Internal Waves in Time-Dependent Stratifications

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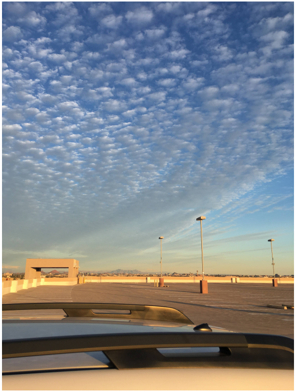
What are internal waves?

Internal Wave: A wave that propagates within a stratified fluid
Stratified Fluid: A fluid whose density varies with height



Why do these waves occur?

Internal waves form in stratified fluids, whose density increases with depth

- In the ocean, density varies due to temperature and salinity
- Perturb water particle $\implies$ oscillations
- Existing theory on internal waves in constant or spatially-dependent stratifications

What if the stratification depends on time?

We can answer this mathematically, by deriving the PDE that governs internal waves and analyzing its solutions!

Note on Necessary Variables

We consider a 2-D problem in x, z, with time variable t

- $\epsilon \ll 1$ scaling parameter
- Stream function $\Psi(x,z,t) = \bar{\Psi}(x,z,t) + \epsilon\Psi'(x,z,t)$
  - $\bar{\Psi}$: large-scale, mean flow $\implies 0$
  - Ψ': fine-scale oscillations, related to velocity field of the wave
- Density $\rho(x,z,t) = \bar{\rho}(z,t) + \epsilon\rho'(x,z,t)$
  - $\bar{\rho}$: large-scale, background stratification
  - ρ': fine-scale variations to this background stratification
- Brunt-Väisälä frequency $N = \sqrt{-\frac{g}{\rho_0}\partial_z\bar{\rho}}$
  - Proportional to background density term $\partial_z\bar{\rho}$
  - Describes frequency of oscillations
  - Stratification parameter
- Buoyancy $b = -\frac{g}{\rho_0}\rho'$
  - Proportional to fine-scale density term ρ'

The PDE

From the inviscid Navier-Stokes equations, mass conservation law, and incompressibility law we can derive the Nonlinear Internal Wave PDE:

$$\begin{cases} \partial_t \Delta\Psi - \partial_x b = -\epsilon J(\Psi, \Delta\Psi) \\ \partial_t b + N^2 \partial_x \Psi = -\epsilon J(\Psi, b) \end{cases}$$

where $\Delta = \partial_x^2 + \partial_z^2$, and $J(f, g) = \partial_x f \partial_z g - \partial_z f \partial_x g$.

The ϵ scale controls the strength of nonlinearity. Setting $\epsilon = 0$, we recover the Linear Internal Wave PDE:

$$\partial_t^2 \Delta\Psi + N^2 \partial_x^2 \Psi = 0$$

How does setting $N = N(t)$ affect the solution Ψ?

Constant Stratification

Suppose $N = N_0$ is constant:

$$\partial_t^2 \Delta \Psi + N_0^2 \partial_x^2 \Psi = 0$$

We can solve using Fourier transform in all variables and find

$$\Psi(x, z, t) = \Psi_0 \exp(i(\omega t - k_x x - k_z z))$$

- Constant amplitude Ψ_0
- Constant temporal and spatial frequencies ω and $\mathbf{k} = (k_x, k_z)$

The wave's temporal and spatial frequencies are related by the dispersion relation:

$$\omega^2 |\mathbf{k}|^2 = N_0^2 k_x^2$$

which tells us that these waves only propagate if $\omega \leq N_0$

Time-dependent Stratification

Suppose our background density varies with time:

$$\partial_t^2 \Delta \Psi + N^2(t)\partial_x^2 \Psi = 0$$

Using spatial Fourier transform and assuming that the stratification is sufficiently slow-varying, we find:

$$\Psi(x, z, t) = \Psi_0(t) \exp\left(-i(k_x x + k_z z)\right)$$

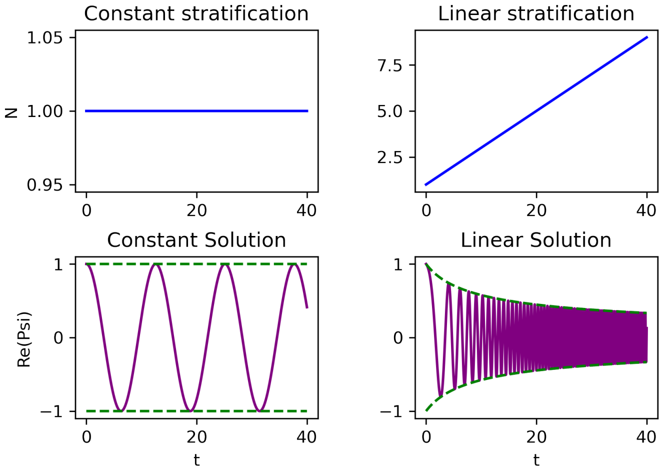
where

$$\Psi_0(t) = \frac{A_0 \sqrt{k}}{\sqrt{N(t)}} \exp\left(i \frac{k_x}{k} \int N(t) \mathrm{d}t\right)$$

- Slow-varying envelope $A(t) \approx \dfrac{1}{\sqrt{N(t)}}$
- Fast-varying oscillations set by $\dfrac{k_x}{k} \displaystyle\int N(t) \mathrm{d}t$

Effect of slow time-dependence

Using simple stratifications $N_0 = 1$ and $N(t) = 0.2 + t$, we can visualize how a time-dependent stratification will affect the shape of the wave:



Mathieu's equation

Suppose our time-dependent stratification has small oscillations, such that $N^2(t) = N_0^2(1 + \epsilon \cos t)$:

$$\partial_t^2 \Delta \Psi + N_0^2(1 + \epsilon \cos t)\partial_x^2 \Psi = 0$$

Then we can study Mathieu's equation on the time-dependent part of the wave:

$$\Psi_0''(t) + \frac{k_x^2}{k^2} N_0^2(1 + \epsilon \cos t)\Psi_0(t) = 0$$

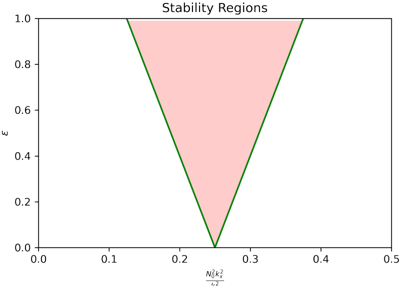
with first-order solution

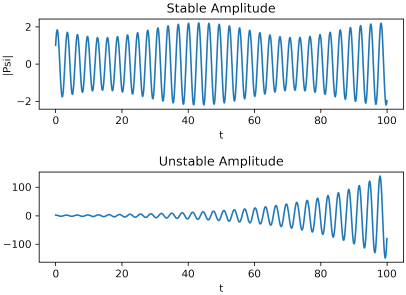
$$\Psi_0(t) = A(t) \cos\left(\frac{k_x N_0}{k} t\right) + B(t) \sin\left(\frac{k_x N_0}{k} t\right)$$

Mathieu's equation allows for parametric resonance:

- $\epsilon \approx 0 \implies \Psi_0$ stable, unless $\frac{k_x N_0}{k} = \frac{1}{4}$
- $\epsilon > 0 \implies$ regions of stability, dependent on the ratio of ϵ and $\frac{k_x N_0}{k}$

Mathieu's: Stability





- Stable amplitude ($\epsilon = 0.2$, $\frac{N_0^2 k_x^2}{k^2} = 0.3$): beating envelope
- Unstable amplitude ($\epsilon = 0.2$, $\frac{N_0^2 k_x^2}{k^2} = 0.25$): exponentially growing envelope

Nonlinear Effect: Self-Interaction

Recall the full nonlinear PDE:

$$\begin{cases} \partial_t b + N^2 \partial_x \Psi = -\epsilon J(\Psi, b) \\ \partial_t \Delta\Psi - \partial_x b = -\epsilon J(\Psi, \Delta\Psi) \end{cases}$$

- We study interactions between two waves Ψ_1 and Ψ_2 by analyzing their sum $\Psi = \Psi_1 + \Psi_2$
- Nonlinear terms $\implies$ self-interaction: a single wave can interact with itself and generate other waves at resonant frequencies

If $N = N_0$ or $N(t) \implies J(\Psi, \Delta\Psi) = J(\Psi, b) = 0$

- No self-interaction in time-dependent stratifications
- The linear solution is a solution of the nonlinear PDE
- To see self-interaction, we need stratifications that depend on space

Time- and space- dependent stratification

Suppose we have a time- and space-dependent stratification $N = N(z, t)$:

$$\partial_t^2 \Delta \Psi + N^2(z, t) \partial_x^2 \Psi = 0$$

If we assume also that $N(z, t) = f(t)g(z)$, then we can use partial Fourier transform and separation of variables to find

$$\Psi(x, z, t) = F(t) G(z) \exp(-ik_x x)$$

where

$$F(t) \approx \frac{1}{\sqrt{f(t)}} \exp\left(i \int f(t) \mathrm{d}t\right)$$

$$G(z) \approx \frac{1}{(g^2(z) - 1)^{1/4}} \exp\left(-i \int \sqrt{g^2(z) - 1} \mathrm{d}z\right)$$

Oscillatory part set by $\Phi = \int f(t) \mathrm{d}t - \int \sqrt{g^2(z) - 1} \mathrm{d}z - k_x x$

Nonlinear Effect: $N(z,t) = f(t)g(z)$

Putting this solution into the nonlinear PDE, we see

$$\begin{aligned}\partial_t^2 \Delta\Psi + N^2(z,t)\partial_x^2\Psi &= \mathcal{G}_0(t,z) + \mathcal{G}_2(t,z)e^{2i\Phi} + \mathcal{G}_\alpha(t,z)e^{i\Phi_t - 2i\Phi_z} + c.c. \\ &\neq 0\end{aligned}$$

where $\Phi = \Phi_t + \Phi_z = \int f(t)\mathrm{d}t - \int \sqrt{g^2(z) - 1}\mathrm{d}z$, and

- $\mathcal{G}_0(t,z) \propto g'(z)$
- $\mathcal{G}_2(t,z) \propto g'(z)$
- $\mathcal{G}_\alpha(t,z) \propto g'(z)f'(t)$

Turning off $z-$dependence $\implies$ no self-interaction
Turning off $t-$dependence $\implies$ existing solution (Baker 2020)

Conclusion & Perspectives

Conclusion:

- Goal: Develop some theory on internal waves in time-dependent stratifications
- Linear dynamics $\implies$ time dependence in wave's amplitude and temporal frequency
- Nonlinear dynamics $\implies$ no self-interaction, unless stratification is also spatially dependent
- Types of nonlinear interactions: self-interaction and generation of superharmonics, multiple waves and triadic resonant instability

Perspectives & Further work:

- More detailed study of nonlinear resonances
- Analyze nonlinearities in Mathieu's case
- Analyze system from energetic point of view

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