The Joint S&P 500/VIX Smile Calibration Puzzle Solved

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Motivation

- Volatility indices, such as the VIX index, are not only used as market-implied indicators of volatility.
- Futures and options on these indices are also widely used as risk-management tools to hedge the volatility exposure of options portfolios.
- The **very high liquidity of S&P 500 (SPX) and VIX derivatives** requires that financial institutions price, hedge, and risk-manage their SPX and VIX options portfolios using **models that perfectly fit market prices of both SPX and VIX futures and options, jointly**.
- Calibration of stochastic volatility models to liquid hedging instruments: SPX options + VIX futures and options.
- Since VIX options started trading in 2006, many researchers and practitioners have tried to build such a jointly calibrating model, but could only, at best, get approximate fits.
- **"Holy Grail of volatility modeling"**
- **Very challenging problem, especially for short maturities**.

Motivation

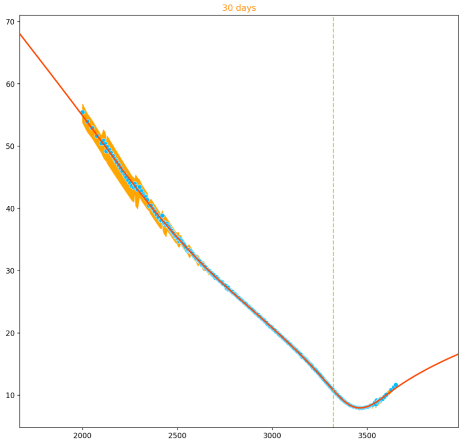


Figure: SPX smile as of January 22, 2020, $T = 30$ days

Motivation

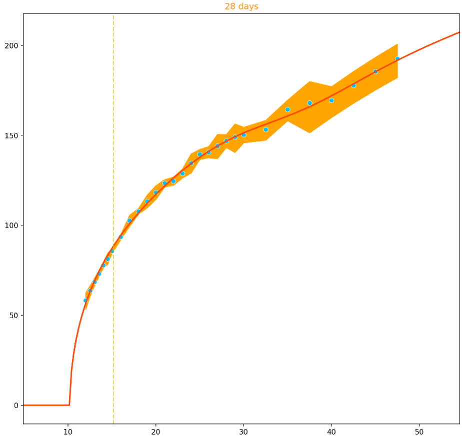


Figure: VIX smile as of January 22, 2020, $T = 28$ days

Motivation

- ATM skew:

$$\begin{aligned} \text{Definition:} \quad & \mathcal{S}_T && = && \left.\frac{d\sigma_{\mathrm{BS}}(K,T)}{\frac{dK}{K}}\right|_{K=F_T} \\ \text{SPX, small } T\text{:} \quad & \mathcal{S}_T && \approx && -1.5 \\ \text{Classical one-factor SV model:} \quad & \mathcal{S}_T && \underset{T\to 0}{\longrightarrow} && \frac{1}{2} \times \text{spot-vol correl} \times \text{vol-of-vol} \end{aligned}$$

- Calibration to short-term ATM SPX skew $\Longrightarrow$

$$\text{vol-of-vol} \geq 3 = 300\% \gg \text{short-term ATM VIX implied vol}$$

The **very large negative skew of short-term SPX options**, which in classical continuous SV models implies a very large volatility of volatility, **seems inconsistent with the comparatively low levels of VIX implied volatilities**.

Gatheral (2008)

Consistent Modeling of SPX and VIX options

Consistent Modeling of SPX and VIX options

Jim Gatheral

Merrill Lynch

The Fifth World Congress of the Bachelier Finance Society
London, July 18, 2008

Double CEV dynamics

- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned} \frac{dS}{S} &= \sqrt{v}\, dW \\ dv &= -\kappa\,(v - v')\, dt + \eta_1\, v^{\alpha}\, dZ_1 \\ dv' &= -c\,(v' - z_3)\, dt + \eta_2\, v'^{\beta}\, dZ_2 \end{aligned} \qquad (2)$$

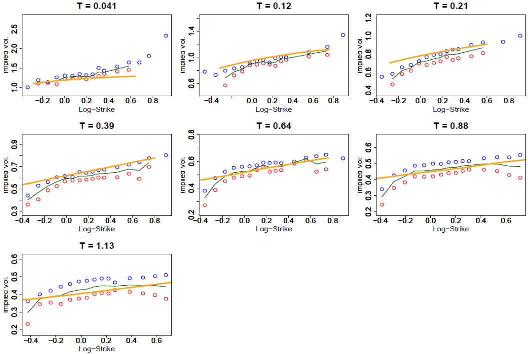
for any choice of α, $\beta \in [1/2, 1]$.

- We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
- the case $\alpha = \beta = 1$ *Double Lognormal*,
- and the general case *Double CEV*.

- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

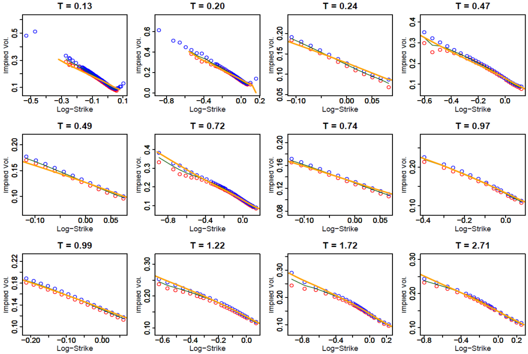
Double CEV fit to VIX options as of 03-Apr-2007

Setting the correlation ρ between volatility factors z_1 and z_2 to its historical average (see later) and iterating on the volatility of volatility parameters ξ_1 and ξ_2 to minimize the differences between model and market VIX option prices, we obtain the following fits (orange lines):

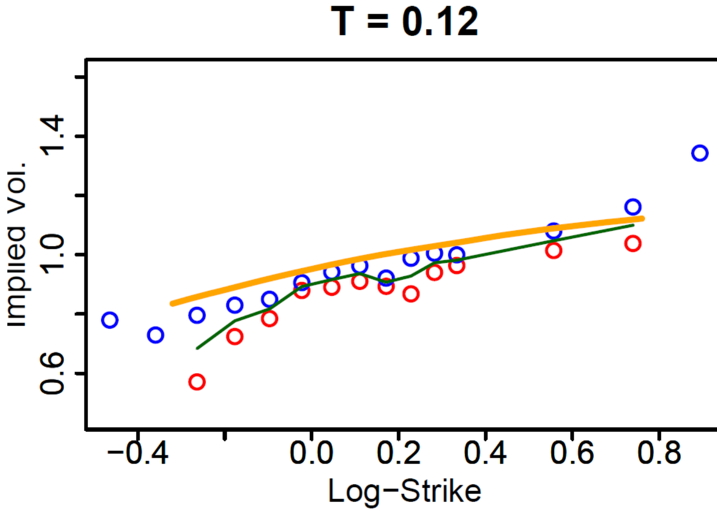


Double CEV fit to SPX options as of 03-Apr-2007

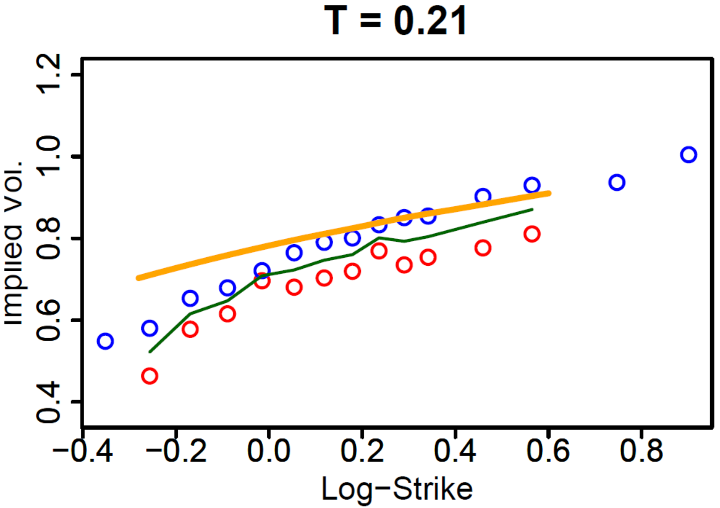
Minimizing the differences between model and market SPX option prices, we find $\rho_1 = -0.9$, $\rho_2 = -0.7$ and obtain the following fits to SPX option prices (orange lines):



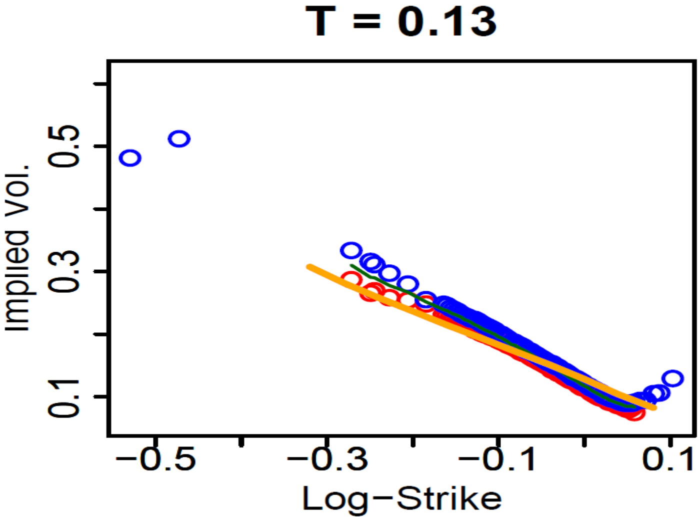
Fit to VIX options



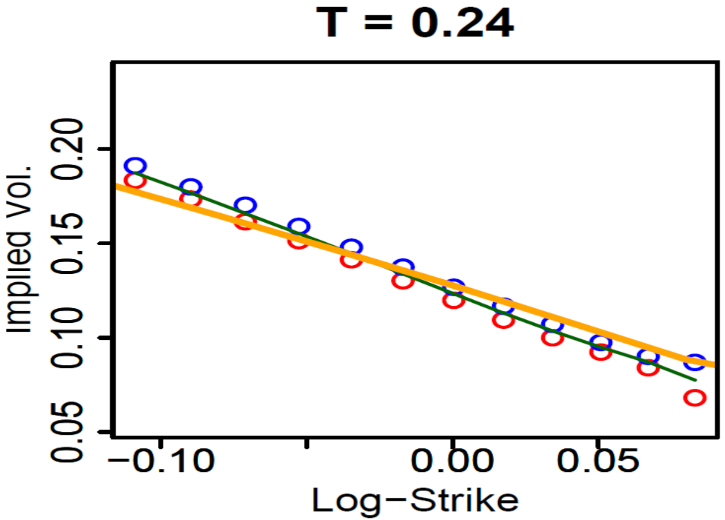
Fit to VIX options



Fit to SPX options



Fit to SPX options



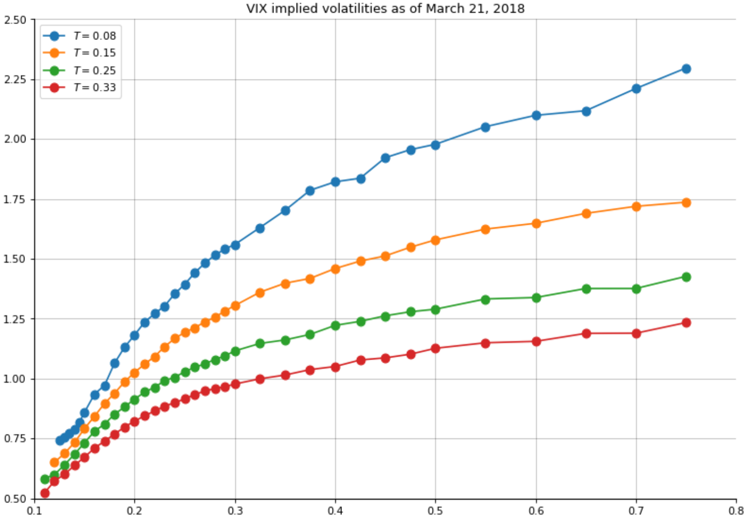
Similar experiments with other models

- Skewed 2-factor Bergomi model (Bergomi 2008)
- Skewed rough Bergomi model (G. 2018, De Marco 2018):

$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\nu_0 \int_0^t (t-s)^{H-\frac{1}{2}} dZ_s\right) + \lambda\mathcal{E}\left(\nu_1 \int_0^t (t-s)^{H-1/2} dZ_s\right)\right)$$

  with $\lambda \in [0,1]$.
- Quadratic rough Heston model (Gatheral Jusselin Rosenbaum 2020)
- **VIX smile well calibrated $\Longrightarrow$ not enough SPX ATM skew**

