Commuting Contractive Operators

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Abstract

We proved that a finite commuting Boyd-Wong type contractive family with equicontinuous words have the approximate common fixed point property. We also proved that given $X \subset \mathbb{R}^n$, compact and convex subset, $F : X \rightrightarrows X$ a compact-and-convex valued Lipschitz correspondence and g an isometry on X, then $gF = Fg$ implies F admits a Lipschitz selection commuting with g.

1 Introduction

This is a summary of what we did in the Summer Undergraduate Research Experience in Courant Institute in 2017. We prove two new results in this report. The formulation of the theorems come from a long line of thoughts. We first present them in natural order.
The famous Banach fixed point theorem was proved in 1922. Since then numerous generalizations of it have been made. David Boyd and James Wong proved a particularly elegant one in 1969. The theorem states that Given a complete metric space (X, d) and an upper semicontinuous mapping ϕ on $[0, \infty)$ with $0 \leqslant \phi(t) < t$ for every $t > 0$, if f is a self-map on X satisfying

$$d(f(x), f(y)) \leqslant \phi(d(x, y)) \text{ for every } x, y \in X$$

then f has a unique fixed point $x \in X$.

It examplifies a trend in metric fixed point theory where people try to replace the $\lambda < 1$ Lipschitz condition in Banach's theorem by some weaker conditions and still have a unique fixed point result. In 2001, the following theorem by Stein and Merryfield provides a new perspective on the way of discovering new theorems [1]:

Theorem 1 (Generalized Banach Contraction Theorem (GBCT))**.** *For a complete metric space (X, d) and a self map T on X, if $\exists m \in \mathbb{N}, \gamma \in (0, 1)$ such that $\forall x, y \in X$, we have*

$$\min\{d(T^k(x), T^k(y)) : 1 \leqslant k \leqslant m\} \leqslant \gamma d(x, y),$$

then T has a unique fixed point.

The proof is combinatorial and relies on Ramsey theorem. It no longer requires the operators to be continuous and the conditions become a restriction on a family of operators. Further, in 2002 Stein conjectured in [2] that: Given (X, d) a complete metric space and $\mathcal{F} = \{T_1, ..., T_n\}$ a finite family of self maps on X. If $\exists \gamma \in (0,1)$ such that $\forall x, y \in X$, $\exists F \in \mathcal{F}$ with $d(F(x), F(y)) \leqslant \gamma d(x, y)$, then some composition of members of $\mathcal{F}$ has a fixed point.

This conjecture in general is disproved by a counterexample in the case $n = 2$ given by Austin in [3] in 2005, however he proved that if members $\mathcal{F}$ are continuous, commuting pairwisely and $|\mathcal{F}| = 2$, then they have a common fixed point. Austin also proved that if the members of $\mathcal{F}$ are uniformly continuous, commuting pairwisedly and X is bounded, then they have a common fixed point. In 2008, Reich and Zaslavski merged the Boyd-Wong theorem with the uniformly continuous version of GBCT and proved that for $\mathcal{F} = \{T, T^2, ..., T^N\}$ with T uniformly continuous on an orbit $T^i x_0$ for some $x_0 \in X$, given the ϕ in Boyd-Wong's theorem, if for any $x, y \in X$, we have $\min\{d(T^i x, T^i y) : i \in \{1, ..., N_0\}\} \leqslant \phi(d(x, y))$, then T has a unique fixed point [4]. Finally, following Austin's method, we aim to generalize Reich and Zaslavski's result to $\mathcal{F} = \{T_1, ..., T_N\}$ with $T_1, ..., T_N$ commuting and the words of them equicontinuous on the orbit of some $x_0 \in X$. The optimal fixed point result is still open, but we are able to show such family exists ϵ-approximate fixed point for any small $\epsilon > 0$ no matter X is complete or not. This is our first result and we prove it in the next section.

