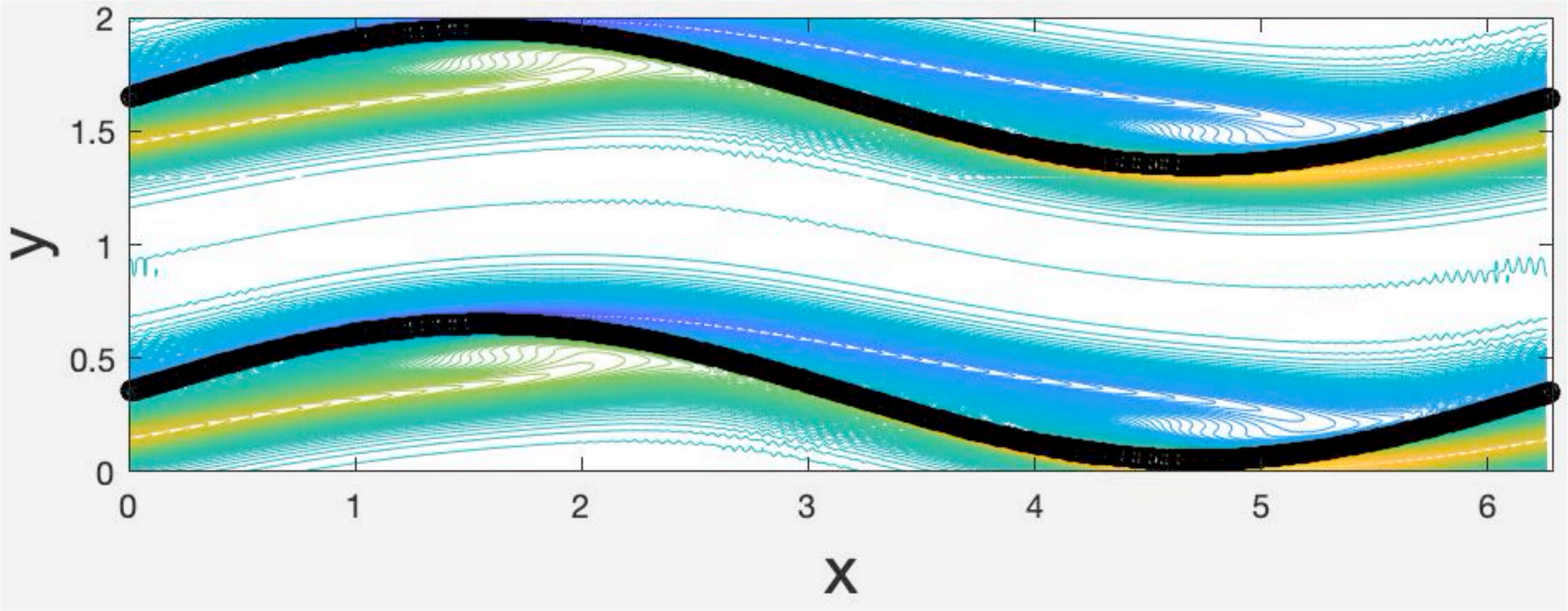
Motivation



Configuration & Equations of motion

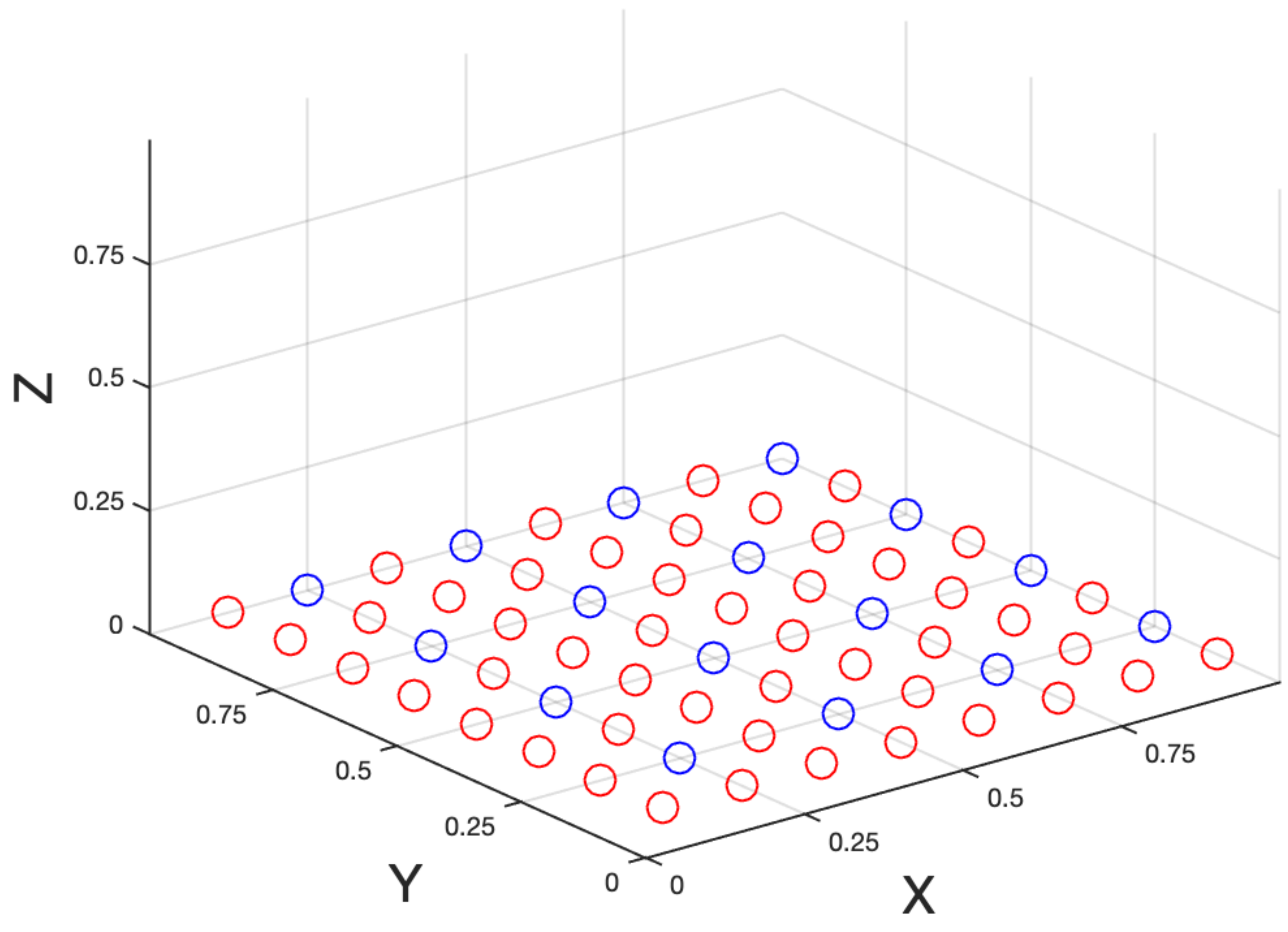


Figure: Boundary grid (red) and fluid grid (blue) at $z = 0$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \mathbf{p} = \mu \Delta \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{f}(\mathbf{x}, t) = \int_{\mathbf{S}} \mathbf{F}(s_1, s_2, t)\delta(\mathbf{x} - \mathbf{X}^0(s_1, s_2))ds_1 ds_2$$

$$\frac{\partial \mathbf{X}}{\partial t}(s_1, s_2, t) = \int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}^0(s_1, s_2))d\mathbf{x}$$

$$\mathbf{F}(s_1, s_2, t) = -K(\mathbf{X}(s_1, s_2, t) - \mathbf{X}^0(s_1, s_2))$$

Notations

s_1, s_2	Material coordinates	K	Spring constant (force/area)
δ	Dirac delta function	$X(s_1, s_2, t)$	Boundary position
F	Feedback force on the boundary	$X^0(s_1, s_2)$	Target point position