Skewed rough Bergomi: Calibration to VIX futures and options (G. 2018)



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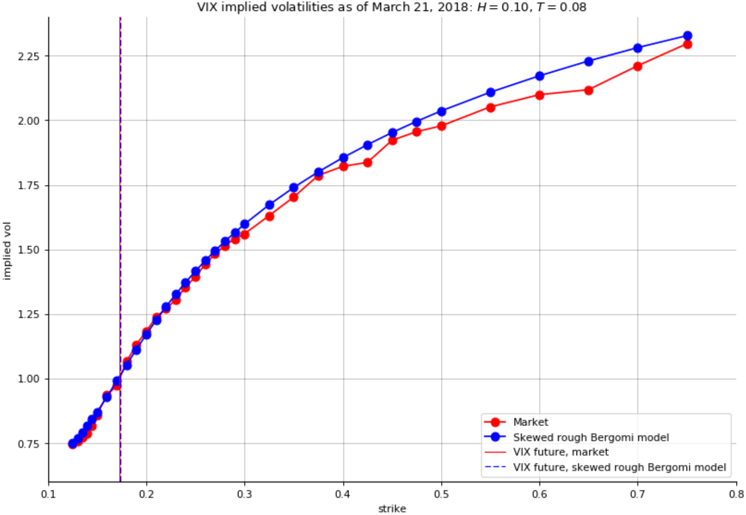

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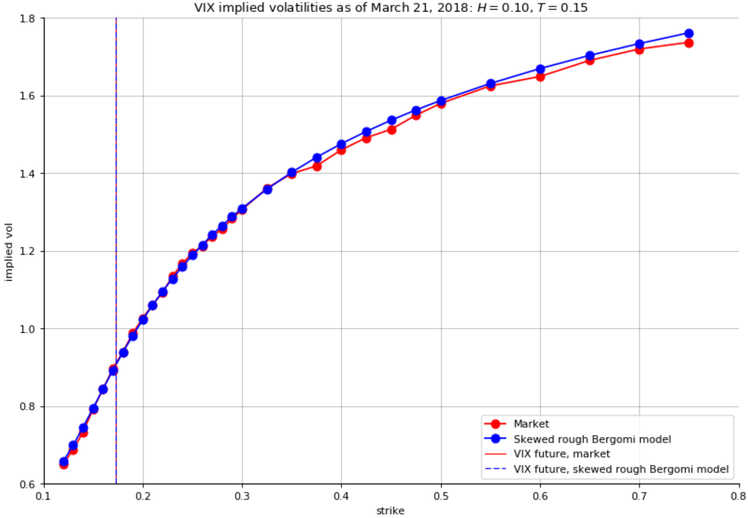

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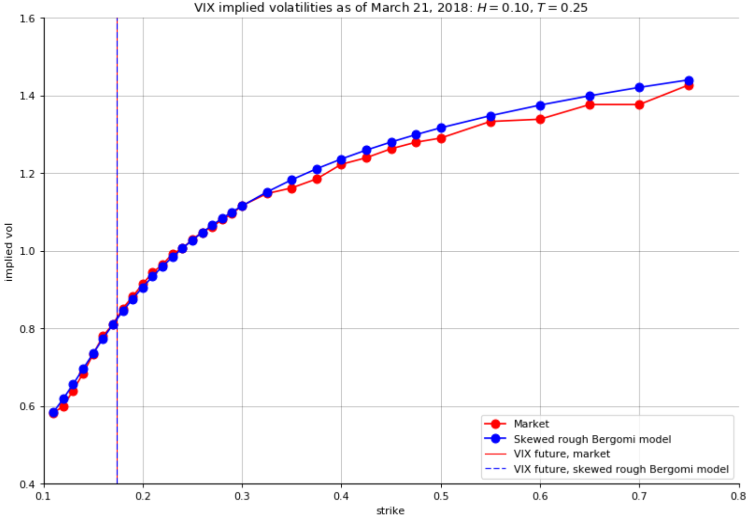

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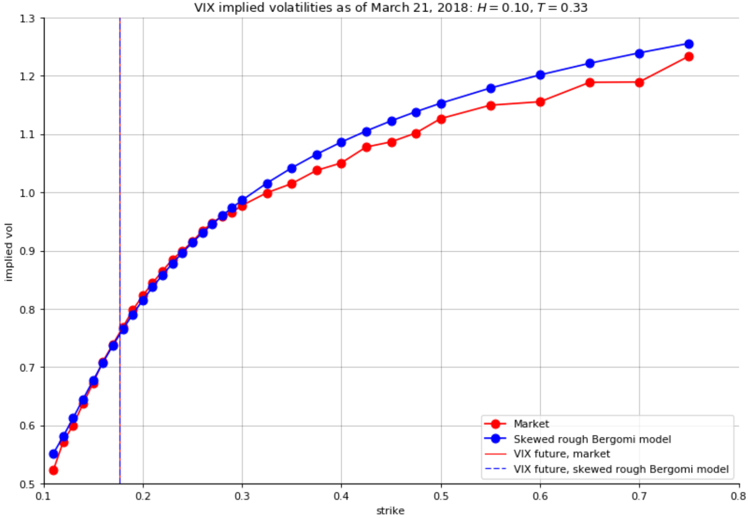
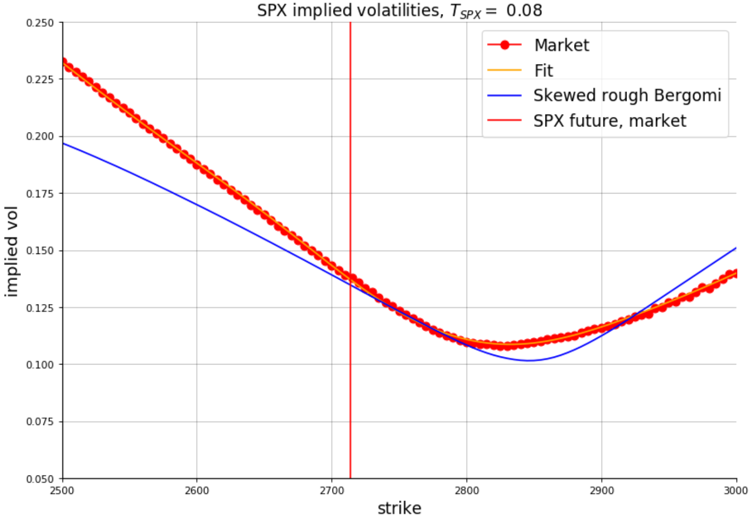
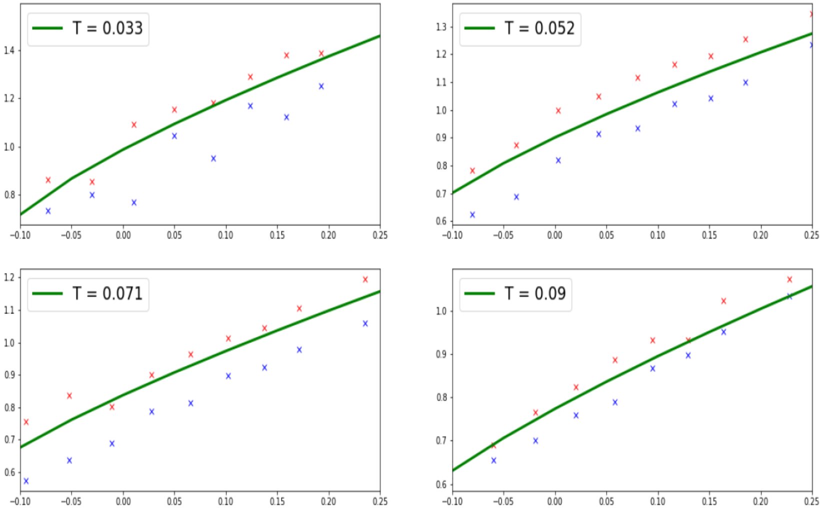


Skewed rough Bergomi calibrated to VIX: SPX smile (G. 2018)



Quadratic rough Heston model (Gatheral Jusselin Rosenbaum 2020)



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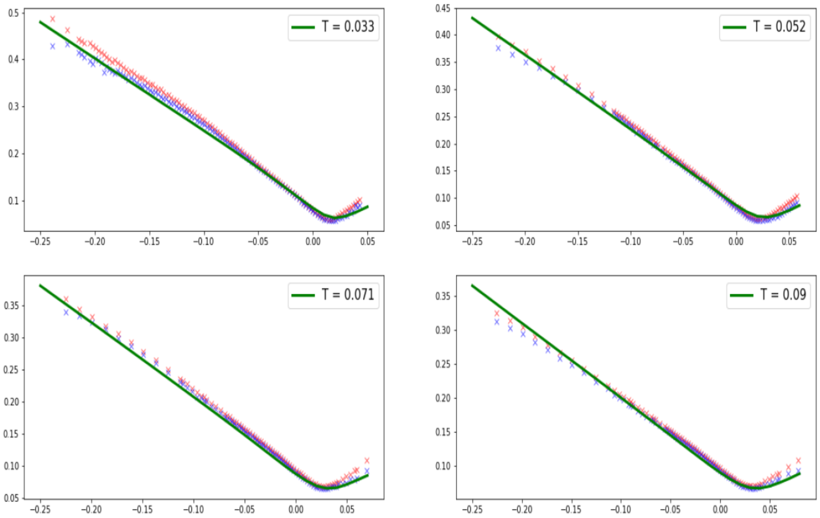

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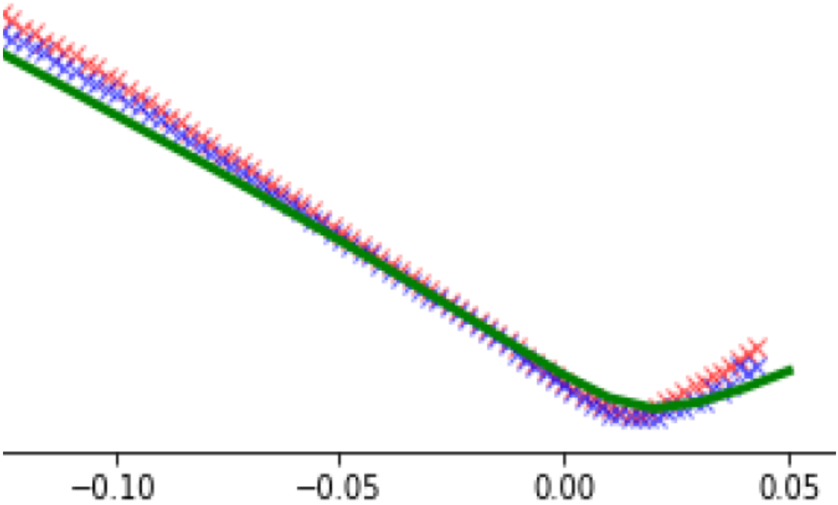

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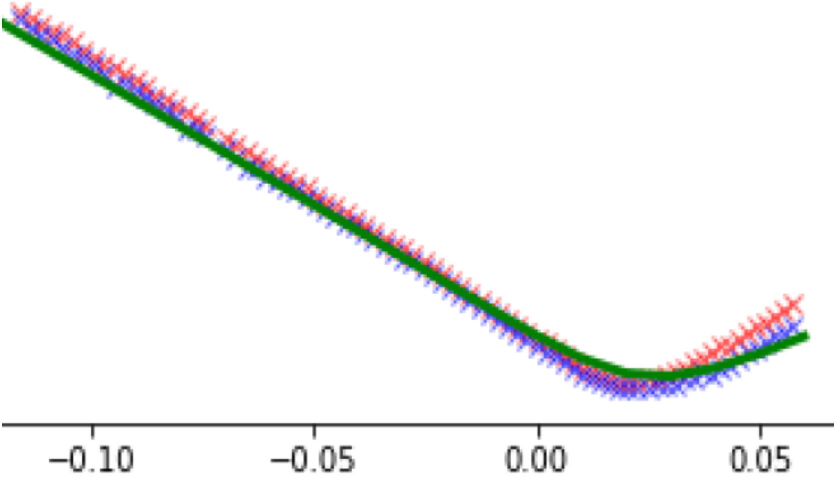

Joint calibration of 2-factor Bergomi model to term-structure of SPX ATM skew and VIX^2 implied vol (G. 2020)

THE VIX FUTURE IN BERGOMI MODELS: ANALYTIC EXPANSIONS AND JOINT CALIBRATION WITH S&P 500 SKEW

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ABSTRACT. We derive the expansion of the price of a VIX future in various Bergomi models at order 6 in small volatility-of-volatility. We introduce the notion of volatility of the VIX squared implied by the VIX future, which we call "VIX^2 implied volatility", expand this quantity at order 5, and show that the implied volatility expansion converges much faster than the price expansion. We cover the one-factor, two-factor, and skewed two-factor Bergomi models and allow for maturity-dependent and/or time-dependent parameters. The expansions allow us to precisely pinpoint the roles of all the model parameters (volatility-of-volatility, mean reversions, correlations, mixing fraction) in the formation of the prices of VIX futures in Bergomi models. The derivation of the expansion naturally involves the (classical or dual bivariate) Hermite polynomials and exploits their orthogonality properties. When the initial term-structure of variance swaps is flat, the expansion is a closed-form expression; otherwise, it involves one-dimensional integrals which are extremely fast to compute. The VIX^2 implied volatility expansion is extremely precise for both the one-factor model and the two-factor model with independent factors, even for the very large values of volatility-of-volatility that are usual in equity derivatives markets, and can virtually be considered an exact formula in those cases. We use the new expansion together with the Bergomi-Guyon expansion of the S&P 500 smile to (instantaneously) calibrate the two-factor Bergomi model jointly to the term-structures of S&P 500 at-the-money skew and VIX^2 implied volatility. Our tests and the new expansion shed more light on the inability of traditional stochastic volatility models to jointly fit S&P 500 and VIX market data. The (imperfect but decent) joint fit requires much larger values of volatility-of-volatility and fast mean reversion than the ones previously reported in [10, 14].