Our second result comes from the set-valued metric fixed point theory. In 1969, Nadler proved that a contractive compact-valued self-correspondence on a complete metric space always has a fixed point. A natural question in our context is whether we can generalize it in the same way as above, i.e. from a single correspondence to a family of them. This question remains open, but it may help us by considering the selection of the correspondence. Therefore we would like to consider for $\Gamma : X \rightrightarrows X$ compact-and-convex valued, $f : X \to X$ both Lispchitz and commuting, whether Γ admits a Lipschitz selection that commutes with f. It seems unlikely for the infinite dimensional case since in [5] it is proved that Γ may not have a Lipschitz selection there. So we restrict X to be a compact and convex subset of $\mathbb{R}^n$ since by the Steiner's point map we know in this case the Lipschitz selection of Γ always exists. Our second result shows that if f is an isometry, then Γ will have a selection g that commutes with f. Further, we know $lip(g) \leqslant n\, lip(\Gamma)$. We prove it in section 3. The proof is essentially an application of Schauder's fixed point theorem on the space of Lipschitz selections.

2 Approximate Common Fixed Point Theorem

Let (X, d) be any metric space, $\mathcal{F} := \{T_1, ..., T_n\}$ any finite family of pairwisely commuting continuous self-maps on X. Let ϕ be an upper semicontinuous mapping ϕ on $[0, \infty)$ with $0 \leqslant \phi(t) < t$ for every $t > 0$. For any $\epsilon > 0$, we say $x \in X$ is an ϵ-approximate fixed point if $d(x, T_i x) < \epsilon$ for $i = 1, .., n$. In this section, we prove the following theorem:

Theorem 2. *If $\forall x, y \in X$, we have $\min\{d(T_i x, T_i y) : i = 1, ..., n\} \leqslant \phi(d(x, y))$ and the set of finite compositions of $T_1, ..., T_n$ is equicontinuous. Then $\mathcal{F}$ has an ϵ-approximate common fixed point for every $\epsilon > 0$.*

Proof. Fix any $\epsilon > 0$, by equicontinuity, we can find a $\delta > 0$ such that $d(Ux, Uy) < \epsilon$ for all U a word generated by elements of $\mathcal{F}$ whenever $d(x, y) < \delta$. Now fix this δ too and let us do an induction on $k \leqslant n$.

The base case $k = 1$ is to find an $x \in X$ such that $d(x, T_1 x) < \epsilon$. To show this, we define a sequence of points inductively. Start with an arbitrary point $a_0 \in X$. Notice for simplicity in this proof we always assume the a_k's are not fixed point of any member of T_j, $j = 1, ..., n$. This is because it is easy to show by commutativity that a fixed point of one operator implies a common fixed point of the whole family. Now suppose we have already chosen up to a_m, we define a_{m+1} as $T_{i(m)} a_m$ where $i(m) := \min\{j \in \{1, ..., n\} : T_j \text{ contract } d(a_m, T_1 a_m)\}$. In this way we have

$$\begin{aligned} d(a_{m+1}, T_1 a_{m+1}) &= d(T_{i(m)} a_m, T_1 T_{i(m)} a_m) = d(T_{i(m)} a_m, T_{i(m)} T_1 a_m) \\ &\leqslant \phi(d(a_m, T_1 a_m)) < d(a_m, T_1 a_m). \end{aligned}$$

Therefore the sequence $(d(a_m, T_1))_{m=1}^{\infty}$ is decreasing and bounded below by 0, hence must converge to some number ξ. Suppose $\xi > 0$, since $... < d(a_{m+2}, T_1 a_{m+2}) < d(a_{m+1}, T_1 a_{m+1}) \leqslant \phi(d(a_m, T_1 a_m))$, taking the limit of left hand side, we have $\xi \leqslant \phi(d(a_m, T_1 a_m))$. Taking limit of right hand side and use the upper semi-continuity of ϕ, we have $\xi \leqslant \phi(\xi)$, contradiction. Therefore $d(a_m, T a_m) \to 0$. So in this sequence we can find an $x \in X$ such that $d(x, T_1 x) < \min\{\epsilon, \delta\}$.

