# graph-theoretic Markov methods for modelling Arctic sea-ice

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# *motivations*

**Note: a chunk of frozen sea water is referred to as an "ice-floe"*

Common methods for modelling sea-ice:

- "Continuous" methods (solving PDE's, etc.)

  references: Hibler (1979), Dansereau et al. (2016 & 2019), and Hunke et al. (many papers)

- "Discrete" methods (discrete element methods, etc.)

  references: West et al. (2020), Hopkins (1991), and Manucharyan et al. (in prep.)

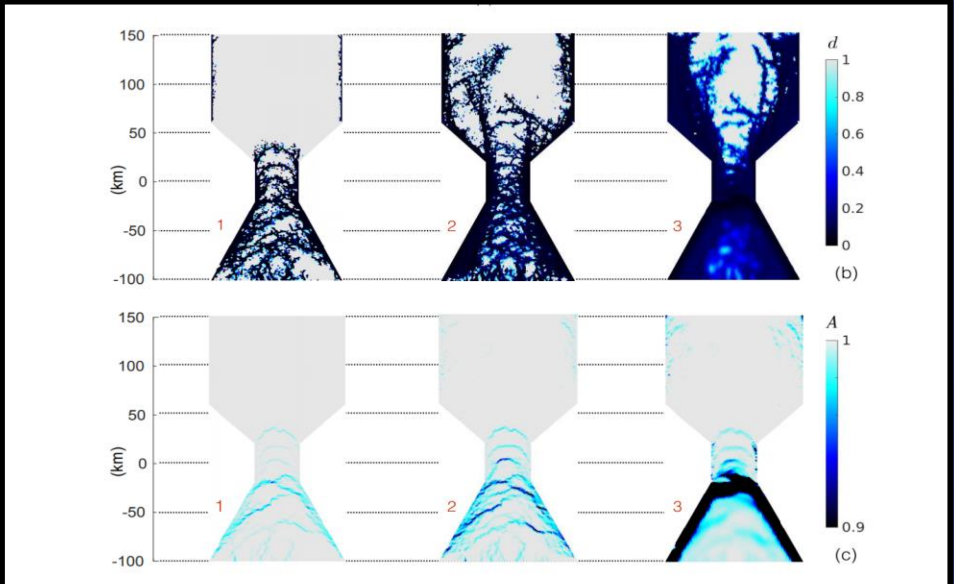
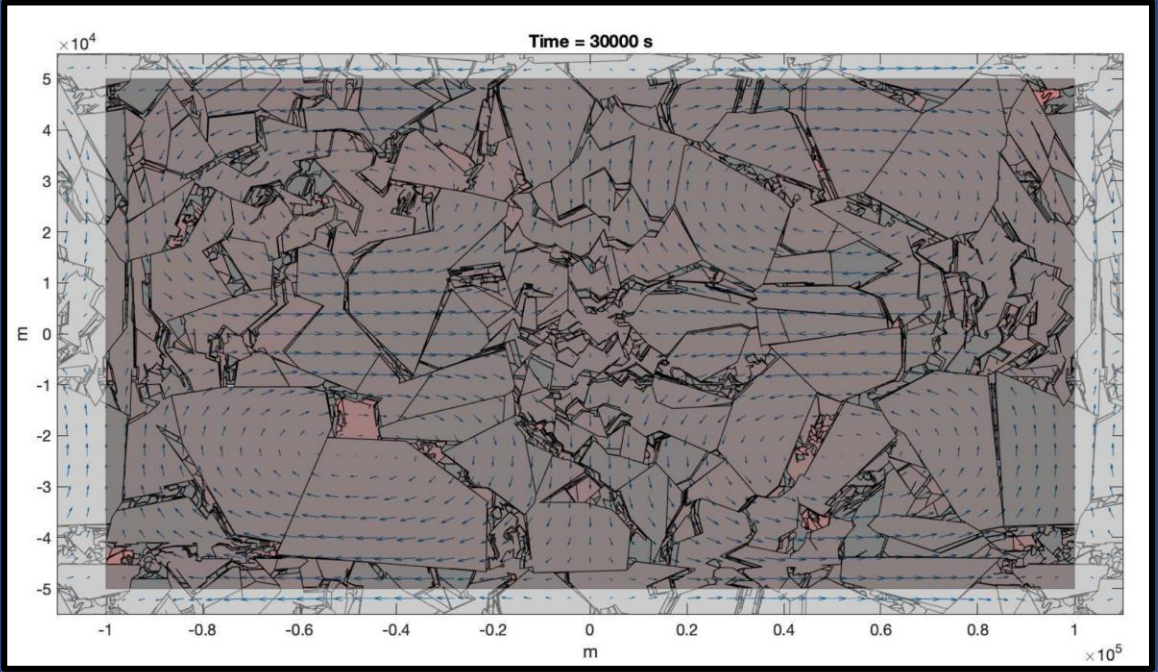