Keywords. VIX, VIX futures, Bergomi models, VIX^2 implied volatility, analytic expansion, small volatlity-of-volatility, at-the-money skew, S&P 500/VIX joint calibration, Hermite polynomials.

1. INTRODUCTION

Closed-form approximations are always very useful in mathematical modeling. They give insights on the structural properties of the models and the precise role of model parameters. They are computed in no time

Joint calibration of 2-factor Bergomi model to term-structure of SPX ATM skew and VIX^2 implied vol (G. 2020)

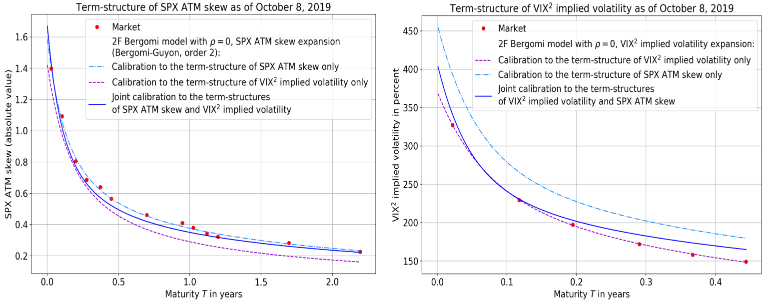


Figure: Left: ATM skew of SPX options as a function of maturity. Right: implied volatility of the squared VIX as a function of maturity. Calibration of the Bergomi-Guyon expansion of the SPX ATM skew and a newly derived expansion of the VIX^2 implied volatility, either jointly or separately. Calibration as of October 8, 2019

Related works with continuous models on the SPX

- Fouque-Saporito (2018), Heston with stochastic vol-of-vol. Problem: their approach does not apply to short maturities (below 4 months), for which VIX derivatives are most liquid and the joint calibration is most difficult.
- Goutte-Ismail-Pham (2017), Heston with parameters driven by a Hidden Markov jump process.
- Jacquier-Martini-Muguruza, *On the VIX futures in the rough Bergomi model* (2017):

  "Interestingly, we observe a 20% difference between the [vol-of-vol] parameter obtained through VIX calibration and the one obtained through SPX. This suggests that the volatility of volatility in the SPX market is 20% higher when compared to VIX, revealing potential data inconsistencies (arbitrage?)."

- Guo-Loeper-Obłoj-Wang (2020): joint calibration via semimartingale optimal transport. **Closely related to VIX-contrained martingale Schrödinger bridges**.

Motivation

- To try to jointly fit the SPX and VIX smiles, many authors have incorporated **jumps** in the dynamics of the SPX: Sepp, Cont-Kokholm, Papanicolaou-Sircar, Baldeaux-Badran, Pacati et al, Kokholm-Stisen, Bardgett et al...
- Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM VIX implied volatility.
- So far all the attempts at solving the joint SPX/VIX smile calibration problem only produced an **approximate fit**.

Exact joint calibration via dispersion-constrained martingale optimal transport

(G. 2019)

Exact joint calibration via dispersion-constrained MOT (G. 2019)

- **A completely different approach**: instead of postulating a parametric continuous-time (jump-)diffusion model on the SPX, we build a **nonparametric discrete-time model**:
  - Help to decouple SPX skew and VIX implied vol.
  - Perfectly fits the smiles.
- Given a VIX future maturity T_1, we build a **joint probability measure on** (S_1, V, S_2) which is **perfectly calibrated** to the SPX smiles at T_1 and $T_2 = T_1 + 30$ days, and the VIX future and VIX smile at T_1.
- S_1: SPX at T_1, V: VIX at T_1, S_2: SPX at T_2.
- Our model satisfies:
  - **Martingality constraint** on the SPX;
  - **Consistency condition**: the VIX at T_1 is the implied volatility of the 30-day log-contract on the SPX.
- Our model is cast as the solution of a **dispersion-constrained martingale transport problem** which is solved using the **Sinkhorn algorithm**, in the spirit of De March and Henry-Labordère (2019).

Risk, April 2020

The joint S&P 500/Vix smile calibration puzzle solved

Since Vix options started trading in 2006, many researchers have tried to build a model that jointly and exactly calibrates to the prices of Standard & Poor's 500 options, Vix futures and Vix options. In this article, Julien Guyon solves this long-standing puzzle by casting it as a discrete-time dispersion-constrained martingale transport problem, which he solves in a non-parametric way using Sinkhorn's algorithm

Volatility indexes, such as the Vix index, do not just serve as market-implied indicators of volatility. Futures and options on these indexes are also widely used as risk management tools to hedge the volatility exposure of options portfolios. The existence of a liquid market for these futures and options has led to the need for models that jointly calibrate to the prices of options on the underlying asset and the prices of volatility derivatives. Without such models, financial institutions could possibly arbitrage each other: even market-making desks within the same institution could do so, eg, the Vix desk could arbitrage the S&P 500 (SPX) desk. By using models that fail to correctly incorporate the prices of the hedging instruments, such as SPX options, Vix futures and Vix options, exotic desks may misprice options, especially (but not only) those with payoffs that involve both the underlying and its volatility index.

For this reason, since Vix options began trading in 2006, many researchers and practitioners have tried to build a model that jointly and exactly calibrates to the prices of SPX futures, SPX options, Vix futures and Vix options. This is known to be a very challenging problem, especially for short maturities. In particular, the very large negative skew of short-term SPX options,

and Vix smiles: that the distribution of the Dupire market local variance be smaller than the distribution of the (instantaneous) Vix squared in the convex order, at all times. He also reported that for short maturities the distribution of the true Vix squared in the market local volatility model is actually larger than the market-implied distribution of the true Vix squared in the convex order. Guyon showed numerically that when the (typically negative) spot-vol correlation is large enough in absolute value, both (a) traditional stochastic volatility models with large mean reversion and (b) rough volatility models with a small Hurst exponent can reproduce this inversion of convex ordering. Acciaio & Guyon (2020) provide a mathematical proof that the inversion of convex ordering can be produced by continuous models. However, the inversion of convex ordering is only a necessary condition for the joint SPX/Vix calibration of continuous models; it is not sufficient.

Since it looks to be very difficult to jointly calibrate the SPX and Vix smiles with continuous models, many authors have incorporated jumps in the dynamics of the SPX: see references in Guyon (2019a). Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM Vix implied volatility. However, the attempts so far have only produced

Setting and notation

- For simplicity: zero interest rates, repos, and dividends.
- μ_1 = risk-neutral distribution of $S_1 \longleftrightarrow$ market smile of SPX at T_1.
- μ_V = risk-neutral distribution of $V \longleftrightarrow$ market smile of VIX at T_1.
- μ_2 = risk-neutral distribution of $S_2 \longleftrightarrow$ market smile of SPX at T_2.
- F_V: value at time 0 of VIX future maturing at T_1.
- We denote $\mathbb{E}^i := \mathbb{E}^{\mu_i}$, $\mathbb{E}^V := \mathbb{E}^{\mu_V}$ and assume

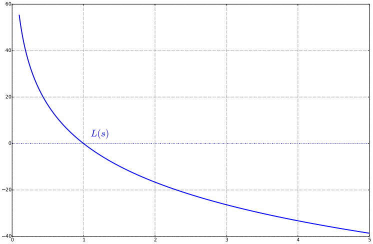
$$\mathbb{E}^i[S_i] = S_0, \quad \mathbb{E}^i[|\ln S_i|] < \infty, \quad i \in \{1,2\}; \qquad \mathbb{E}^V[V] = F_V, \quad \mathbb{E}^V[V^2] < \infty.$$

- No calendar arbitrage $\Longleftrightarrow \mu_1 \leq_c \mu_2$ (convex order)

Setting and notation

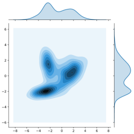
$$V^2 := (\mathrm{VIX}_{T_1})^2 := -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right] = \mathrm{Price}_{T_1}\left[L\left(\frac{S_2}{S_1}\right)\right]$$

- $\tau := 30$ days.
- $L(x) := -\frac{2}{\tau}\ln x$: convex, decreasing.



Superreplication, duality

Superreplication of forward-starting options



- The knowledge of μ_1 and μ_2 gives little information on the prices $\mathbb{E}^{\mu}[g(S_1, S_2)]$, e.g., prices of forward-starting options $\mathbb{E}^{\mu}[f(S_2/S_1)]$.
- Computing upper and lower bounds of these prices:
  Optimal transport (Monge, 1781; Kantorovich)
- Adding the no-arbitrage constraint that (S_1, S_2) is a martingale leads to more precise bounds, as this provides information on the conditional average of S_2/S_1 given S_1:
  Martingale optimal transport (Henry-Labordère, 2017)
- When $S = \text{SPX}$: Adding VIX market data information produces even more precise bounds, as it gives information on the conditional dispersion of S_2/S_1, which is controlled by the VIX V:
  Dispersion-constrained martingale optimal transport

Classical optimal transport

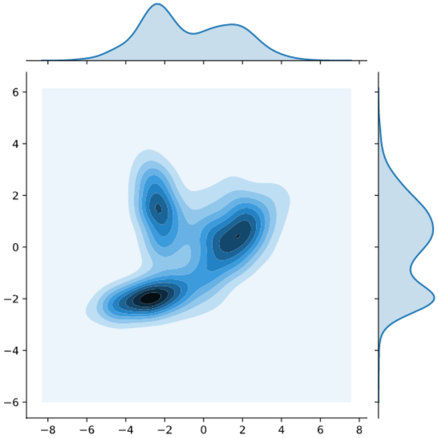


Figure: Example of a transport plan. Source: Wikipedia

Superreplication: primal problem

Fundamental principle: Upper bound for the price of payoff $f(S_1, V, S_2)$ = smallest price at time 0 of a superreplicating portfolio.

Following De Marco-Henry-Labordère (2015), G.-Menegaux-Nutz (2017), the available instruments for superreplication are:

- At time 0:
  - $u_1(S_1)$: SPX vanilla payoff maturity T_1 (including cash)
  - $u_2(S_2)$: SPX vanilla payoff maturity T_2
  - $u_V(V)$: **VIX vanilla payoff maturity** T_1

  Cost: $\text{MktPrice}[u_1(S_1)] + \text{MktPrice}[u_2(S_2)] + \text{MktPrice}[u_V(V)]$
  $= \quad \mathbb{E}^1[u_1(S_1)] \quad + \quad \mathbb{E}^2[u_2(S_2)] \quad + \quad \mathbb{E}^V[u_V(V)]$
- At time T_1:
  - $\Delta_S(S_1, V)(S_2 - S_1)$: delta hedge
  - $\Delta_L(S_1, V)(L(S_2/S_1) - V^2)$: buy $\Delta_L(S_1, V)$ log-contracts

  Cost: 0

Shorthand notation:

$$\Delta^{(S)}(s_1, v, s_2) := \Delta(s_1, v)(s_2 - s_1), \quad \Delta^{(L)}(s_1, v, s_2) := \Delta(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right)$$

Superreplication: primal problem

- The model-independent no-arbitrage upper bound for the derivative with payoff $f(S_1, V, S_2)$ is the smallest price at time 0 of a superreplicating portfolio:

$$P_f := \inf_{\mathcal{U}_f} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \right\}.$$

- $\mathcal{U}_f$: set of superreplicating portfolios, i.e., the set of all functions $(u_1, u_V, u_2, \Delta_S, \Delta_L)$ that satisfy the superreplication constraint:

$$u_1(s_1) + u_V(v) + u_2(s_2) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2) \geq f(s_1, v, s_2).$$

- Linear program.