Now build from this x a sequence inductively as follows. Let $y_0 = x$, suppose we have defined up to y_m, let $y_{m+1} = T_{j(m)} y_m$ where j is the smallest index of $\mathcal{F}$ such that T_j contracts y_m and $T_2 y_m$, that is, we just build from x a sequence like above but replace T_1 with T_2. Running the same argument again we know $d(T_2 y_m, y_m) \to 0$ as $m \to \infty$. Therefore we can pick a y in this sequence such that $d(y, T_2) < \min\{\delta, \epsilon\}$. Notice $y = U_2 x$ for some finite word U_2 constructed as above, hence we also have

$$d(T_1 y, y) = d(T_1 U_2 x, U_2 x) = d(U_2 T_1 x, U_2 x) < \epsilon,$$

since we made $d(T_1 x, x) < \delta$. Therefore y is an ϵ-approximate common fixed point of $\{T_1, T_2\}$.

Proceeding just like above for each T_k, $k = 3, ..., n$, start from a point α_{k-1} which satisfies $d(\alpha_{k-1}, T\alpha_{k-1}) < \min\{\epsilon, \delta\}$, construct a sequence by prefixing an operator that contract it and its image under T_k, in this sequence we can find our α_k with $d(\alpha_k, T\alpha_k) < \min\{\epsilon, \delta\}$. This α_k will be an ϵ-approximate common fixed point of $\{T_1, ..., T_k\}$. Continuing until $k = n$, we are done. $\square$

2.1 Discussion

As we mentioned in the introduction, it is proved in [4] that if X is complete, T is uniform continuous and $\mathcal{F}$ is of the form $\{T, T^2, ..., T^N\}$ for some N, then T has a unique fixed point. If we can strengthen our result above to a fixed point result by requiring the metric space to be complete, we can generalize their result.

Reason. Since T is uniformly continuous, so does $T^2, ..., T^N$. Therefore, for any $\epsilon > 0$, we can choose $\delta > 0$ such that $d(T^i x, T^i y) < \epsilon, i = 1, ..., N$ whenever $d(x, y) < \delta$. To show the equicontinuity of their $\mathcal{F}$, i.e the finite words generated by the elements in their $\mathcal{F}$, we just need to show under the same condition we have $d(T^j x, T^j y) < \epsilon$ for all other $j \in \mathbb{N}$. This is true since for $d(T^i x, T^i y) < \epsilon$, $i = 1, ..., N$, there is at least one of them that contracts x and y, that is, $\exists a \in \{1, ..., N\}$ such that $d(T^a x, T^a y) < \delta$. Therefore we have

$$d(Tx, Ty) < \epsilon, d(T^2 x, T^2 y) < \epsilon, ..., d(T^a x, T^a y) < \min\{\epsilon, \delta\},$$
$$d(T^{a+1} x, T^{a+1} y) < \epsilon, ..., d(T^{a+N} x, T^{a+N} y) < \epsilon.$$

Repeating this procedure, we can prove $d(T^j x, T^j y) < \epsilon$ for any $j \in \mathbb{N}$. ☐

However, it is still unclear for us how to prove that having all ϵ-approximate common fixed point implies a common fixed point in our case even with an extra condition of completeness. In [3] it is proved that if $\phi(x) = \gamma x$ for some $\gamma < 1$, then approximate common fixed points implies fixed points. That is because for those ϕ's, pick x_i to be the 2^{-i}-approximate fixed point, then the triangular inequality gives a nice $d(x_n, x_m) \leqslant d(x_n, Tx_n) + d(Tx_n, Tx_m) + d(Tx_m, x_m) \leqslant 2^{-n} + 2^{-m} + \gamma d(x_n, x_m)$, which implies (x_i) is Cauchy hence converges to some point. However, when γ is replaced by our more general ϕ function, it may not be invertible and $\phi(x)$ can be much closer to x than γx is. Ideally if we know ϕ is invertible and increasing, then we still will have a fixed point, but this restriction is quite strong. To really push this line of thought about these forced conditions we can have the following result. Let $\mathcal{F} = \{S, T\}$ family of commuting continuous operators on complete X.