**Figure 4.** **(a)** Time series of the wind forcing (dashed curve) and of the damage rate (solid grey curve) in an idealized channel simulation using $C_{min} = 20$ kPa. Instantaneous spatial distribution of **(b)** the level of damage and **(c)** ice concentration at the times indicated by the numbers 1, 2 and 3 on the time series of panel **(a)**.

*"Continuum" model, Dansereau et al. (2016)*




*Discrete element method, Sea Ice MURI*

# *domain discretization (using a graph):*

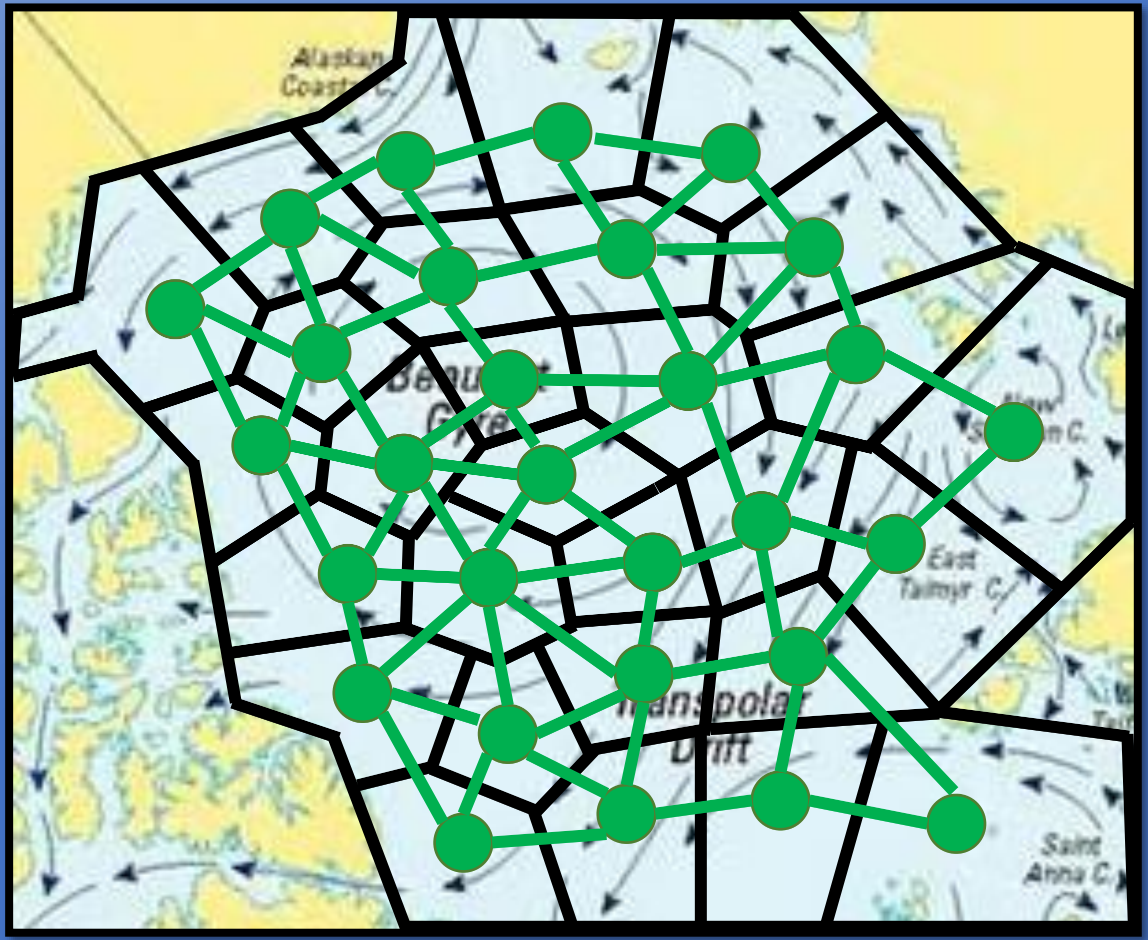
**definition:** *graph* – a collection of nodes and edges