Superreplication: dual problem

- $\mathcal{P}(\mu_1, \mu_V, \mu_2)$: set of all the probability measures μ on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ such that

$$S_1 \sim \mu_1, \quad V \sim \mu_V, \quad S_2 \sim \mu_2, \quad \mathbb{E}^\mu[S_2|S_1, V] = S_1, \quad \mathbb{E}^\mu\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1, V\right] = V^2.$$

- Dual problem:

$$D_f := \sup_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} \mathbb{E}^\mu[f(S_1, V, S_2)].$$

- **Dispersion-constrained martingale optimal transport problem**.
- $\mathbb{E}^\mu[S_2|S_1, V] = S_1$: martingality condition of the SPX index, condition on the average of the distribution of S_2 given S_1 and V.
- $\mathbb{E}^\mu[L(S_2/S_1)|S_1, V] = V^2$: consistency condition, condition on dispersion around the average.

Superreplication: strong duality theorem (absence of a duality gap)

Theorem (G. 2020)

Let $f : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}$ *be upper semicontinuous and satisfy*

$$|f(s_1, v, s_2)| \leq C\big(1 + s_1 + s_2 + |L(s_1)| + |L(s_2)| + v^2\big)$$

for some constant $C > 0$. *Then*

$$P_f := \inf_{\mathcal{U}_f} \Big\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \Big\} = \sup_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} \mathbb{E}^\mu[f(S_1, V, S_2)] =: D_f.$$

Moreover, $D_f \neq -\infty$ *if and only if* $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$, *and in that case the supremum is attained.*

Superreplication of forward-starting options

- The knowledge of μ_1 and μ_2 gives little information on the prices $\mathbb{E}^{\mu}[g(S_1, S_2)]$, e.g., prices of forward-starting options $\mathbb{E}^{\mu}[f(S_2/S_1)]$.
- Computing upper and lower bounds of these prices: **Optimal transport** (Monge, 1781; Kantorovich)
- Adding the no-arbitrage constraint that (S_1, S_2) is a martingale leads to more precise bounds, as this provides information on the conditional average of S_2/S_1 given S_1: **Martingale optimal transport** (Henry-Labordère, 2017)
- When $S = \text{SPX}$: Adding VIX market data information produces even more precise bounds, as it information on the conditional dispersion of S_2/S_1, which is controlled by the VIX V: **Dispersion-constrained martingale optimal transport**
- **Adding VIX market data may possibly reveal a joint SPX/VIX arbitrage. Corresponds to $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \emptyset$** (see next slides).
- In the limiting case where $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \{\mu_0\}$ is a singleton, the joint SPX/VIX market data information completely specifies the joint distribution of (S_1, S_2), hence the price of forward starting options.

Joint SPX/VIX arbitrage

Joint SPX/VIX arbitrage

- $\mathcal{U}_0$ = the portfolios $(u_1, u_2, u_V, \Delta^S, \Delta^L)$ superreplicating 0:

$$u_1(s_1)+u_2(s_2)+u_V(v)+\Delta^S(s_1,v)(s_2-s_1)+\Delta^L(s_1,v)\left(L\left(\frac{s_2}{s_1}\right)-v^2\right) \geq 0$$

- An (S_1, S_2, V)-arbitrage is an element of $\mathcal{U}_0$ with negative price:

$$\mathsf{MktPrice}[u_1(S_1)] + \mathsf{MktPrice}[u_2(S_2)] + \mathsf{MktPrice}[u_V(V)] < 0$$

- Equivalently, there is an (S_1, S_2, V)-arbitrage if and only if

$$\inf_{\mathcal{U}_0}\left\{\mathsf{MktPrice}[u_1(S_1)] + \mathsf{MktPrice}[u_2(S_2)] + \mathsf{MktPrice}[u_V(V)]\right\} = -\infty$$

Consistent extrapolation of SPX and VIX smiles

- If $\mathbb{E}^V[V^2] \neq \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]$, there is a trivial (S_1, S_2, V)-arbitrage. For instance, if $\mathbb{E}^V[V^2] < \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]$, pick

$$u_1(s_1) = L(s_1), \quad u_2(s_2) = -L(s_2), \quad u_V(v) = v^2, \quad \Delta_S(s_1, v) = 0, \quad \Delta_L(s_1, v) = 1.$$

- $\Longrightarrow$ We assume that

$$\mathbb{E}^V[V^2] = \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]. \tag{2.1}$$

- Violations of (2.1) in the market have been reported, suggesting arbitrage opportunities, see, e.g., Section 7.7.4 in Bergomi (2016).
- However, the quantities in (2.1) do not purely depend on market data. They depend on smile extrapolations.
- The reported violations of (2.1) actually rely on some arbitrary smile extrapolations.
- G. (2018) explains how to build **consistent extrapolations of the VIX and SPX smiles** so that (2.1) holds.

Joint SPX/VIX arbitrage

Theorem (G. 2020)

The following assertions are equivalent:

(i) *The market is free of (S_1, S_2, V)-arbitrage,*

(ii) $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$,

(iii) *There exists a coupling ν of μ_1 and μ_V such that $\mathrm{Law}_\nu(S_1, L(S_1) + V^2)$ and $\mathrm{Law}_{\mu_2}(S_2, L(S_2))$ are in convex order, i.e., for any convex function $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}^\nu[f(S_1, L(S_1) + V^2)] \leq \mathbb{E}^2[f(S_2, L(S_2))].$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- Recall $\mathcal{P}(\mu_1, \mu_V, \mu_2)$:= probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ s.t.

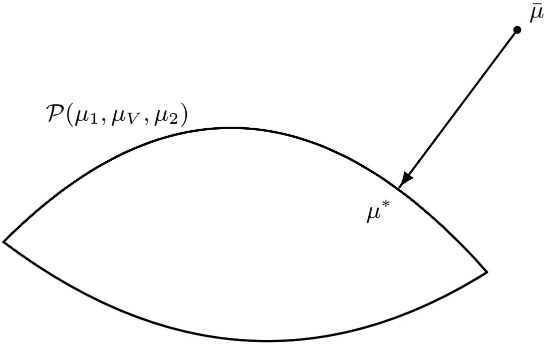
$$S_1 \sim \mu_1, \quad V \sim \mu_V, \quad S_2 \sim \mu_2, \quad \mathbb{E}^\mu\left[S_2 | S_1, V\right] = S_1, \quad \mathbb{E}^\mu\left[L\left(\frac{S_2}{S_1}\right)\middle| S_1, V\right] = V^2.$$

- **Build a model $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ = solve the joint calibration puzzle**.
- Our strategy is inspired by Avellaneda (1998, 2001) and De March and Henry-Labordère (2019).
- We assume that $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$ and try to build an element μ in this set. To this end, we fix a **reference probability measure $\bar{\mu}$** on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ and look for the measure $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ that **minimizes the relative entropy $H(\mu, \bar{\mu})$** of μ w.r.t. $\bar{\mu}$, also known as the Kullback-Leibler divergence:

$$D_{\bar{\mu}} := \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}), \quad H(\mu, \bar{\mu}) := \begin{cases} \mathbb{E}^\mu\left[\ln \frac{d\mu}{d\bar{\mu}}\right] = \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu}{d\bar{\mu}} \ln \frac{d\mu}{d\bar{\mu}}\right] & \text{if } \mu \ll \bar{\mu}, \\ +\infty & \text{otherwise.} \end{cases}$$

- This is a **strictly convex problem that can be solved after dualization using, e.g., Sinkhorn's fixed point iteration** (Sinkhorn, 1967).

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$



A technique inspired by Marco Avellaneda's ideas (NYU)

- **Minimum relative entropy approach for calibration purposes was pioneered by Avellaneda** at the end of the 90s.
- Marco Avellaneda: *Minimum-relative-entropy calibration of asset pricing models*. International Journal of Theoretical and Applied Finance, 1(4):447–472, 1998.
- Marco Avellaneda, Robert Buff, Craig Friedman, Nicolas Grandchamp, Lukasz Kruk, and Joshua Newman: *Weighted Monte Carlo: a new technique for calibrating asset-pricing models*. International Journal of Theoretical and Applied Finance, 4(1):91–119, 2001.
- Our approach is very much inspired by Marco's ideas.
- Here we have added (a) martingality constraint on the SPX and (b) constraint on prices of VIX options.

Reminder on Lagrange multipliers

$$\inf_{g(x,y)=c} f(x,y) = \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y) - \lambda(g(x,y)-c)\}$$
$$= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y) - \lambda(g(x,y)-c)\}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
- Constraint $g(x,y) = c \iff \frac{\partial}{\partial\lambda}\{f(x,y) - \lambda(g(x,y)-c)\} = 0$.

$$\inf_{\mu \text{ s.t. } S_1\sim\mu_1} H(\mu,\bar\mu) = \inf_{\mu} \sup_{u_1(\cdot)} \left\{ H(\mu,\bar\mu) + \mathbb{E}^1[u_1(S_1)] - \mathbb{E}^\mu[u_1(S_1)] \right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^\mu[S_2|S_1,V]=S_1} H(\mu,\bar\mu) = \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{ H(\mu,\bar\mu) - \mathbb{E}^\mu[\Delta_S(S_1,V)(S_2-S_1)] \right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^\mu\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1,V\right]=V^2} H(\mu,\bar\mu) = \inf_{\mu} \sup_{\Delta_L(\cdot,\cdot)} \left\{ H(\mu,\bar\mu) - \mathbb{E}^\mu\left[\Delta_L(S_1,V)\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \right\}$$

Reminder on Lagrange multipliers

$$\begin{aligned}\inf_{g(x,y)=c} f(x,y) &= \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y) - \lambda(g(x,y)-c)\} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y) - \lambda(g(x,y)-c)\}\end{aligned}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
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- Constraint $g(x,y) = c \iff \frac{\partial}{\partial\lambda}\{f(x,y) - \lambda(g(x,y)-c)\} = 0$.

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$$\inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}[S_2|S_1,V]=S_1} H(\mu,\bar\mu) = \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{ H(\mu,\bar\mu) - \mathbb{E}^{\mu}[\Delta_S(S_1,V)(S_2-S_1)] \right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}\left[L\left(\frac{S_2}{S_1}\right)\middle|S_1,V\right]=V^2} H(\mu,\bar\mu) = \inf_{\mu} \sup_{\Delta_L(\cdot,\cdot)} \left\{ H(\mu,\bar\mu) - \mathbb{E}^{\mu}\left[\Delta_L(S_1,V)\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \right\}$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar\mu} &:= \inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu, \bar\mu) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar\mu) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu\Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar\mu) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
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$$
\begin{aligned}
D_{\bar\mu} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar\mu) \\
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$$
\begin{aligned}
D_{\bar{\mu}} \quad &:= \quad \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
&= \quad \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
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Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

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\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
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\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

$$\begin{aligned} D_{\bar\mu} \quad = \quad & \sup_{u\in\mathcal{U}} \inf_{\mu\in\mathcal{M}_1} \Big\{ H(\mu,\bar\mu) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\ & -\mathbb{E}^\mu\left[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1,V,S_2) + \Delta_L^{(L)}(S_1,V,S_2)\right] \Big\} \end{aligned}$$

- **Remarkable fact: The inner infimum can be exactly computed:**

$$\inf_{\mu\in\mathcal{M}_1} \{H(\mu,\bar\mu) - \mathbb{E}^\mu[X]\} = -\ln \mathbb{E}^{\bar\mu}\left[e^X\right]$$

and **the infimum is attained at** $\mu = \bar\mu_X$ **defined by** (Gibbs type)

$$\frac{d\bar\mu_X}{d\bar\mu} = \frac{e^X}{\mathbb{E}^{\bar\mu}[e^X]}.$$

- That is why we like (and chose) the “distance” $H(\mu,\bar\mu)$!