Theorem 3. *Suppose*

1. *$\forall x, y \in X$, $\exists F \in \mathcal{F}$ with $d(F(x), F(y)) \leqslant \phi(d(x, y))$ and $\exists$ a small $q > 0$ such that $(id - \phi)\chi_{\{t<q\}}$ is increasing.*
2. *$\exists x \in X$ such that both S and T are uniformly continuous on the set $\{S^i T^j x : i, j \in \mathbb{N}\}$, and*
3. *$\exists M \in \mathbb{N}$ such that $\forall x \in X$, $\exists p(x) \in \mathbb{N}, 0 < p(x) < M$ such that $d(ST^{p-1}x, T^{p+1}x) \leqslant \psi(d(Tx, x))$ for some $\psi \in \mathbb{R}_+^{\mathbb{R}_+}$ strictly increasing,*

then S and T have a unique common fixed point.

Proof. As we noted before with condition 1, the approximate common fixed points can imply a common fixed point– just notice $d(x_n, x_m) \leqslant (id-\phi)^{-1}(2^{-n} + 2^{-m}) \to 0$ as $m, n \to \infty$. Therefore we just need to find the approximate common fixed points.

Starting from the $x \in X$ in the assumption, we construct the sequence as before. In that way we have $d(x_n, Tx_n) \leqslant \phi(d(x_{n-1}, Tx_{n-1}))$ a decreasing sequence with a lower bound. If $d(x_n, Tx_n) \to \epsilon' > 0$, then $\forall N \in \mathbb{N}, \exists n(N) > N$ such that $d(x_{n(N)}, Tx_{n(N)}) > \epsilon'$. Therefore $\epsilon' < d(x_{n(N)}, Tx_{n(N)}) \leqslant \phi(d(x_{n(N)-1}, Tx_{n(N)-1}))$, letting $n \to \infty$, we have $\phi(\epsilon') > \epsilon'$ a contradiction. So $d(x_n, Tx_n) \to 0$. Now fix a small $\eta > 0$, let the sequence start from $d(x_1, Tx_1) < \eta$.

To show $d(Sy_i, y_i) \to 0$ too, notice $d(y_i, Sy_i) = d(STx_{n_i-1}, Tx_{n_i-1}) \leqslant d(STx_{n_i-1}, Sx_{n_i-1}) + d(Sx_{n_i-1}, Tx_{n_i-1})$. Given any $\epsilon > 0$ by uniform continuity, there exists an $\eta' > 0$, such that if $d(Tx_{n_i-1}, x_{n_i-1}) < \eta'$, then $d(STx_{n_i-1}, Sx_{n_i-1}) < \epsilon$, by construction, we can always find a large n_i such that $d(Tx_{n_i-1}, x_{n_i-1}) < \eta'$, so the $d(STx_{n_i-1}, Sx_{n_i-1})$ part is done.

We just need to show $d(Sx_{n_i-1}, Tx_{n_i-1})$ will also go to zero. Now we have two cases, the first case is that we can always find infinitely many consecutive y_i and y_{i+1} such that $|n_i - n_{i+1}| > M$; the second case is that we cannot do so.