Stability analysis via discrete Fourier transform

DFT on the fluid grid

$$\widehat{\mathbf{u}}(\boldsymbol{\xi}) = \sum_{\mathbf{j} \in Z_N^3} e^{-i\frac{2\pi}{N}\mathbf{j}\cdot\boldsymbol{\xi}} \mathbf{u}(\mathbf{x}_\mathbf{j})$$

$$\mathbf{u}(\mathbf{x}_\mathbf{j}) = \frac{1}{N^3} \sum_{\boldsymbol{\xi} \in Z_N^3} e^{i\frac{2\pi}{N}\mathbf{j}\cdot\boldsymbol{\xi}} \widehat{\mathbf{u}}(\boldsymbol{\xi})$$

DFT on the boundary grid

$$\widetilde{\mathbf{F}}(m_1, m_2) = \sum_{(k_1,k_2) \in Z_{NP}^2} e^{-i\frac{2\pi}{NP}(k_1 m_1 + k_2 m_2)} \mathbf{F}_{k_1,k_2}$$

$$\mathbf{F}_{k_1,k_2} = \frac{1}{N^2 P^2} \sum_{(m_1,m_2) \in Z_{NP}^2} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} \widetilde{\mathbf{F}}(m_1, m_2)$$

What connects these two different discrete Fourier transforms is the smoothed Delta function(IB 4-point delta function).

Please ask me for more details if you are interested!