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

$$D_{\bar{\mu}} := \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) = \sup_{u \in \mathcal{U}} \Psi_{\bar{\mu}}(u) =: P_{\bar{\mu}}$$

$$\Psi_{\bar{\mu}}(u) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - \ln \mathbb{E}^{\bar{\mu}}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S^{(S)}(S_1,V,S_2)+\Delta_L^{(L)}(S_1,V,S_2)}\right].$$

- $\inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)}$: **constrained** optimization, **difficult**.
- $\sup_{u \in \mathcal{U}}$: **unconstrained** optimization, **easy!** To find the optimum $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$, simply cancel the gradient of $\Psi_{\bar{\mu}}$.
- Most important, $\inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu})$ is reached at

$$\mu^*(ds_1, dv, ds_2) = \bar{\mu}(ds_1, dv, ds_2) \frac{e^{u_1^*(s_1)+u_V^*(v)+u_2^*(s_2)+\Delta_S^{*(S)}(s_1,v,s_2)+\Delta_L^{*(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar{\mu}}\left[e^{u_1^*(S_1)+u_V^*(V)+u_2^*(S_2)+\Delta_S^{*(S)}(S_1,V,S_2)+\Delta_L^{*(L)}(S_1,V,S_2)}\right]}.$$

- **Problem solved:** $\mu^* \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$**!**

Minimum entropy strong duality theorem

Theorem (G. 2020)

Let $\bar{\mu} \in \mathcal{M}_1$. *Then*

$$D_{\bar{\mu}} := \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \;=\; \sup_{u \in \mathcal{U}} \Psi_{\bar{\mu}}(u) =: P_{\bar{\mu}}$$

where $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$ *and*

$$\Psi_{\bar{\mu}}(u) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - \ln \mathbb{E}^{\bar{\mu}}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S(S_1,V)(S_2-S_1)+\Delta_L(S_1,V)\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right].$$

Moreover, when $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$, *the infimum is attained. This is in particular the case when the above quantity is finite.*

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

$$\mu^*(ds_1, dv, ds_2) = \bar{\mu}(ds_1, dv, ds_2) \frac{e^{u_1^*(s_1)+u_V^*(v)+u_2^*(s_2)+\Delta_S^{*(S)}(s_1,v,s_2)+\Delta_L^{*(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar{\mu}}\left[e^{u_1^*(S_1)+u_V^*(V)+u_2^*(S_2)+\Delta_S^{*(S)}(S_1,V,S_2)+\Delta_L^{*(L)}(S_1,V,S_2)}\right]}.$$

- $\Psi_{\bar{\mu}}$ is invariant by translation of u_1, u_V, and u_2: for any constant $c \in \mathbb{R}$, $\Psi_{\bar{\mu}}(u_1 + c, u_V, u_2, \Delta_S, \Delta_L) = \Psi_{\bar{\mu}}(u_1, u_V, u_2, \Delta_S, \Delta_L)$ (and similarly with u_V and u_2); $c =$ cash position $\Longrightarrow$ We will always work with a normalized version of $u^* \in \mathcal{U}$ s.t.

$$\mathbb{E}^{\bar{\mu}}\left[e^{u_1^*(S_1)+u_V^*(V)+u_2^*(S_2)+\Delta_S^{*(S)}(S_1,V,S_2)+\Delta_L^{*(L)}(S_1,V,S_2)}\right] = 1. \quad (2.2)$$

- **By duality, the initial, difficult problem of minimizing over $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ (constrained) has been reduced to the simpler problem of maximizing the strictly concave function $\Psi_{\bar{\mu}}$ over $u \in \mathcal{U}$ (unconstrained)**. If it exists, the optimum u^* cancels the gradient of $\Psi_{\bar{\mu}}$: $\frac{\partial \Psi_{\bar{\mu}}}{\partial u_1(s_1)} = \frac{\partial \Psi_{\bar{\mu}}}{\partial u_V(v)} = \frac{\partial \Psi_{\bar{\mu}}}{\partial u_2(s_2)} = \frac{\partial \Psi_{\bar{\mu}}}{\partial \Delta_S(s_1,v)} = \frac{\partial \Psi_{\bar{\mu}}}{\partial \Delta_L(s_1,v)} = 0.$

Equations for $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$

$$
\begin{aligned}
\frac{\partial \Psi_{\bar\mu}}{\partial u_1(s_1)}=0: &\quad \forall s_1>0, & u_1(s_1) &= \Phi_1(s_1; u_V, u_2, \Delta_S, \Delta_L)\\
\frac{\partial \Psi_{\bar\mu}}{\partial u_V(v)}=0: &\quad \forall v\geq 0, & u_V(v) &= \Phi_V(v; u_1, u_2, \Delta_S, \Delta_L)\\
\frac{\partial \Psi_{\bar\mu}}{\partial u_2(s_2)}=0: &\quad \forall s_2>0, & u_2(s_2) &= \Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L)\\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_S(s_1,v)}=0: &\quad \forall s_1>0,\ \forall v\geq 0, & 0 &= \Phi_{\Delta_S}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v))\\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_L(s_1,v)}=0: &\quad \forall s_1>0,\ \forall v\geq 0, & 0 &= \Phi_{\Delta_L}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v))
\end{aligned}
$$

- We could have simply postulated a model of the form

$$
\mu(ds_1, dv, ds_2) = \bar\mu(ds_1, dv, ds_2)\frac{e^{u_1(s_1)+u_V(v)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar\mu}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S^{(S)}(S_1,V,S_2)+\Delta_L^{(L)}(S_1,V,S_2)}\right]}.
$$

- Then the 5 conditions defining $\mathcal{P}(\mu_1, \mu_V, \mu_2)$ translate into the 5 above equations.
- The system of equations is solved using **Sinkhorn's algorithm**.
- **If the algorithm diverges**, then $P_{\bar\mu} = +\infty$, so $D_{\bar\mu} = +\infty$, i.e., $\mathcal{P}(\mu_1, \mu_V, \mu_2) \cap \{\mu \in \mathcal{M}_1 | \mu \ll \bar\mu\} = \emptyset$. In practice, when $\bar\mu$ has full support, this is a sign that **there likely exists a joint SPX/VIX arbitrage**.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^*.**
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$\begin{aligned}
\forall s_1 > 0, \quad & u_1^{(n+1)}(s_1) &=& \ \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad & u_V^{(n+1)}(v) &=& \ \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
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\forall s_1 > 0, \ \forall v \geq 0, \quad & 0 &=& \ \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
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\end{aligned}$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

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- Each of the above five lines corresponds to a **Bregman projection** in the space of measures.
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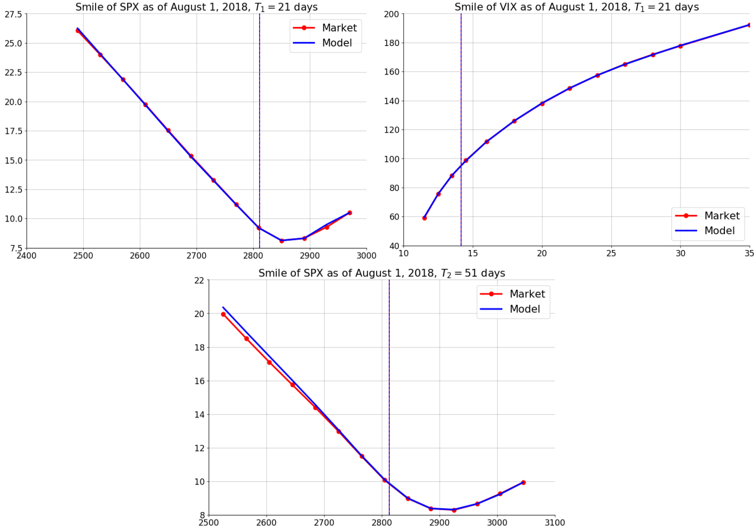
Numerical experiments

Implementation details

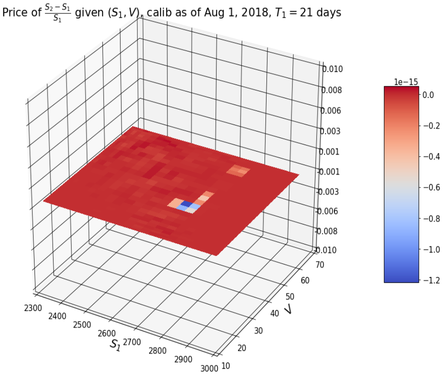
- Choice of $\bar{\mu}$:
  - $S_1 \sim \mu_1$ and $V \sim \mu_V$ independent;
  - Conditional on (S_1, V), S_2 lognormal with mean S_1 and variance V.

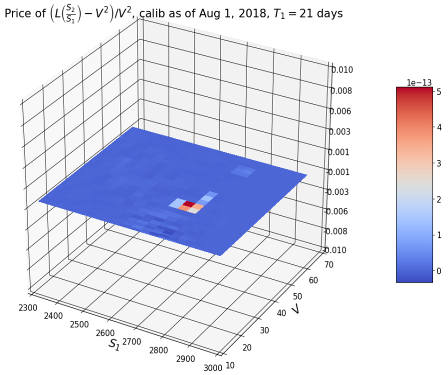
  Under $\bar{\mu}$, $S_2 \not\sim \mu_2$.
- Instead of abstract payoffs u_1, u_V, u_2, we work with market strikes and market prices of vanilla options on S_1, V, and S_2.
- Canceling the gradient of $\Psi_{\bar{\mu}} \to$ system of equations solved using Sinkhorn's algorithm.
- Enough accuracy is typically reached after ≈ 100 iterations.

August 1, 2018, $T_1 = 21$ days



August 1, 2018, $T_1 = 21$ days





August 1, 2018, $T_1 = 21$ days

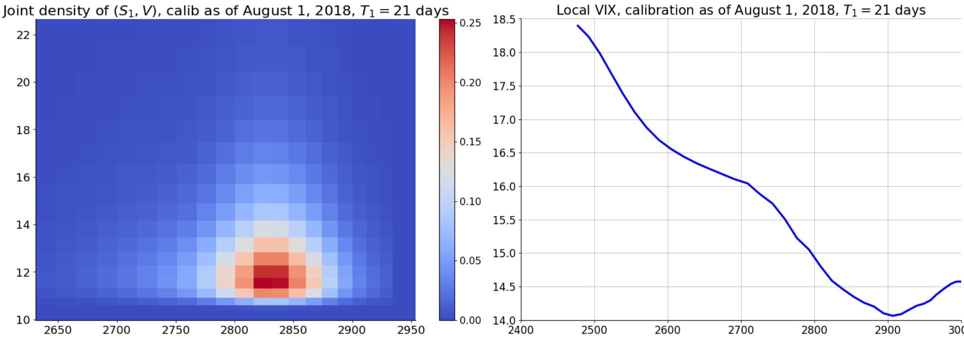


Figure: Joint distribution of (S_1, V) and local VIX function $\mathrm{VIX}_{\mathrm{loc}}(s_1)$

$$\mathrm{VIX}^2_{\mathrm{loc}}(S_1) := \mathbb{E}^{\mu^*}\left[V^2 \middle| S_1\right]$$

August 1, 2018, $T_1 = 21$ days

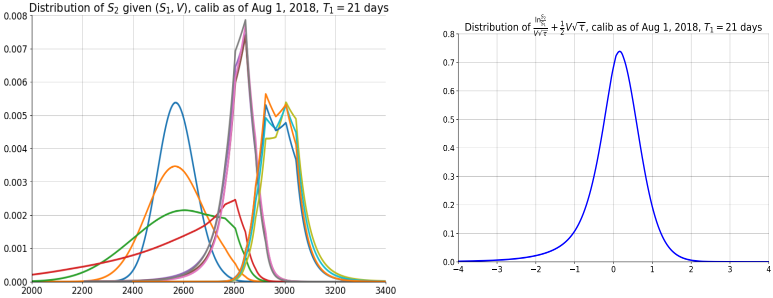
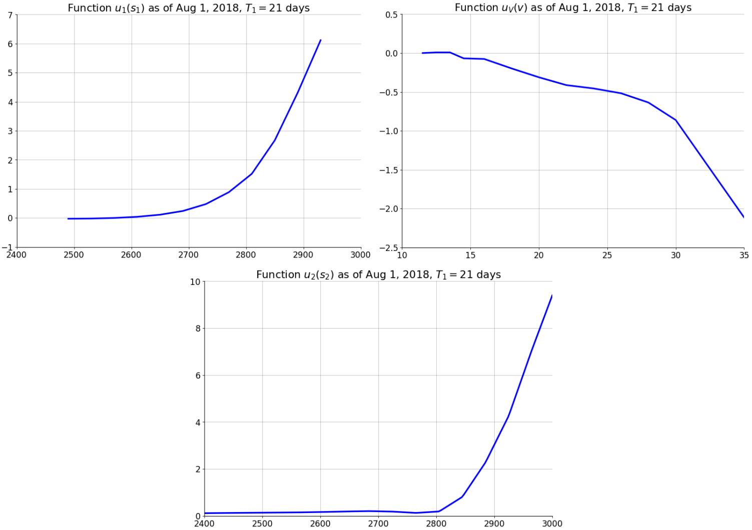