In the first case, pass to the subsequence of such y_i's, we have $x_{n_i-1} = T^{p(x_{n_i-1})-1}x_{n_i-p(x_{n_i-1})}$ since $p < M$ by assumption, so

$$\begin{aligned}
&d(Sx_{n_i-1}, Tx_{n_i-1})\\
=&d(ST^{p(x_{n_i-1})-1}x_{n_i-p(x_{n_i-1})}, TT^{p(x_{n_i-1})-1}x_{n_i-p(x_{n_i-1})})\\
\leqslant&d(ST^{p(x_{n_i-1})-1}x_{n_i-p(x_{n_i-1})}, T^{p(x_{n_i-1})+1}x_{n_i-p(x_{n_i-1})}) + d(T^{p(x_{n_i-1})+1}x_{n_i-p(x_{n_i-1})}, T^{p(x_{n_i-1})}x_{n_i-p(x_{n_i-1})})\\
\leqslant&\psi(d(Tx_{n_i-1}, x_{n_i-1})) + d(Tx_{n_i}, x_{n_i})
\end{aligned}$$

Therefore, for $\forall\epsilon > 0$, we can always find an n_i large enough such that $d(Tx_{n_i-1}, x_{n_i-1}) < \psi^{-1}(\frac{\epsilon}{2})$ and $d(Tx_{n_i}, x_{n_i}) < \frac{\epsilon}{2}$ and we are done.

The second case is similar to Austin's proof in [3]: in this case, we can truncate the sequence x_n from a very large n such that after the truncation every y_i and y_{i+1} are separated at most $M\eta$ for all i. Then we are still interested in the y_i's. Notice $d(y_i, Sy_i) \leqslant d(y_i, y_{i+1}) + d(y_{i+1}, Sy_i) < M\eta + d(y_{i+1}, y_i)$. For any pair y_i and y_{i+1}, let $L_i := d(y_i, y_{i+1})$, then further we have two cases: either T contracts L_i for some i or not.

If T contracts L_i for some i, we have

$$\begin{aligned}
L_i = d(y_i, y_{i+1}) &\leqslant d(y_i, Ty_i) + d(Ty_i, Ty_{i+1}) + d(Ty_{i+1}, y_{i+1})\\
&< \eta + \phi(L_i) + \eta
\end{aligned}$$

Therefore, $(id - \phi)(L_k) < 2\eta$. When ϵ is small enough $\epsilon < q$, we can let $2\eta < (id - \phi)(\epsilon)$ and have $L_k < \epsilon$ and we are done for this part.

If T does not contract L_i for any i, then S does, so fix any i

$$\begin{aligned}
L_{i+1} \leqslant d(y_{i+1}, y_{i+2}) &\leqslant d(S(y_i), y_{i+1}) + d(S(y_i), S(y_{i+1})) + d(S(y_{i+1}), y_{i+2})\\
&\leqslant M\eta + \phi(L_i) + M\eta
\end{aligned}$$

Therefore if $(id-\phi)(L_i) > 3M\eta$, then $L_{i+1} + M\eta \leqslant 3M\eta + \phi(L_i) = L_i + 3M\eta + \phi(L_i) - L_i < L_i$. If still $(id - \phi)(L_{i+1}) > 3M\eta$, we will have $L_{i+2} + M\eta < L_{i+1}$. This decreasing sequence will terminate some time since the difference $M\eta$ is fixed, which means that $(id - \phi)(L_i) \leqslant 3M\eta$ someday. When ϵ is small enough, we can let $3M\eta \leqslant (id - \phi)(\epsilon)$ and this gives us $L_i \leqslant \epsilon$ and we are finally done. $\square$

Although in this theorem we can get rid of the equicontinuity condition of an infinite family and have fixed point result, the conditions imposed are really forced. We just remark that there are also other ways to produce similar results, namely changing the ψ in third condition to act differently on the orbit.