- Place nodes according to some placement scheme

- Generate a partition via Voronoi tessellation,

- Deduce edges from the partition.

*(for simplicity's sake, we're going to model a square domain and lattice)*



# ***background: Markov models***

**definition:** *Markov model* – a sequence of random variables (that we call *states*)

our state? *sea-ice mass distribution*, written $\theta_t$ for $t = \{0,1,2,\ldots\}$

$\theta_t = m\vec{p_t}$ , where $m$ is the total ice mass and $\vec{p_t}$ is the ice mass probability density at time $t$.

**non-linear transition:**

$$\theta_{t+1} = \boldsymbol{K}(\theta_t, \xi)$$

$\boldsymbol{K}$ is called a *transition kernel.*

**linear transition:**

$$\theta_{t+1} = \theta_t \mathbf{T}_\xi$$

$\mathbf{T}$ is called a *transition matrix*.

$\xi$ – *fixed model parameters*

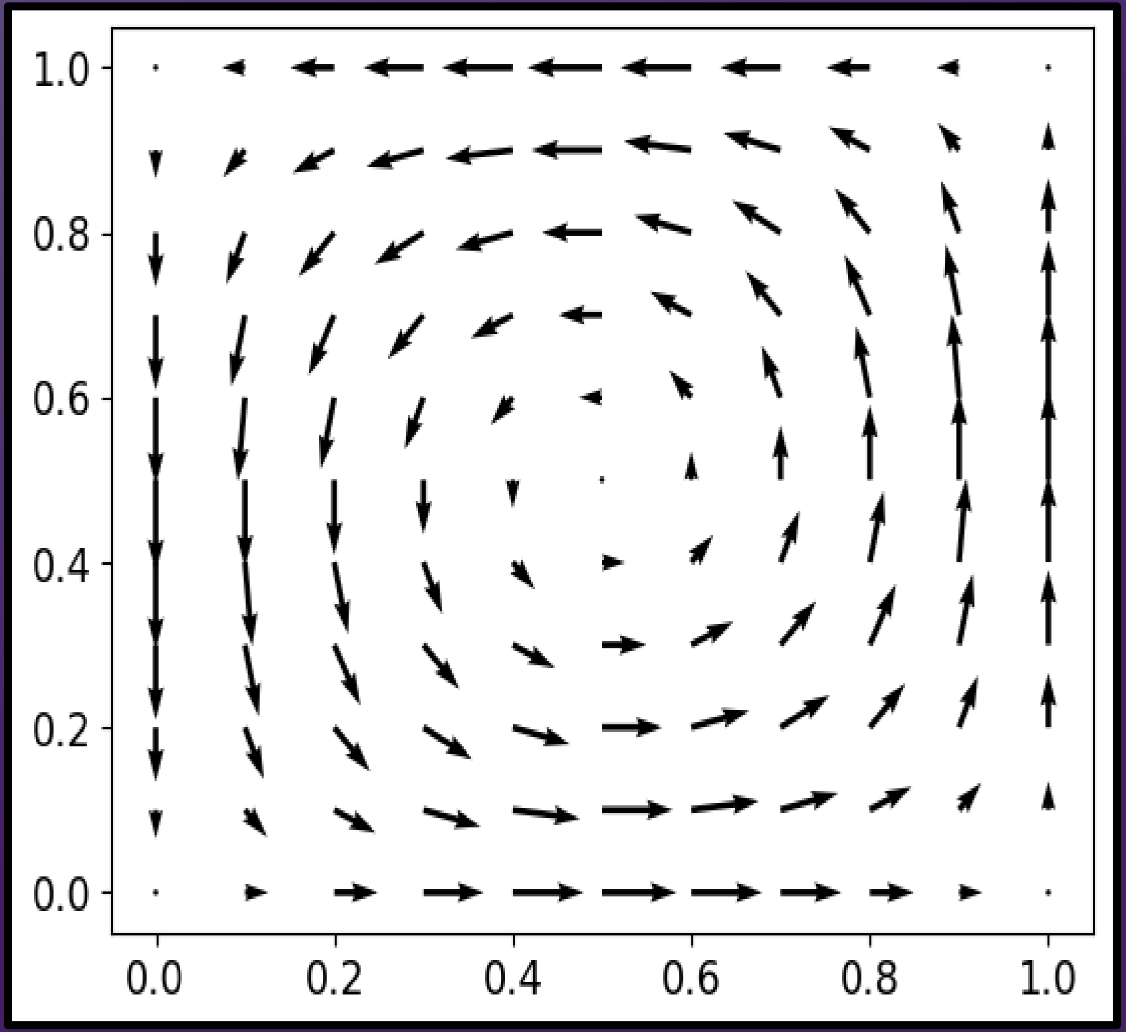
# *advective (external) forces*

- Wind and ocean currents drive much of the advection of sea ice.

- On the domain $\Omega = [0,1] \times [0,1]$, we define an external forcing field due to wind/ocean currents:

$$\vec{V} : \Omega \to \mathbb{R}^2$$

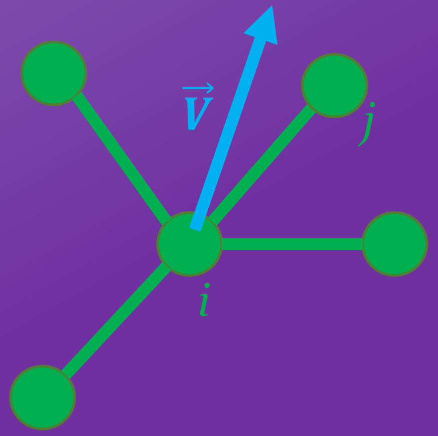
*Vector field $\vec{V}$ should be sufficiently smooth and well behaved!*




*a counter-clockwise gyre*

# *generating the transition probabilities*

**definition:** *transition probability* – the probability of ice moving from node *i* to node *j* ($p_{i,j}$); elements of the *transition matrix*.



