Controlling a Stochastic Harmonic Oscillator

Joonsoo Lee, Georg Stadler, Shanyin Tong

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Abstract

We develop a method to optimize the probability of extreme events associated with stochastic systems. Specifically, we examine methods for a stochastically forced harmonic oscillator to hit extreme event regions. We proceed to focus on strategies that are scalable, i.e. their efficiency does not degrade upon spatial and temporal refinement.

1 Introduction

Our study is motivated by chance constrained optimization problems of the following form.

$$\min_u J(u) \tag{1}$$

$$\text{s.t. } \mathbb{P}[F(u,\xi) \geq z] \leq \alpha. \tag{2}$$

There exists a method to solve this for low dimensional randomness, as seen in [11], but we aim to provide a scalable framework to tackle the optimization. The most common high dimensional randomness one would encounter is Brownian motion. For this project, we consider a stochastically forced harmonic oscillator with damping, where $m > 0$ is the mass, u the parameters we wish to control, and $\gamma > 0$ the damping coefficient:

$$m\phi'' + \gamma\phi' + G(\phi, u) = \sigma\eta(t). \tag{3}$$

Here, $G(\phi, u)$ is a combination of the nonlinear and deterministic forcing term. The initial conditions $\phi(0)$ and $\phi'(0)$ can also depend on u.

2 Probability Approximation

We wish to compute the probability (2) using the scalable path space probability estimation from [9]:

$$\mathbb{P}[F(\phi,\psi) \geq z] \underset{\sigma\to 0}{\longrightarrow} (2\pi)^{-1/2} C_F(z) \exp\left(-I_F(z)\right). \tag{4}$$

The real valued functions $C_F(z)$ is called the prefactor and I_F is called the rate function. In our case, these functions are defined

$$I_F(z) = \frac{1}{2}\|\eta_z(t)\|^2_{L^2}, \tag{5}$$

$$C_F(z) = [2I_F(z)\det\left(1_{N\times N} - \lambda_z \mathrm{pr}_{\eta_z^\perp}\nabla^2 F(\eta_z)\mathrm{pr}_{\eta_z^\perp}\right)]^{-1/2}, \tag{6}$$

$$\mathrm{pr}_{\eta_z^\perp} = 1_{N\times N} - \frac{\eta_z \otimes \eta_z}{||\eta_z||^2}. \tag{7}$$

The first step to obtain the values of the prefactor and rate function is to find the optimal forcing $\eta_z(t)$, which can be found by solving a PDE constrained optimization problem. $\eta_z(t)$ represents the least costly forcing that results in $F(u,\xi) \geq z$ being satisfied. Next, we use a random svd algorithm, shown in [4], to compute the determinant of the operator in (6). Then we are able to calculate an accurate probability estimation for small probabilities. Finally, using jax autodiff, we are able to also compute the gradient of these probability estimations.

3 Results

In Figure 1, we illustrate a comparison of the probability approximations and probabilities found through sampling. The blue shading shows the 95% confidence interval of the true probability.

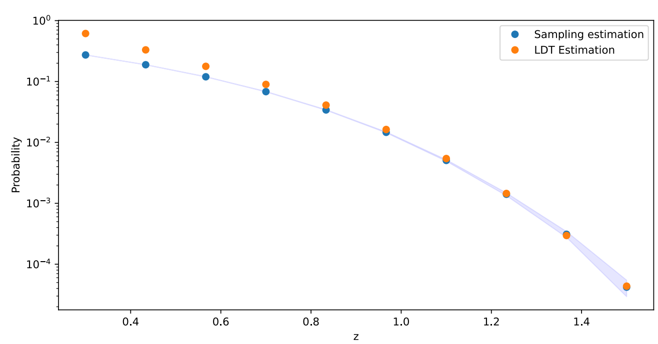

Figure 1: LDT approximation and sampling probability values with $\gamma = 1.5$, $m = 1$, $\sigma = 1$, $N = 500$, $T = 5$.

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