Equilibrium Stat Mech
of Semi-flexible Fibers
Worm-like Chains
(e.g. polymers, DNA, actin, microtubules)

\[ l_j = \| x_{j+1} - x_j \| \quad \quad \quad t_j = \frac{x_{j+1} - x_j}{l_j} \]
2. Energy due to stretching:

\[ E = \frac{k}{2} \sum_{j=0}^{n-1} l_j^2 \]

Spring constant called "freely-jointed chain".

Statistical mechanics dictates the equilibrium distribution (invariant measure) at the thermodynamic equilibrium at temperature T. This is the Gibbs-Boltzmann distribution.
\[ \mathcal{X} = \{ \mathbf{x}_j \} \text{ (configuration)} \]

\[ P_{eq}(\mathcal{X}) = Z^{-1} \exp \left( - \frac{E(\mathcal{X})}{k_B T} \right) \]

But really GB is a **Gibbs-Boltzmann weight** and to make it into a distribution or measure we need a *reference* or *base measure* \( M_0 \) :
Today we discuss an example (not unique!) where it is difficult if not impossible to define the base measure.
Part I

Bead-Link Chain

Often to a very good approximation the links of the chain are of fixed length.

\[ \| \mathbf{x}_{j+1} - \mathbf{x}_j \| = l_j = l \quad \text{(inextensibility)} \]

Inextensible freely-jointed chain

What is \( \lim_{k \to \infty} d_{\text{H} \text{eq}}(\overline{X}) \)?
Now $d\nu_\rho(Y)$ is a measure on a manifold

$$M = \big\{ \frac{1}{2} \times \frac{1}{2} \big\} \ell_j = \ell, \#j \geq 3$$

If there is only stretching energy (freely-jointed chain) then $\mu_{eq} = \mu_0$

So what is the base measure?
Two options:

\[ 0 \, \rho \mu_0 (\bar{x}) = \int \rho \mu (\bar{x}) \, \text{Hausdorff measure (area)} \]

\[ \rho \mu_0 (\bar{x}) = \prod \delta (\varepsilon_j (\bar{x}) - \varepsilon) \]

\[ \delta (q (\bar{x})) \]

where \( M = \{ \bar{x} | q (\bar{x}) = 0 \} \)

\( q \mu \) area. "Thickness" of manifold
The two are related by the co-area formula

\[ \delta(q(x)) \cdot \| \nabla q(x) \| \, d\mathcal{X} = d\mu(x) \]

Turns out the correct measurable limit is

\[ \int d\mu_0 = \delta(q(\mathcal{X})) \, d\mathcal{X} \]

\[ q_i = e_j^2 - e_i^2 \]
If there is an additional energy $U$ (elastic, steric, electrostatic, etc.) then

$$d\mu_{eq}(\mathbf{x}) = e^{-U/k_BT} \, d\mu_0(\mathbf{x})$$

This has become complicated. Why not just avoid constraints?
If we know tangent vectors \( t_j \) then we know \( X \) if \( X_0 = 0 \).

So why not use \( t_j \)'s instead of \( x_i \)’s?

\[
t_j = \frac{1}{2} \cos \theta_j, \sin \theta_j \quad \theta_j \in [0, \pi]
\]

and \( \theta_i \)’s are unconstrained.
At equilibrium, $\theta_j$'s are uncorrelated and uniformly distributed in $[0, 2\pi)$.

$$d\mu_0(\vec{\theta}) = \prod_{i} \frac{d\theta_i}{2\pi}$$

This is simple and physically intuitive — we could have just started here (?)
Part 2

Worm-like chain

(based in part on notes by Eric Vandenbogaert)

Let's now consider a chain that resists bending (most do, e.g., DNA).