3 Set-Valued Result

Recall that a correspondence F is a multivalued function and a function f is called a selection of F if $f \in F$ pointwisely. In a metric space, we define the continuity and Lipschitzness of a correspondence using the Hausdorff metric, that is, $d_H(A, B) := \max\{\sup_{x\in X} \inf_{y\in Y} d(x, y), \sup_{y\in Y} \inf_{x\in X} d(x, y)\}$. We ask the question whether a pair of commuting Lipschitz correspondence and Lipschitz function would lead to a pair of commuting Lipschitz selection and Lipschitz function. The partial answer that we have now is given in the theorem below.

Theorem 4. *Given $X \subset \mathbb{R}^n$ a compact and convex subset, if $F : X \rightrightarrows X$ a Lipschitz compact-and-convex-valued correspondence commutes with $g \in C(X, X)$ an isometry, then F admits a Lipschitz selection f such that $fg = gf$. Further, we have $lip(f) \leqslant nlip(F)$.*

Proof. The proof relies on an application of Schauder's fixed point theorem: every continuous self-map on a nonempty compact and convex subset of a normed linear space has a fixed point [6] .

First we refer to the classical theorem about Lipschitz selection [7]: A Lipschitz set-valued map F from a metric space to nonempty closed convex subsets of $\mathbb{R}^n$ has a Lipschitz selection f. The selection is constructed using the Steiner point $s_n(K)$ of a compact set $K \subset \mathbb{R}^n$

$$s_n(K) = \frac{1}{Vol(B^n)} \int_{B^n} m(\partial\sigma(K, p))dp$$

where $\partial\sigma(K, p) := \{x \in K | < p, x >= \sigma(K, p)\}$ and σ is the support function. It can be shown that $s_n(K) \in K$ and $||s_n(K) - s_n(L)|| \leqslant nd_H(K, L)$.

This theorem tells us that the set $\mathcal{F} := \{f$ is a Lipschitz selection of $F, lip(f) \leqslant n\, lip(F)\}$ is a nonempty subset of $C(X, X) \subset C(X, \mathbb{R}^n)$. Equip $\mathcal{F}$ with the sup norm from $C(X, \mathbb{R}^n)$, $\mathcal{F}$ is convex since F is convex, so the convex linear combinations of selections are still selections and since $lip : C(X, X) \to \mathbb{R}_+$ is homogeneous, we still have $lig(f) \leqslant n\, lip(F)$. We are going to show that $\mathcal{F}$ is compact by first showing it is complete and then use Arzelà-Ascoli. Then we are going to define an operator on $\mathcal{F}$ with a fixed point, which is the selection that commutes with g.

$\mathcal{F}$ is complete: Consider $(f_i) \in \mathcal{F}^\infty$ Cauchy, for each $x \in X$, $(f_i(x))$ is a Cauchy sequence, since X is complete, we can define $f(x) := f_i(x)$. Now we need to show (1) $f(x) \in F(x)$, (2)the convergence $f_i \to f$ is uniform and (3) f

is Lipschitz with $lip(f) \leqslant n\, lip(F)$.

(1) is true since F is closed valued, $f_i(x) \in F(x)$, $f_i(x) \to f(x)$ implies that $f(x) \in F(x)$. (2) is true since $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| \leqslant ||f_n - f_m|| < \epsilon, \forall m, n > N, \forall x \in X$. Sending $m \to \infty$, we have $\forall x \in X, |f_n(x) - f(x)| < \epsilon$. Also for $x_n \to x$ we have $|f(x_n) - f(x)| \leqslant |f(x_n) - f_i(x_n)| + |f_i(x_n) - f_i(x)| + |f_i(x) - f(x)| \to 0 \Rightarrow f$ is continuous $\Rightarrow ||f||$ makes sense. Therefore $||f_n - f|| \to 0$.

(3) is true since $\forall x, y \in X, |f_i(x) - f_i(y)| \leqslant n\, lip(F)|x - y|$ for all $i \in \mathbb{N}$. since $|f_i(x) - f_i(y) - f(x) + f(y)| \to 0$ as $i \to \infty$, we have $|f(x) - f(y)| \leqslant n\, lip(F)|x - y|$.