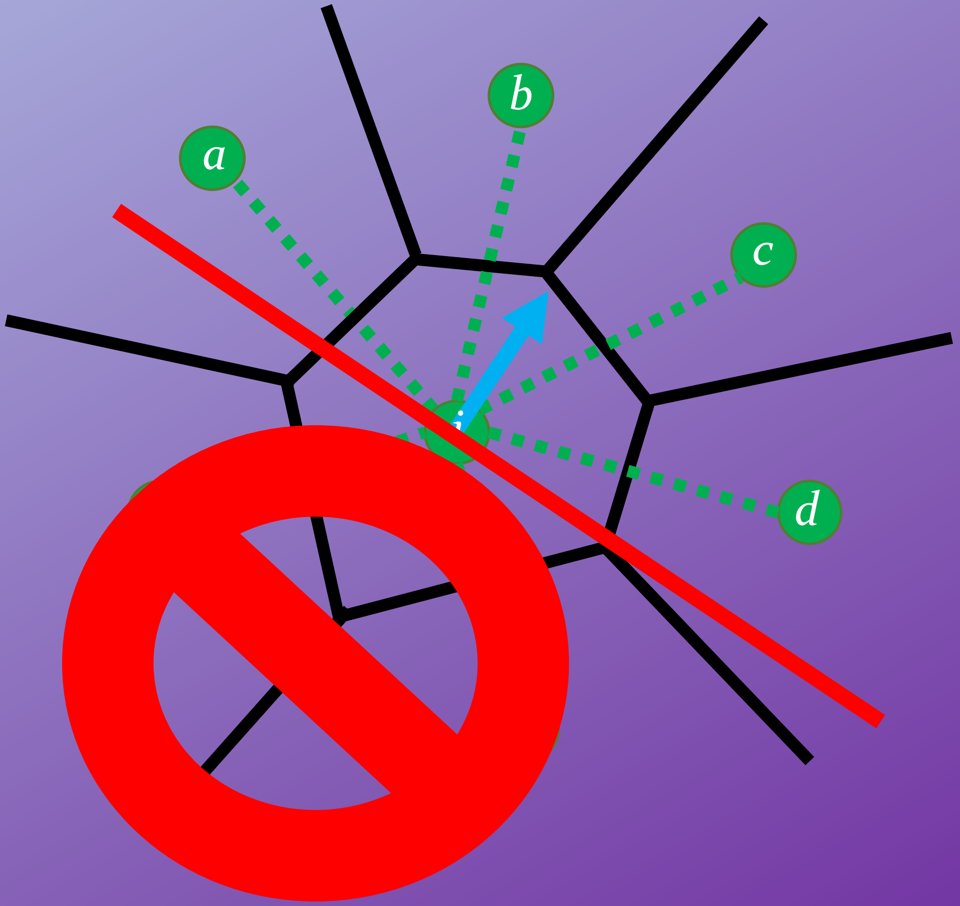

- *Diagonal entries $p_{i,i}$ indicate the probability of mass staying at node i. We want $0 \leq p_{i,i} \leq 1$ :*
  - *As* $\vec{V} \rightarrow \mathbf{0}$, $p_{i,i} \rightarrow 1$ *and vice versa.*
  - *As* $\Delta t \rightarrow \mathbf{0}$, $p_{i,i} \rightarrow 1$ *and vice versa.*
  - *Normalize by edge lengths ($p_i$)*

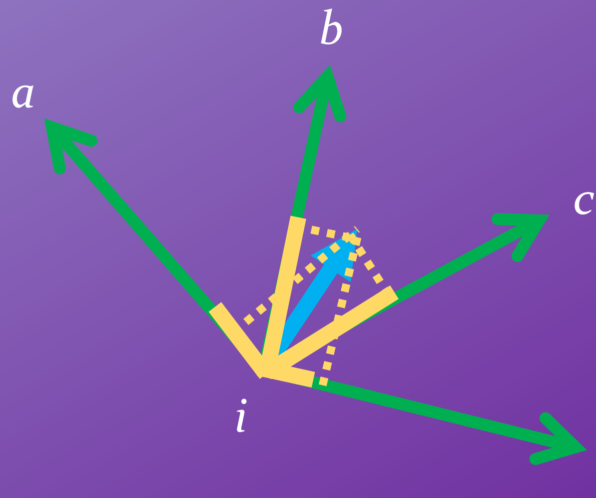
$$p_{i,i} = \frac{1}{1 + p_i \Delta t ||\vec{V}||}$$

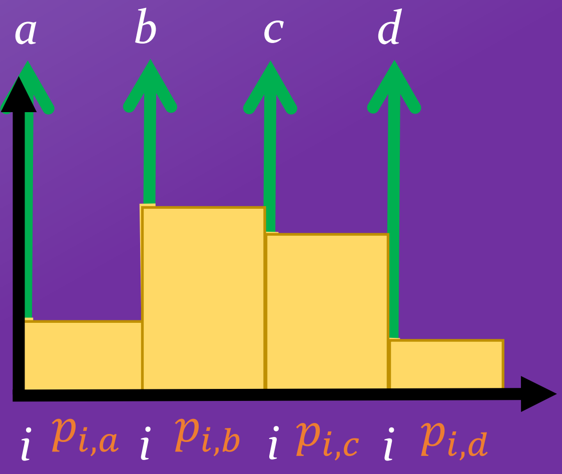
- *Non-diagonal entries $p_{i,j}$ in the transition matrix indicate the probability of mass leaving node i to node j.*

$$p_{i,j} = \left(1 - p_{stay}\right) P(\text{mass moving from i } \rightarrow \text{j } \textit{given} \text{ mass is not staying})$$