Elastic bending energy:

\[ U = U_{\text{bend}} = \frac{2K}{\ell} \sum_{i=1}^{n-1} s m^2 \left( \frac{\ell_i}{2} \right) \]

(semi-flexible fiber/worm-like chain)
\[ \cos \alpha_j = t_j \cdot t_{j-1} \]

\[ \alpha_j = \Theta_j - \Theta_{j-1} \]

So we can use \( \alpha_j \)'s as configuration variables

\((x_0, t_0, z) \leftrightarrow \bar{x} \)

Base measure is

\[ d\mu_0(z) = \bigwedge_{i=0}^{n} \frac{dx_i}{2\pi} \]
\[ d\text{Meq}(\mathbf{z}) = \prod_i \left( 1 + \exp \left( -\frac{2K}{le_k} \sin^2 \left( \frac{z_i}{2} \right) \right) \right) d\mathbf{z}. \]

The \( z_i \)'s are i.i.d. (product distribution).

If chain is "stiff" \( \sin^2 \frac{z_i}{2} = \left( \frac{z_i}{2} \right)^2 \) and \( z_i \)'s are Gaussian i.i.d.

What does it mean for a chain to be stiff? (Dimensionless parameter)
\[ \langle t_j \cdot t_{i+j} \rangle_{\text{Meq}} = \langle \cos \left( \sum_{k=1}^{i} \alpha_k \right) \rangle_{\text{Meq}} \]

can be done using \( \alpha \ll 1 \) approx.

\[ \langle t_j \cdot t_{i+j} \rangle \approx e^{-\frac{j^2}{2S^2}} = e^{-\frac{j^2}{2S^2}} \]

\[ S_j = j \ell = \text{arc length} \]

\[ S = \frac{2k}{k_B T} = \text{persistence length} \]

\( \approx 50 \text{ nm for A-DNA} \)
Stiff chain means 
\[ l << \xi \]
In this case we can take the continuum limit \( \xi \to 0 \)

\[ X \to \overline{X}(s) \in \mathbb{R}^3 \]
\( s \in [0, L] = \) arc length

Recall \( \lambda_j = \theta_j - \theta_{j-1} \)
\( \frac{dx_i}{c} \rightarrow \theta'(s_i) \)

\( t(s) = \overline{x}_s(s) = \{ \cos(\theta(s)), \sin(\theta(s)) \} \)

Is \( \theta(s) \) is (almost surely) continuous for \( vs \)?

For a stiff chain

\( \sin^2 \left( \frac{\alpha_0}{2} \right) \approx \left( \frac{\alpha_0}{2} \right)^2 \)
(18)

\[ U_{\text{end}} = \frac{k}{2 \ell} \sum_{j=1}^{n-1} \alpha_j^2 \]

\[ \xrightarrow{\ell \to 0} \quad \frac{k}{2} \int_0^L (\varphi'(s))^2 \, ds \]

\[ s=0 \quad \Rightarrow \quad = \frac{k}{2} \int_0^s \| x''(s) \|^2 \, ds \]

\[ U \left[ x(x) \right] = \frac{k}{2} \int_0^L \| x''(s) \|^2 \, ds \quad (2D/3D) \]

(\text{inverse curvature})
Part 3

Continuum Brownian chains

What is the continuum limit of $\text{Meq}(X)$ for a worm-like chain?

Physicists often (naively!) write

$$\text{Meq}[X(t)] \sim \exp\left(-\frac{u[X(t)]}{k_B T}\right) dX$$

"nonsense" notation (formal)
20) But Lebesgue measure cannot be generalized to infinite dimensions (theorem), so $DX$ is just formal notation with no clear meaning. Functional distributions don't usually make sense, except when they are Gaussian (we are in luck!)
Let's work with $\Theta(s)$ since $X(s)$ is constrained by $\|X_s\| = 1$ (mextensibility).

What does

$$P_{eq}[\Theta(.)] \sim \exp \left[ -\frac{L}{4} \int_0^\infty (\theta'(s))^2 ds \right]$$

persistence length mean precisely?
Derivation #1

discrete $\rightarrow$ continuum

(more physical for modeling)

Recall $x_j$'s are i.i.d.