Figure: Conditional distribution of S_2 given (s_1, v) under μ^* for different vales of (s_1, v): $s_1 \in \{2571, 2808, 3000\}$, $v \in \{10.10, 15.30, 23.20, 35.72\}\%$, and distribution of the normalized return $R := \frac{\ln(S_2/S_1)}{V\sqrt{\tau}} + \frac{1}{2}V\sqrt{\tau}$

August 1, 2018, $T_1 = 21$ days



August 1, 2018, $T_1 = 21$ days

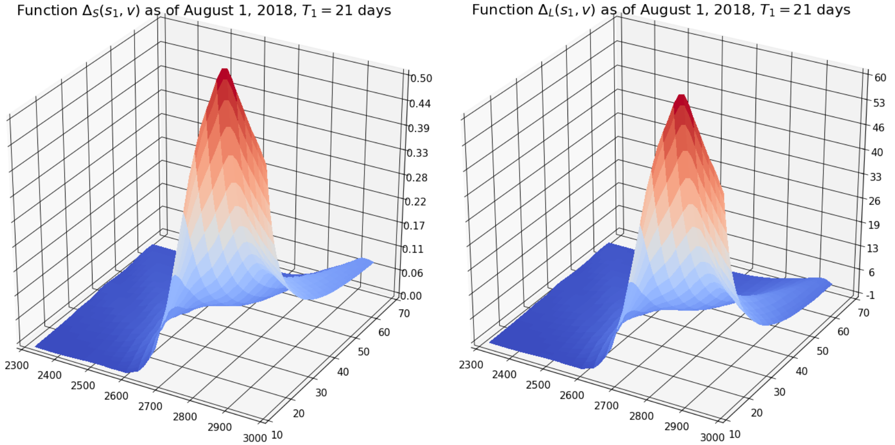
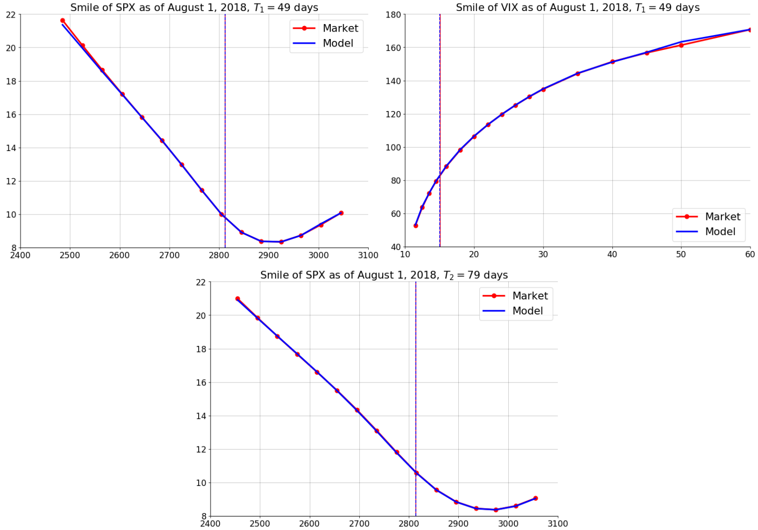


Figure: Optimal functions $\Delta_S^*(s_1, v)$ and $\Delta_L^*(s_1, v)$ for (s_1, v) in the quadrature grid

August 1, 2018, $T_1 = 49$ days



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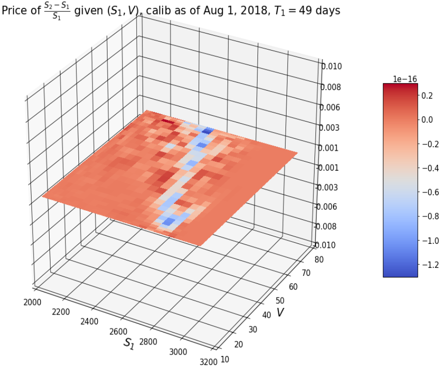
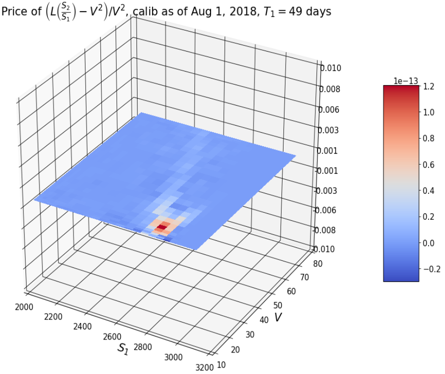
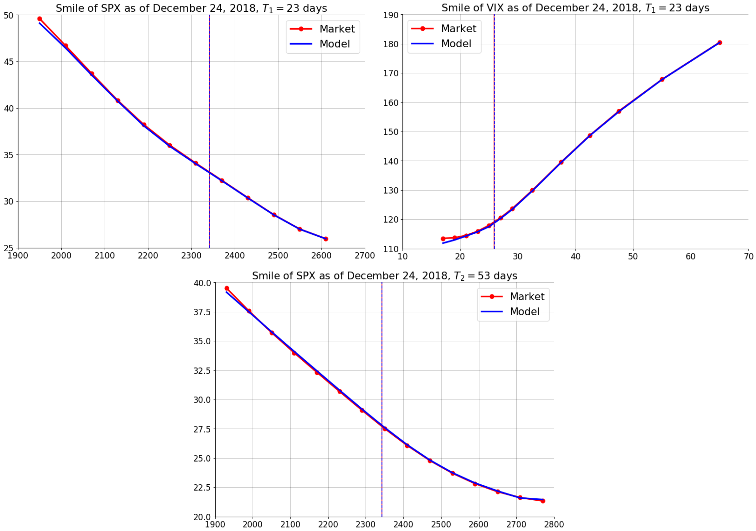


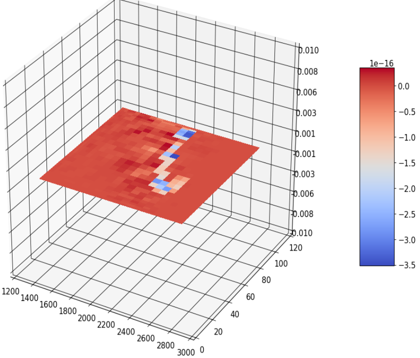


December 24, 2018, $T_1 = 23$ days: large VIX, $F_V \approx 26\%$

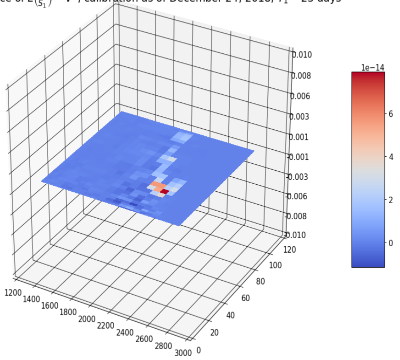


December 24, 2018, $T_1 = 23$ days

Model price of $S_2 - S_1$, calibration as of Dec 24, 2018, $T_1 = 23$ days



Model price of $L\left(\frac{S_2}{S_1}\right) - V^2$, calibration as of December 24, 2018, $T_1 = 23$ days



MOT in continuous time: Exact joint calibration via VIX-constrained martingale Schrödinger bridges

(G. 2020)

Martingale optimal transport approach in continuous time

- Same point of view as the discrete-time model: Pick a reference measure $\mathbb{P}_0 \longleftrightarrow$ a particular SV model:

$$\begin{aligned} \frac{dS_t}{S_t} &= a_t \, dW_t^0 \\ da_t &= b(a_t)\, dt + \sigma(a_t) \left(\rho \, dW_t^0 + \sqrt{1-\rho^2} dW_t^{0,\perp} \right) \end{aligned}$$

- We want to prove that $\mathcal{P} \neq \emptyset$ and build $\mathbb{P} \in \mathcal{P}$, where

$$\mathcal{P} := \{\mathbb{P} \ll \mathbb{P}_0 | S_1 \sim \mu_1, S_2 \sim \mu_2, \sqrt{\mathbb{E}^{\mathbb{P}}[L(S_2/S_1)|\mathcal{F}_1]} \sim \mu_V, S \text{ is a } \mathbb{P}\text{-martingale}\}.$$

- No need to introduce a new r.v. for the VIX: VIX $= \sqrt{\mathbb{E}^{\mathbb{P}}[L(S_2/S_1)|\mathcal{F}_1]}$.
- We look for $\mathbb{P} \in \mathcal{P}$ that minimizes the relative entropy w.r.t. $\mathbb{P}_0$:

$$D := \inf_{\mathbb{P} \in \mathcal{P}} H(\mathbb{P}, \mathbb{P}_0)$$

- Inspired by Henry-Labordère 2019: *From (Martingale) Schrödinger Bridges to a New Class of Stochastic Volatility Models* (calib to SPX smiles)
- Follows closely the construction of **Schrödinger bridges**

Simple Schrödinger bridge (à la Follmer, Saint-Flour 1988)

$$\begin{aligned} dX_t &= dW_t^0, \qquad X_0 = x_0 \\ \mathcal{P} &:= \{\mathbb{P} \ll \mathbb{P}_0 \,|\, X_1 \sim \mu_1\} \end{aligned}$$

$$\begin{aligned} D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\ &= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1)} \left\{ H(\mathbb{P},\mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} \\ &= \sup_{u_1\in L^1(\mu_1)} \inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P},\mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} \end{aligned}$$

Recall the remarkable fact about the inner infimum:

$$\inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P},\mathbb{P}_0) - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} = -\ln \mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]$$

and the infimum is reached at $\mathbb{P}^*$ defined by $\dfrac{d\mathbb{P}^*}{d\mathbb{P}_0} = \dfrac{e^{u_1(X_1)}}{\mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]}$.

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\end{aligned}
$$

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P}, \mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1)} \left\{ H(\mathbb{P}, \mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} \\
&= \sup_{u_1\in L^1(\mu_1)} \inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}, \mathbb{P}_0) + \mathbb{E}^{\mu_1}[u_1(X_1)] - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\}
\end{aligned}
$$

Recall the remarkable fact about the inner infimum:

$$
\inf_{\mathbb{P}\in\mathcal{M}_1} \left\{ H(\mathbb{P}, \mathbb{P}_0) - \mathbb{E}^{\mathbb{P}}[u_1(X_1)] \right\} = -\ln \mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)} \right]
$$

and the infimum is reached at $\mathbb{P}^*$ defined by $\frac{d\mathbb{P}^*}{d\mathbb{P}_0} = \frac{e^{u_1(X_1)}}{\mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right]}$.

Simple Schrödinger bridge (à la Follmer, Saint-Flour 1988)

$$D := \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) = \sup_{u_1\in L^1(\mu_1)} \left\{ \mathbb{E}^{\mu_1}\left[u_1(X_1)\right] - \ln \mathbb{E}^{\mathbb{P}_0}\left[e^{u_1(X_1)}\right] \right\} =: P$$

- Assume $P < +\infty$ and the sup is reached at u_1^*. Then

$$M_{T_1} := \frac{d\mathbb{P}^*}{d\mathbb{P}_0} = e^{u_1^*(X_1)} \qquad (Z = 1 \text{ by cash adjustment of } u_1^*)$$

- Let $M_t := \mathbb{E}^{\mathbb{P}_0}[M_{T_1}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}_0}[e^{u_1^*(X_1)}|\mathcal{F}_t]$. Then $M_t = U^*(t, X_t)$ where

$$\partial_t U^* + \frac{1}{2}\partial_x^2 U^* = 0, \qquad U^*(T_1, x) = e^{u_1^*(x)}.$$

- By Girsanov, $W_t^* := W_t^0 - \int_0^t \partial_x \ln U^*(s, X_s)\,ds$ is a $\mathbb{P}^*$-Brownian motion,

$$dX_t = \partial_x \ln U^*(t, X_t)\,dt + dW_t^* = \partial_x \ln \mathbb{E}^{\mathbb{P}_0}[e^{u_1^*(X_1)}|X_t = x]_{|X_t}\,dt + dW_t^*$$

- Brownian motion with drift, which is explicitly known.
- In practice, $u_1(X_1)$ is replaced by $\sum_{K\in\mathcal{K}} \alpha_K (X_1 - K)_+$. The gradient of

$$\mathbb{E}^{\mu_1}\left[\sum_{K\in\mathcal{K}} \alpha_K (X_1 - K)_+\right] - \ln \mathbb{E}^{\mathbb{P}_0}\left[e^{\sum_{K\in\mathcal{K}} \alpha_K (X_1-K)_+}\right]$$

is simply the vector of differences between model and market call prices.