# *projection method for mass transition*



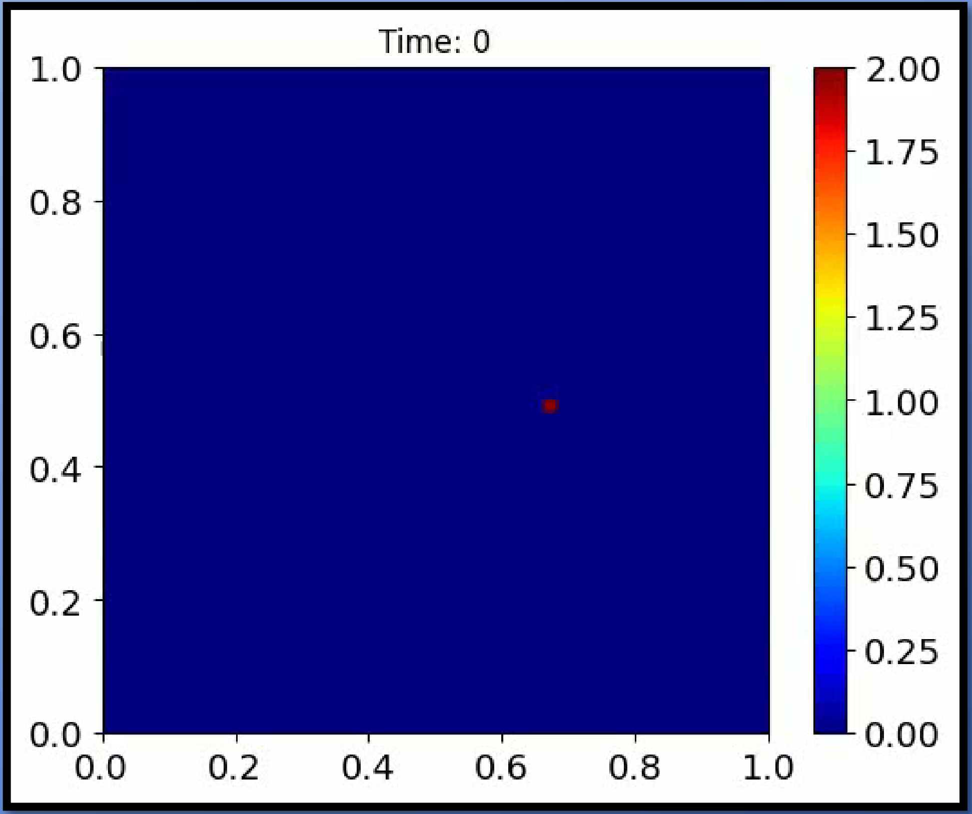

*Non-diagonal entries $p_{i,j}$ indicate the probability of mass leaving node i.*

$$p'_{i,j} = \frac{\max(0, \overrightarrow{e_{i,j}} \cdot \vec{V})}{\left\|\overrightarrow{e_{i,j}}\right\|^2}, \quad p_{i,j} = (1 - p_{i,i}) \frac{{p_{i,j}}'}{\sum_{k \in N(i)} {p_{i,k}}'},$$