Stability analysis via discrete Fourier transform

Form of the solution

$$\widehat{\mathbf{u}}^n(\boldsymbol{\xi}) = z^n \widehat{\mathbf{u}}^0(\boldsymbol{\xi})$$

$$\widetilde{\mathbf{F}}^{n+\frac{1}{2}}(m_1, m_2) = z^n \widetilde{\mathbf{F}}^{\frac{1}{2}}(m_1, m_2)$$

$$\widehat{\mathbf{f}}^{n+\frac{1}{2}}(\boldsymbol{\xi}) = z^n \widehat{\mathbf{f}}^{\frac{1}{2}}(\boldsymbol{\xi})$$

Stable if z lies inside the unit circle

$(|z| < 1)$

→

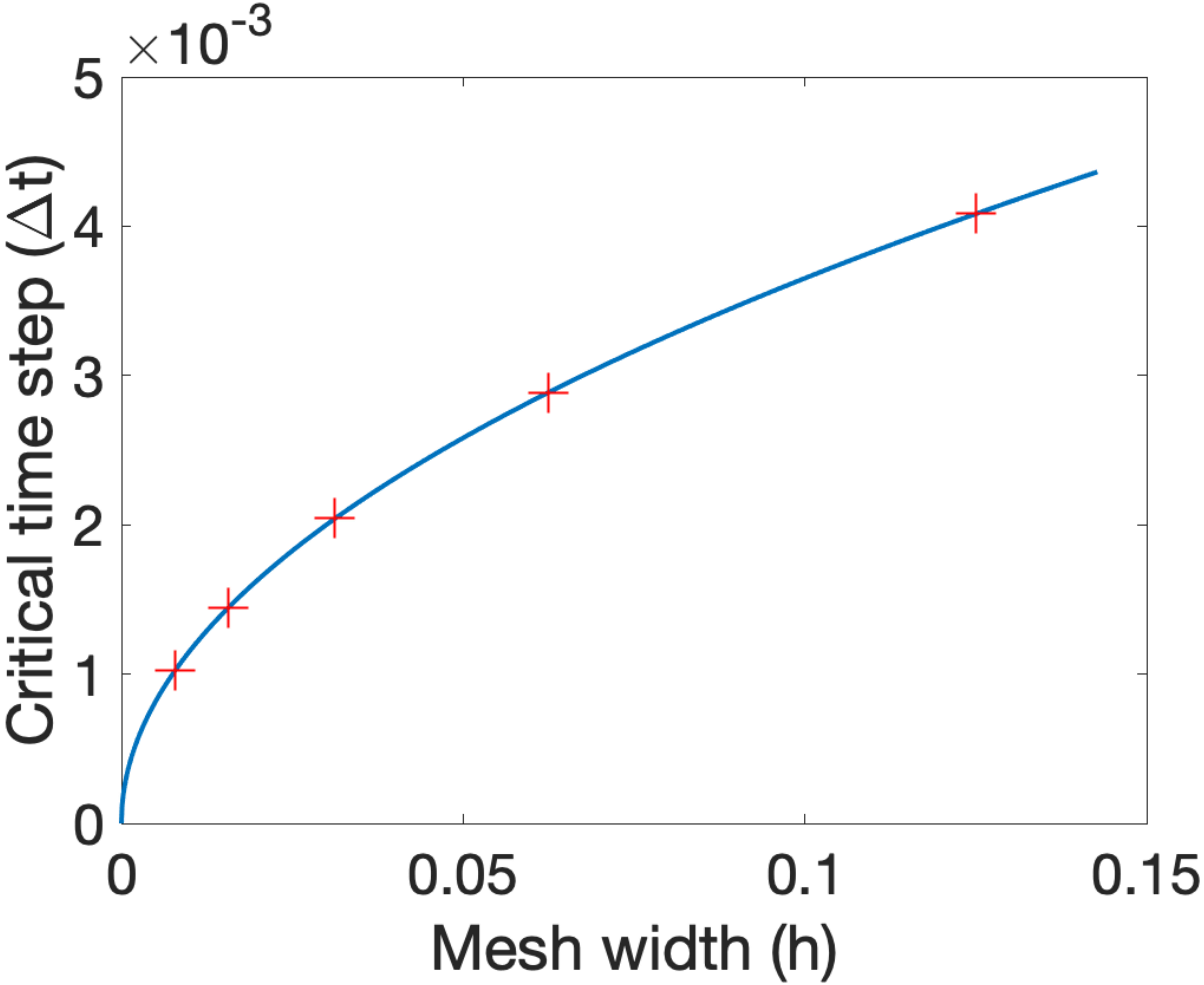
Stability criterion:

$$\frac{\Delta t^2 K}{\rho h} \leq \frac{32}{3}$$

Consequence: we can achieve the continuum limit and the no-slip limit simultaneously by letting

$$\Delta t \to 0, K \propto \frac{1}{t}, h \propto t \, .$$

Numerical Results



Generalizations of the framework

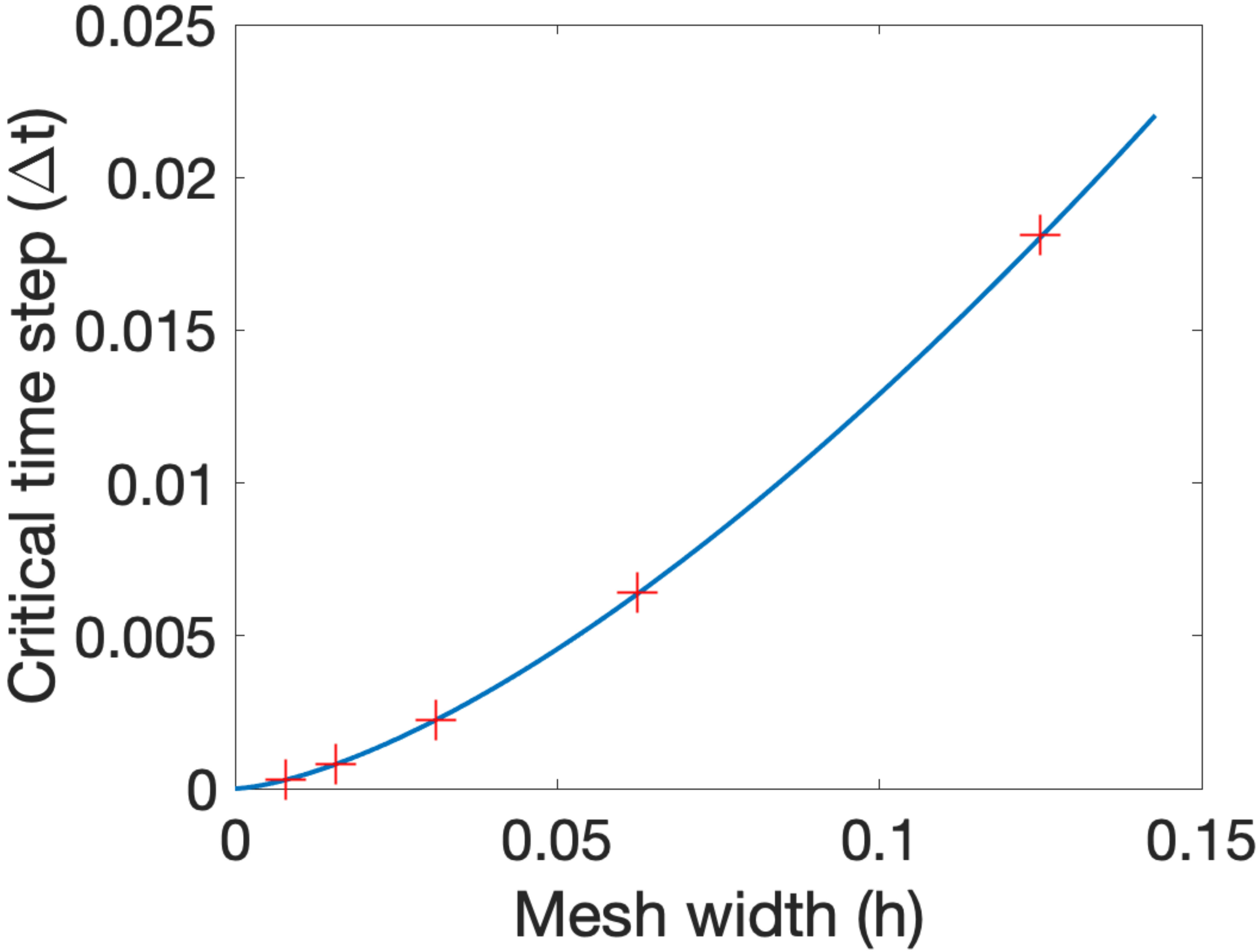
- Although we only did analysis for the linearized Navier-Stokes equations, the numerical tests show that the analysis also perfectly aligns with the observed stability behaviors of the Naiver-Stokes equations.

- We also generalized the analysis to the case where the boundary is shifted.

- The boundary has not to be a no-slip boundary. For example, It can be an elastic membrane with $F = K\Delta X$ and we have also done the analysis for this case. The stability issue gets worse when the order of the force increases.

Results about the elastic membrane case

$$\frac{P^2 \Delta t^2 K}{h^3 \rho} \leq \frac{50}{3}$$



Questions?