VIX-constrained martingale Schrödinger bridge

$$\begin{aligned}\frac{dS_t}{S_t} &= a_t\, dW_t^*\\ da_t &= \left(b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a u^*(t,S_t,a_t)\right)dt+\sigma(a_t)\left(\rho\, dW_t^*+\sqrt{1-\rho^2}dW_t^{*,\perp}\right)\end{aligned}$$

- Let $P := \sup_{u_1,u_V,u_2}\left\{\sum_{i\in\{1,2,V\}}(\mu_i,u_i)-u(0,S_0,a_0)\right\}$ where u is solution to a nonlinear Hamilton-Jacobi-Bellman PDE:

$$\begin{aligned}u(T_2,s,a;\delta^L) &= u_2(s)+\delta^L L(s),\\ \partial_t u+\mathcal{L}^0 u+\frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t\in(T_1,T_2),\\ \Phi(s,a) := \sup_{v\geq 0}\inf_{\delta^L\in\mathbb{R}}\Big\{u_V(v) &- \delta^L(L(s)+v^2)\ +\ u(T_1,s,a;\delta^L)\Big\},\\ u(T_1,s,a) &= u_1(s)+\Phi(s,a),\\ \partial_t u+\mathcal{L}^0 u+\frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t\in[0,T_1).\end{aligned}$$

- Assume $P<+\infty$ and (u_1^*,u_V^*,u_2^*) maximizes $P\longrightarrow u^*$

VIX-constrained martingale Schrödinger bridge

$$\frac{dS_t}{S_t} = a_t\,dW_t^*$$
$$da_t = \left(b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a u^*(t,S_t,a_t)\right)dt + \sigma(a_t)\left(\rho\,dW_t^* + \sqrt{1-\rho^2}\,dW_t^{*,\perp}\right)$$

- Optimal deltas:

$$\Delta_t^* = -\partial_s u^*(t,S_t,a_t) - \rho\frac{\sigma(a_t)}{a_t S_t}\partial_a u^*(t,S_t,a_t); \quad \Delta^{*,L} = \delta^{*,L}(S_1,a_1)$$

- The drift of (a_t) under $\mathbb{P}^*$ also reads as

$$b(a_t) + (1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_1^*(S_1)+\int_t^{T_1}\Delta^*(r,S_r,a_r)dS_r+\Phi^*(S_1,a_1)}|S_t,a_t], \quad t\in[0,T_1],$$
$$b(a_t) + (1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_2^*(S_2)+\int_t^{T_2}\Delta^*(r,S_r,a_r)dS_r+\delta^{*,L}(S_1,a_1)L(S_2)}|S_t,a_t], \quad t\in[T_1,T_2].$$

- It is path-dependent on $[T_1,T_2]$ so as to match the market VIX smile.
- If $P=+\infty$, then $\mathcal{P}=\emptyset$.

Martingale optimal transport approach in continuous time

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\Big|\mathcal{F}_1\right]}\right) + \int_0^{T_2} \Delta_t\,dS_t \right] \Big\} \\
&\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\,dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \Big\} \\
&\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\,dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \Big\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \right] \Big\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Lagrange multipliers u_1, u_2, u_V

$$D := \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0)$$

$$= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i)$$

$$-\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\middle|\mathcal{F}_1\right]}\right)+\int_0^{T_2}\Delta_t\,dS_t\right]\Bigg\}$$

$$\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i)$$

$$-\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Bigg\}$$

$$\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i)$$

$$-\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Bigg\}$$

$$= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i)$$

$$-\ln\mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right]\Bigg\}$$

$$= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}$$

Lagrange multipliers Δ_t: martingality of S

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\middle|\mathcal{F}_1\right]}\right) + \int_0^{T_2} \Delta_t\, dS_t \right] \Bigg\} \\
&\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \Bigg\} \\
&\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right] \Bigg\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \right] \Bigg\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Relaxation

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\mathcal{F}\text{-adapted}} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\Big|\mathcal{F}_1\right]}\right) + \int_0^{T_2}\Delta_t\,dS_t\right]\Big\} \\
&\overset{(\text{relax})}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Big\} \\
&\overset{(\text{dual})}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Big\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\ln\mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right]\Big\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Relaxation

$$\mathbb{E}^{\mathbb{P}}\left[u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\middle|\mathcal{F}_1\right]}\right)\right] = \inf_{V\in\mathcal{F}_1}\sup_{\Delta^L\in\mathcal{F}_1}\mathbb{E}^{\mathbb{P}}\left[u_V(V) + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right)\right]$$

Relaxation

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\Big[u_1(S_1) + u_2(S_2) + u_V\Big(\sqrt{\mathbb{E}^{\mathbb{P}}\Big[L\Big(\frac{S_2}{S_1}\Big)\Big|\mathcal{F}_1\Big]}\Big) + \int_0^{T_2} \Delta_t\, dS_t \Big] \Big\} \\
&\overset{(\text{relax})}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\Big[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\Big(L\Big(\frac{S_2}{S_1}\Big) - V^2\Big)\Big] \Big\} \\
&\overset{(\text{dual})}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\Big[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\Big(L\Big(\frac{S_2}{S_1}\Big) - V^2\Big)\Big] \Big\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad - \ln \mathbb{E}^0\Big[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\, dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \Big] \Big\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Duality

$$
\begin{aligned}
D \quad &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right) \middle| \mathcal{F}_1 \right]}\right) + \int_0^{T_2} \Delta_t \, dS_t \right] \Bigg\} \\
&\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t \, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2 \right) \right] \Bigg\} \\
&\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Bigg\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t \, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2 \right) \right] \Bigg\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \right] \Bigg\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Remarkable fact: inner inf is explicit

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\,\mathcal{F}\text{-adapted}} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\Big|\mathcal{F}_1\right]}\right)+\int_0^{T_2}\Delta_t\,dS_t\right]\Big\} \\
&\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Big\} \\
&\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\mathbb{E}^{\mathbb{P}}\left[u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\right]\Big\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) \\
&\quad -\ln\mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\,dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right]\Big\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Optimizing first over $[T_1, T_2]$, then T_1, then $[T_0, T_1]$, then T_0

$$
\begin{aligned}
D &:= \inf_{\mathbb{P}\in\mathcal{P}} H(\mathbb{P},\mathbb{P}_0) \\
&= \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1\in L^1(\mu_1),u_2\in L^1(\mu_2),u_V\in L^1(\mu_V),(\Delta_t)\mathcal{F}\text{-adapted}} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V\left(\sqrt{\mathbb{E}^{\mathbb{P}}\left[L\left(\frac{S_2}{S_1}\right)\middle|\mathcal{F}_1\right]}\right) + \int_0^{T_2} \Delta_t\, dS_t \right] \Big\} \\
&\overset{\text{(relax)}}{=} \inf_{\mathbb{P}\in\mathcal{M}_1} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right) \right] \Big\} \\
&\overset{\text{(dual)}}{=} \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \inf_{\mathbb{P}\in\mathcal{M}_1} \Big\{ H(\mathbb{P},\mathbb{P}_0) + \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad - \mathbb{E}^{\mathbb{P}}\left[u_1(S_1) + u_2(S_2) + u_V(V) + \int_0^{T_2} \Delta_t\, dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right) - V^2\right) \right] \Big\} \\
&= \sup_{u_1,u_2,u_V,(\Delta_t)} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Big\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\
&\quad - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t\, dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \right] \Big\} \\
&= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \{\cdots\}
\end{aligned}
$$

Optimize over $[T_1, T_2]$

- The inner $\inf_{\mathbb{P}\in\mathcal{M}_1}$ is reached at $\mathbb{P}^*$ defined by (renorm. $Z = 1$ by cash adjustment of vanilla payoffs)

$$\frac{d\mathbb{P}^*}{d\mathbb{P}_0} = e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}$$

$$\begin{aligned} D &= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \sup_{(\Delta_t)_{t\in[T_1,T_2]}} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\ &\quad - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right]\Bigg\} \\ &= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\ &\quad - \ln \inf_{(\Delta_t)_{t\in[T_1,T_2]}} \mathbb{E}^0\left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2}\Delta_t dS_t+\Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)}\right]\Bigg\} \end{aligned}$$

Optimize over $[T_1, T_2]$: stochastic control

$$\begin{aligned} D &= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\ &\quad - \ln \inf_{(\Delta_t)_{t\in[T_1,T_2]}} \mathbb{E}^0 \left[e^{u_1(S_1)+u_2(S_2)+u_V(V)+\int_0^{T_2} \Delta_t dS_t + \Delta^L\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)} \right] \Bigg\} \\ &\overset{\text{(DPP)}}{=} \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \Bigg\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) \\ &\quad - \ln \mathbb{E}^0 \left[e^{u_1(S_1)+u_V(V)+\int_0^{T_1} \Delta_t dS_t - \Delta^L(L(S_1)+V^2)} U(T_1, S_1, a_1; \Delta^L) \right] \Bigg\} \end{aligned}$$

- Stochastic control:

$$U(t, S_t, a_t; \Delta^L) := \inf_{(\Delta_r)_{r\in[t,T_2]}} \mathbb{E}^0 \left[e^{u_2(S_2)+\int_t^{T_2} \Delta_r dS_r + \Delta^L L(S_2)} \middle| S_t, a_t, \Delta^L \right], \quad t \in [T_1, T_2].$$

Optimize over $[T_1, T_2]$: stochastic control

- U is solution to the HJB PDE

$$\begin{aligned}\partial_t U + \mathcal{L}^0 U + \inf_{\Delta}\left\{\frac{1}{2}\Delta^2 a^2 s^2 U + \Delta a s\left(as\partial_s U + \rho\sigma(a)\partial_a U\right)\right\} &= 0,\\ U(T_2, s, a; \delta^L) &= e^{u_2(s)+\delta^L L(s)}.\end{aligned}$$

- Optimal delta:

$$\Delta_t^* = -\frac{\partial_s U(t, S_t, a_t) + \rho\frac{\sigma(a_t)}{a_t S_t}\partial_a U(t, S_t, a_t)}{U(t, S_t, a_t)},$$

- U satisfies

$$\partial_t U + \mathcal{L}^0 U - \frac{\left(as\partial_s U + \rho\sigma(a)\partial_a U\right)^2}{2U} = 0, \qquad U(T_2, s, a; \delta^L) = e^{u_2(s)+\delta^L L(s)}.$$

- $u := \ln U$ satisfies

$$\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 = 0, \qquad u(T_2, s, a; \delta^L) = u_2(s) + \delta^L L(s).$$

Optimize at T_1: simply pathwise

$$D = \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_V(V)+\int_0^{T_1}\Delta_t dS_t - \Delta^L(L(S_1)+V^2)+u(T_1,S_1,a_1;\Delta^L)} \right] \right\}$$

Since S_1, a_1, and $\int_0^{T_1}\Delta_t dS_t$ are $\mathcal{F}_1$-measurable,

$$\begin{aligned}
&\inf_{V\in\mathcal{F}_1} \sup_{\Delta^L\in\mathcal{F}_1} \left\{ - \ln \mathbb{E}^0\left[e^{u_1(S_1)+u_V(V)+\int_0^{T_1}\Delta_t dS_t - \Delta^L(L(S_1)+V^2)+u(T_1,S_1,a_1;\Delta^L)} \right] \right. \\
&= -\ln \sup_{V\in\mathcal{F}_1} \inf_{\Delta^L\in\mathcal{F}_1} \mathbb{E}^0\left[e^{u_1(S_1)+u_V(V)+\int_0^{T_1}\Delta_t dS_t - \Delta^L(L(S_1)+V^2)+u(T_1,S_1,a_1;\Delta^L)} \right] \\
&= -\ln \mathbb{E}^0\left[e^{u_1(S_1)+\int_0^{T_1}\Delta_t dS_t + \Phi(S_1,a_1)} \right]
\end{aligned}$$

$$\Phi(s,a) := \sup_{v\geq 0} \inf_{\delta^L\in\mathbb{R}} \left\{ u_V(v) - \delta^L(L(s)+v^2) + u(T_1,s,a;\delta^L) \right\}.$$

The optimal V and Δ^L are functions of (S_1,a_1): $v^*(S_1,a_1)$, $\delta_L^*(S_1,a_1)$.