Now to show $\mathcal{F}$ is compact, we only need to show it is totally bounded. By Arzelà-Ascoli, we need to show $\mathcal{F}$ is pointwise bounded and equicontinuous: $\forall x \in X, \{f(x) : f \in \mathcal{F}\} \subset F(x)$ is bounded, so $\mathcal{F}$ is pointwise bounded. It is equicontinuous since given any ϵ, the δ can be taken as $\epsilon/(n\, lip(F))$.

Now we know $\mathcal{F}$ is compact, we define an operator $c_g : \mathcal{F} \to \mathcal{F}$ by $c_g(f) := g^{-1} \circ f \circ g$. To show c_g is well-defined, we need to show $c_g(f) \in \mathcal{F}$, that is (1) $c_g(f)(x) \in F(x)$ for all $x \in X$ and (2) $c_g(f)$ is still Lipschitz with $lip(c_g(f)) \leqslant n\, lip(F)$.

(1) is true since $c_g(f)(x) \in g^{-1} \circ F \circ g(x) = g^{-1} \circ g \circ F(x) = F(x), \forall x \in X$ since $gF = Fg$.

(2) is true since

$$\begin{aligned}
|c_g(f)(x) - c_g(f)(y)| &\leqslant lip(g^{-1})|fg(x) - fg(y)| \\
&\leqslant lip(g^{-1}) lip(f)|g(y) - g(y)| \\
&\leqslant lip(g^{-1}) lip(f) lip(g)|x - y| \\
&= lip(f)|x - y|,
\end{aligned}$$

therefore $lip(c_g(f)) = lip(f)$.

Finally to use Schauder's fixed point theorem, we only need to show c_g is continuous. So consider a sequence f_i in $\mathcal{F}$ that converges to f in the sup norm. $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $\forall n > N, \forall x \in X, |f_i g(x) - fg(x)| < \epsilon$. Since X is compact, we know g^{-1} is uniformly continuous, i.e, $\forall \epsilon > 0, \exists \delta(\epsilon)$ such that $\forall |x_1 - x_2| < \delta(\epsilon), |g^{-1}(x_1) - g^{-1}(x_2)| < \epsilon$. Now fix any $\epsilon > 0$, just take $N(\delta(\epsilon))$ and we are done.

c_g has a fixed point tells us that there is a selection f of F with $lip(f) \leqslant n\, lip(F)$ such that $c_g(f) = f$, which means that $fg = gf$. $\square$

Remark

An easy generalization is that we can relax the assumption for g to let it be bi-Lipschitz and $lip(g) lip(g^{-1}) \leqslant 1$ since we only used these conditions in the proof.

3.1 Discussion

As we can tell, the result above is still quite naïve and far from ideal. We can see there are at least two steps from here to an ideal result. The first one is about g being isometry, if we can relax this requirement such that g is just Lipschitz, we can arrive at some interesting result in its own right, that is, if a Lipschitz function commutes with a Lipschitz correspondence, then the Lipschitz correspondence admits a Lipschitz selection that commutes with that function. We can not do this relaxation in our proof above since if g is not invertible then the pivotal c_g will be ill-defined. Even if we only require g to be bi-Lipschitz, $c_g(f)$ may still be outside $\mathcal{F}$ since $lip(g)lip(g^{-1})$ may be large. The second step is to relax the requirement about Lipschitzness of both operators to mimic the situation in the first part, that is, given any pair of points, either the correspondence F or the function g will contract them. If the F still admits a selection f satisfying this property and commuting with g, then from [3] we know f and g have a common fixed point, hence F and g has a common fixed point.

Acknowledgements

The above work was done in 2017 during the Summer Undergraduate Research Experience in Courant Institute, NYU. My thanks go to Dr. Efe. Ok for his supervision of this project and many helpful insights, and to Courant Institute for their support.

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