*where $N(i)$ are the neighbors of node i.*

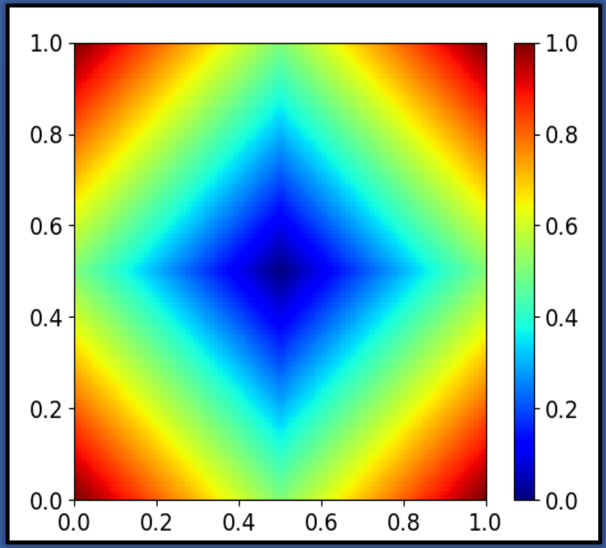
# *results:*

**definition**: *steady state* – if the system meets certain requirements (called *ergodicity*), then a unique steady state exists; this is the described as $\vec{\pi} = \lim_{t\to\infty} \theta_t = \lim_{n\to\infty} \theta_0 \mathbf{T}^n$ for any initial state $\theta_0$.



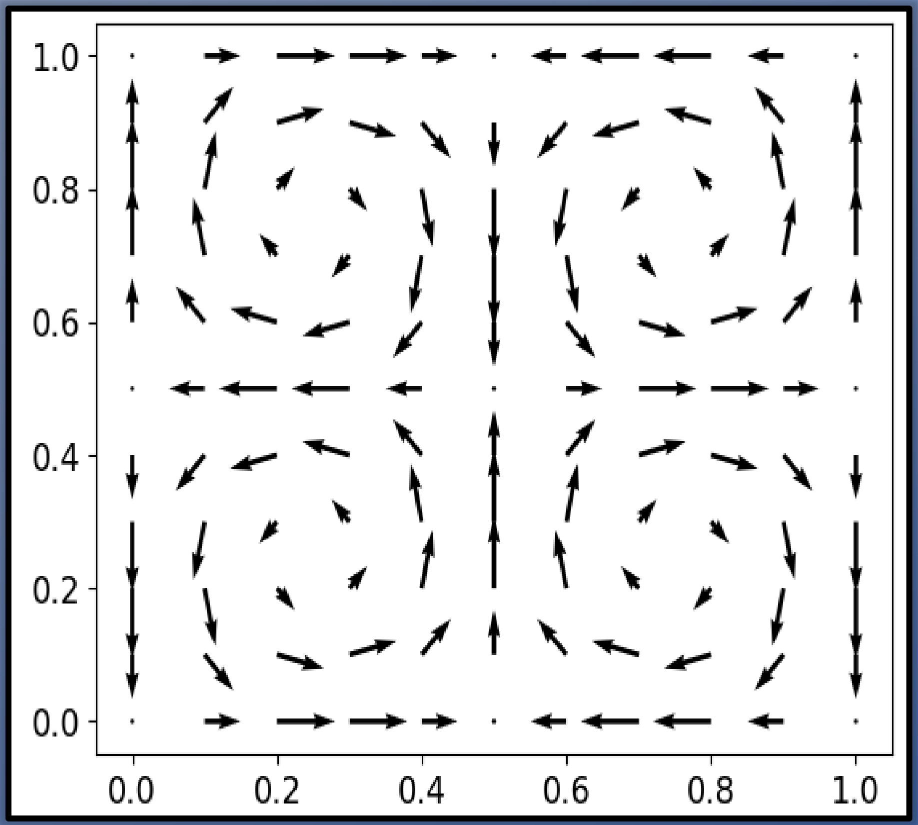

## note:

from the definition, we see that $\vec{\pi} = \vec{\pi}\mathbf{T}$ must be true, and thus the steady state is an eigen - vector of **T**! We can use this to predict steady-states without needed to simulate for long time-frames.