Gaussian random variables

with mean $\mu$ and variance $\frac{3}{4\ell}$.

\[
d\mu_{eq}(x_j) = 2^{-1} \exp\left(-\frac{3}{4\ell} x_j^2\right) dx_j
\]

\[
d\mu_{eq}(z^2) = \prod_j d\mu_{eq}(x_j)
\]
\[ \Theta_{i+1} = \Theta_i + \sqrt{\frac{2 \varepsilon}{3}} \sqrt{N(0, 1)} \]

\[ \Theta(s + ds) = \Theta(s) + \sqrt{\frac{2d_s}{3}} \sqrt{N(0, 1)} \]

\[
\frac{1}{3} \Theta(s) = \sqrt{\frac{2}{3}} \int \frac{dB(s)}{\sqrt{s}} \quad (SDE) \]

Brownian motion

(periodic)

\[ \Theta(s) = \Theta_0 + B \left( \frac{2s}{3} \right) \]

is Brownian motion on \([0, 2\pi]) \]
This works in 3D also:

The tangent vector $\mathbf{t}(s)$ performs Brownian motion on the unit circle (2D) / sphere (3D) with diffusion coefficient $\xi^{-1}$.

This has a clear and precise mathematical meaning.
But observe that $s \to 0$
(freely-jointed chain) limit
does not make sense
(almost) nowhere differentiable
therefore we cannot write
\[ \delta \mu_p = e^{-U}/h \]

makes sense

(m continuum \( \ell \ll s \))
Derivation #2

(continuum from the start, math not modeling)

For simplicity, take a ring polymer, \( k_b T = 1 \), \( S = 2 \), so \( \alpha \) (see \( E_0, 2\pi \)) is periodic

Let \( Z \Theta = -\Theta \)

Hermitian SPD operator
\[ U [\theta (\cdot)] = \frac{1}{2} \int_0^{2\pi} (\theta'(s))^2 \, ds \]

\[ = -\frac{1}{2} \int_0^{2\pi} \theta(s) \theta''(s) \, ds \]

\[ = \frac{1}{2} \left( \Theta, \Delta \Theta \right)_{L^2 : [0, 2\pi]} \]

Go to Fourier space

\[ \Theta(s) \rightarrow \{ \hat{\Theta}_k \} \]
\[ U(\hat{\theta}) = \sum_{k=-\infty}^{\infty} \frac{k^2 \hat{\theta}_k^2}{2} \quad \text{(Parseval's theorem)} \]

\[ d\mu_{eq}(\hat{\theta}) = e^{-U(\hat{\theta})} d\hat{\theta} \]

\[ = \prod_k e^{-\frac{k^2 \hat{\theta}_k^2}{2}} d\hat{\theta}_k \]

which makes sense even though infinite-dimensional product distribution
\[ \hat{\Theta}_k \overset{d}{=} \mathcal{N}(0, k^{-2}) \]

which is the spectrum of periodic Brownian motion!

So we get \( \Theta(s) \equiv \text{Brownian motion (periodic)} \)

**Note:** \( \frac{d \hat{\Theta}}{ds} = \mathcal{W} \) (white noise)

\[ \hat{W}_k \overset{d}{=} \mathcal{N}(0, 1) \Rightarrow \hat{\beta}_k \overset{d}{=} \mathcal{N}(0, k^{-2}) \]

(complex)
The advantage of the "math" derivation is that it is obvious how to generalize to any quadratic energy:

$$U[\theta] = \frac{1}{2} \langle \theta, \theta \rangle$$

$$\Theta(s) = \sum c_j \Psi_j(s)$$

random coefficients

eigenfunctions of $\mathbf{K}$
3) \[ x_i \Psi_i = \lambda_i \Psi_i, \quad \lambda_i > 0 \]

\[ (\Psi_i, \Psi_j) = \delta_{ij} \]

\[ \Rightarrow (0, x^0) = \sum \lambda_i c_i^2 \Rightarrow \]

\[ \text{d}M_{eq} (\vec{c}) = \prod_j \exp \left( -\frac{\lambda_j c_j^2}{2k_B T} \right) \text{d}c_j \]

Gaussian functional distributions can be made sense of.
But we still don't know how to add dynamics (non-equilibrium stat mech).

Is there a continuum limit of Brownian dynamics, i.e. functional overdamped (multiplicative noise) Langevin eqs?