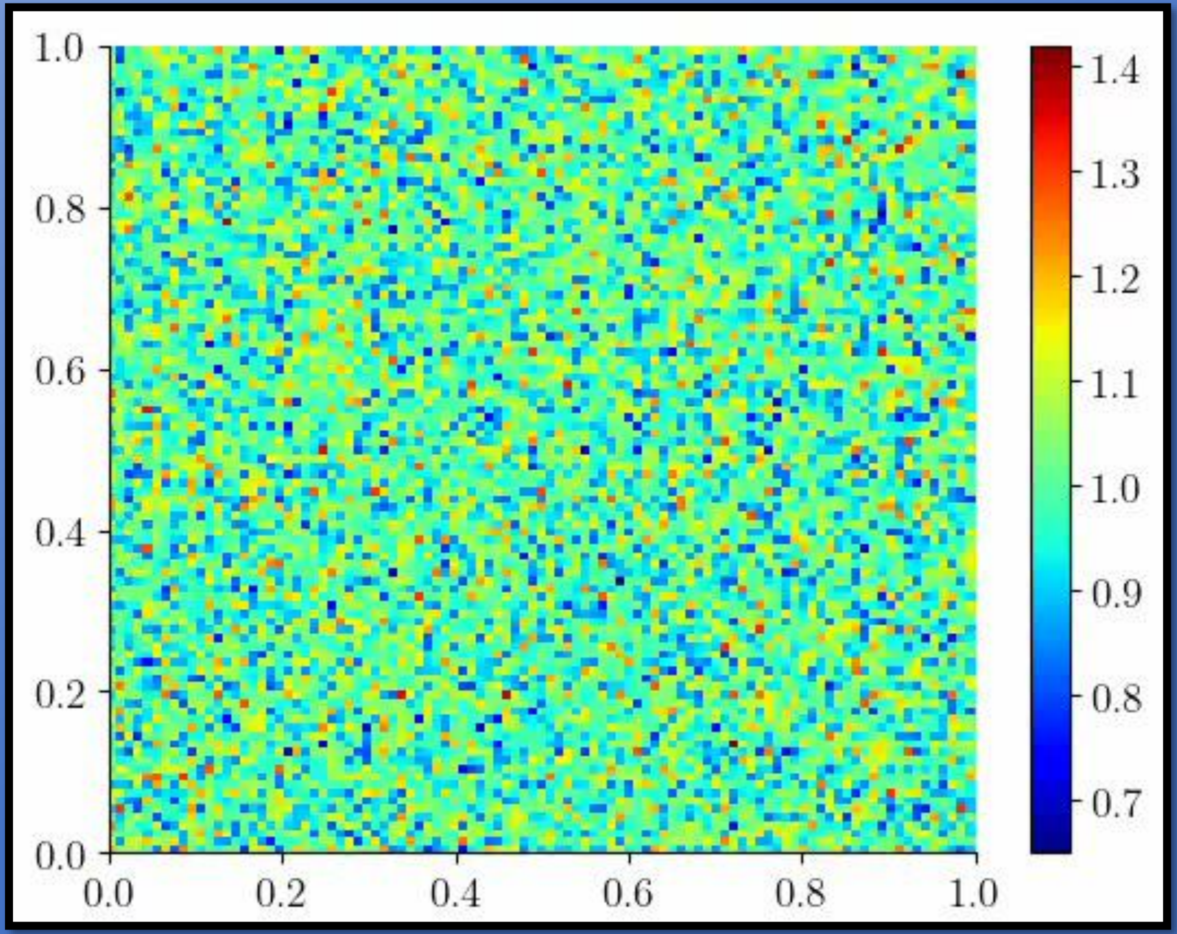



*predicted eigenstate for a gyre*

# *more results: exploring particle methods*




particle simulation with
100,000 particles in advection
according to a 4-gyre



# conclusions:

- We created a Markov model that simulates the advection of sea-ice mass in the Arctic.
- Using this model, we can make predictions about the long-term behavior/ergodicity of sea ice mass given a time-invariant external forcing field.
- This is a smaller part of a larger collaboration on modelling sea-ice as a particle system.

# future work:

- Use kinetic theory to inspire a large scale model.
- Use particle methods to inform new fields that give interesting insight and control over the model.
- Tweak parameters and non-linear transition kernel $\boldsymbol{K}(\theta_t, \xi)$ to better reflect the dynamics of sea-ice.

# thanks for listening!

any questions?

my sweet baby, (mini) cooper

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