Optimize over $[T_0, T_1]$: same stochastic control

$$D = \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln \mathbb{E}^0 \left[e^{u_1(S_1)+\int_0^{T_1} \Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\}$$

$$= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln \inf_{(\Delta_t)_{t\in[0,T_1]}} \mathbb{E}^0 \left[e^{u_1(S_1)+\int_0^{T_1} \Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\}$$

$$= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln U(0, S_0, a_0) \right\}$$

$$= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - u(0, S_0, a_0) \right\} =: P$$

where $U(t, S_t, a_t) := \inf_{(\Delta_r)_{r\in[t,T_1]}} \mathbb{E}^0 \left[e^{u_1(S_1)+\int_t^{T_1} \Delta_r dS_r + \Phi(S_1,a_1)} \middle| S_t, a_t \right]$ satisfies

$$\partial_t U + \mathcal{L}^0 U - \frac{(as\partial_s U + \rho\sigma(a)\partial_a U)^2}{2U} = 0, \ t \in [0, T_1), \quad U(T_1, s, a) = e^{u_1(s)+\Phi(s,a)}$$

and $u := \ln U$ satisfies

$$\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 = 0, \ t \in [0, T_1), \quad u(T_1, s, a) = u_1(s) + \Phi(s, a).$$

Optimize over $[T_0, T_1]$: same stochastic control

$$
\begin{aligned}
D &= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln \mathbb{E}^0 \left[e^{u_1(S_1)+\int_0^{T_1} \Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln \inf_{(\Delta_t)_{t\in[0,T_1]}} \mathbb{E}^0 \left[e^{u_1(S_1)+\int_0^{T_1} \Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - \ln U(0, S_0, a_0) \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i, u_i) - u(0, S_0, a_0) \right\} =: P
\end{aligned}
$$

where $U(t, S_t, a_t) := \inf_{(\Delta_r)_{r\in[t,T_1]}} \mathbb{E}^0 \left[e^{u_1(S_1)+\int_t^{T_1} \Delta_r dS_r + \Phi(S_1,a_1)} \middle| S_t, a_t \right]$ satisfies

$$
\partial_t U + \mathcal{L}^0 U - \frac{(as\partial_s U + \rho\sigma(a)\partial_a U)^2}{2U} = 0, \ t \in [0, T_1), \quad U(T_1, s, a) = e^{u_1(s)+\Phi(s,a)}
$$

and $u := \ln U$ satisfies

$$
\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 = 0, \ t \in [0, T_1), \quad u(T_1, s, a) = u_1(s) + \Phi(s, a).
$$

Final dual representation

$$
\begin{aligned}
D &= \sup_{u_1,u_2,u_V} \sup_{(\Delta_t)_{t\in[0,T_1]}} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) - \ln \mathbb{E}^0\left[e^{u_1(S_1)+\int_0^{T_1}\Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) - \ln \inf_{(\Delta_t)_{t\in[0,T_1]}} \mathbb{E}^0\left[e^{u_1(S_1)+\int_0^{T_1}\Delta_t dS_t + \Phi(S_1,a_1)} \right] \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) - \ln U(0,S_0,a_0) \right\} \\
&= \sup_{u_1,u_2,u_V} \left\{ \sum_{i\in\{1,2,V\}} (\mu_i,u_i) - u(0,S_0,a_0) \right\} =: P
\end{aligned}
$$

where $U(t,S_t,a_t) := \inf_{(\Delta_r)_{r\in[t,T_1]}} \mathbb{E}^0\left[e^{u_1(S_1)+\int_t^{T_1}\Delta_r dS_r + \Phi(S_1,a_1)} \middle| S_t, a_t \right]$ satisfies

$$
\partial_t U + \mathcal{L}^0 U - \frac{(as\partial_s U + \rho\sigma(a)\partial_a U)^2}{2U} = 0,\ t\in[0,T_1), \quad U(T_1,s,a) = e^{u_1(s)+\Phi(s,a)}
$$

and $u := \ln U$ satisfies

$$
\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 = 0,\ t\in[0,T_1), \quad u(T_1,s,a) = u_1(s) + \Phi(s,a).
$$

Calibrated model = reference model with modified drift

- Assume $P < +\infty$ and (u_1^*, u_V^*, u_2^*) maximizes P. The probability $\mathbb{P}^*$ that minimizes $H(\mathbb{P}, \mathbb{P}_0)$ satisfies $(Z = 1)$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}_0} = e^{u_1^*(S_1)+u_2^*(S_2)+u_V^*(V^*)+\int_0^{T_2} \Delta_t^* dS_t + \Delta^{*,L}\left(L\left(\frac{S_2}{S_1}\right)-(V^*)^2\right)} =: M_{T_2}.$$

- Let $M_t := \mathbb{E}^0[M_{T_2}|\mathcal{F}_t]$. It is easy to check that $M_t = \mathcal{E}(L)_t$ with

$$dL_t = \sqrt{1-\rho^2}\sigma(a_t)\partial_a u^*(t, S_t, a_t)\, dW_t^{0,\perp}$$

- Girsanov $\Longrightarrow$ $(W^*, W^{*,\perp})$ is a standard $\mathbb{P}^*$-Brownian motion, where

$$W_t^* = W_t^0, \qquad W_t^{*,\perp} = W_t^{0,\perp} - \sqrt{1-\rho^2}\int_0^t \sigma(a_r)\partial_a u^*(r, S_r, a_r)\, dr.$$

- The model dynamics reads

$$\begin{aligned} \frac{dS_t}{S_t} &= a_t\, dW_t^* \\ da_t &= \left(b(a_t) + (1-\rho^2)\sigma(a_t)^2\partial_a u^*(t, S_t, a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2}dW_t^{*,\perp}\right) \end{aligned}$$

Recap: VIX-constrained martingale Schrödinger bridge

$$
\begin{aligned}
\frac{dS_t}{S_t} &= a_t\, dW_t^* \\
da_t &= \left(b(a_t) + (1-\rho^2)\sigma(a_t)^2 \partial_a u^*(t, S_t, a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2} dW_t^{*,\perp}\right)
\end{aligned}
$$

- Let $P := \sup_{u_1, u_V, u_2} \left\{ \sum_{i \in \{1,2,V\}} (\mu_i, u_i) - u(0, S_0, a_0) \right\}$ where u is solution to a nonlinear Hamilton-Jacobi-Bellman PDE:

$$
\begin{aligned}
u(T_2, s, a; \delta^L) &= u_2(s) + \delta^L L(s), \\
\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in (T_1, T_2), \\
\Phi(s, a) := \sup_{v \geq 0} \inf_{\delta^L \in \mathbb{R}} \Big\{ u_V(v) &- \delta^L (L(s) + v^2) + u(T_1, s, a; \delta^L) \Big\}, \\
u(T_1, s, a) &= u_1(s) + \Phi(s, a), \\
\partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in [0, T_1).
\end{aligned}
$$

- Assume $P < +\infty$ and (u_1^*, u_V^*, u_2^*) maximizes $P \longrightarrow u^*$

Recap: VIX-constrained martingale Schrödinger bridge

$$\frac{dS_t}{S_t} = a_t\, dW_t^*$$
$$da_t = \left(b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a u^*(t,S_t,a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2}dW_t^{*,\perp}\right)$$

- Optimal deltas:

$$\Delta_t^* = -\partial_s u^*(t,S_t,a_t) - \rho\frac{\sigma(a_t)}{a_t S_t}\partial_a u^*(t,S_t,a_t); \quad \Delta^{*,L} = \delta^{*,L}(S_1,a_1)$$

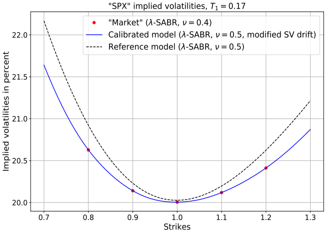
- The drift of (a_t) under $\mathbb{P}^*$ also reads as

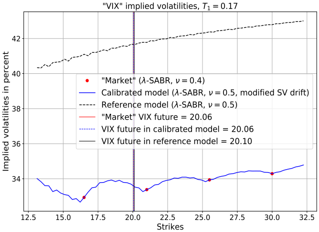
$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_1^*(S_1)+\int_t^{T_1}\Delta^*(r,S_r,a_r)dS_r+\Phi^*(S_1,a_1)}|S_t,a_t], \quad t\in[0,T_1],$$
$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_2^*(S_2)+\int_t^{T_2}\Delta^*(r,S_r,a_r)dS_r+\delta^{*,L}(S_1,a_1)L(S_2)}|S_t,a_t], \quad t\in[T_1,T_2].$$

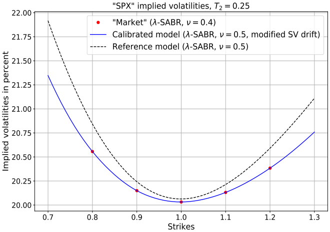
- It is path-dependent on $[T_1,T_2]$ so as to match the market VIX smile.
- If $P=+\infty$, then $\mathcal{P}=\emptyset$.

$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.4$, $\mathbb{P}_0 : \nu = 0.5$

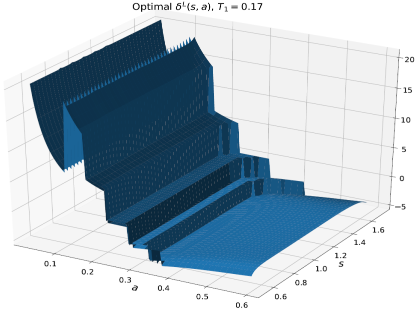
$$k = 1.5, \quad a_0 = m = 0.2, \quad \rho = 0$$

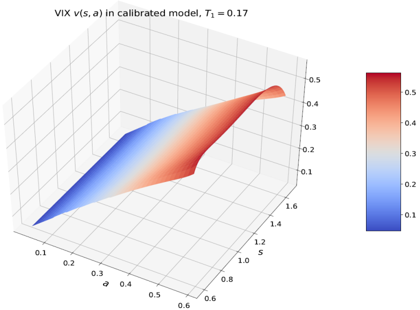




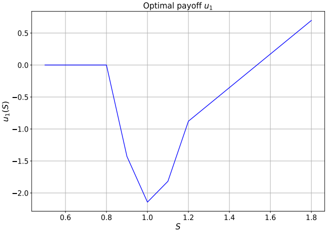


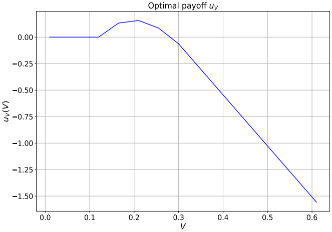
$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.4$, $\mathbb{P}_0 : \nu = 0.5$

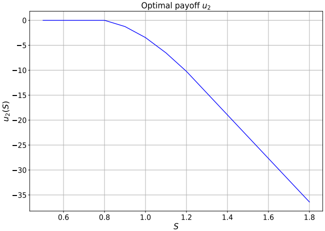




$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.4$, $\mathbb{P}_0 : \nu = 0.5$

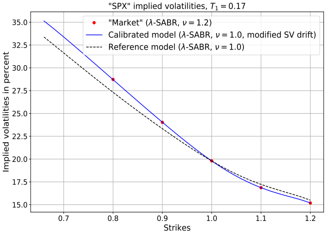


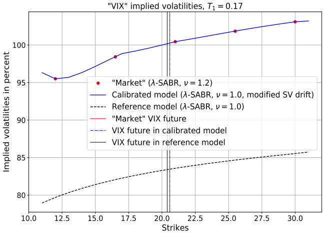


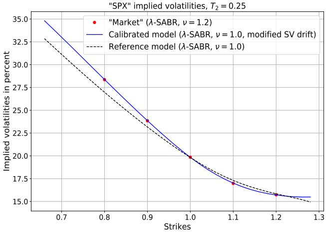


$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$

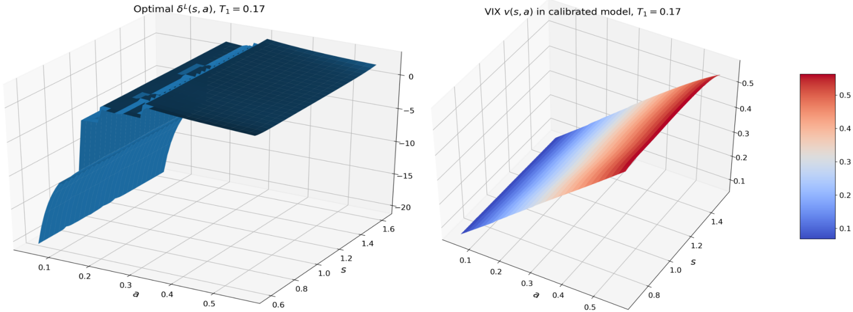
$$k = 1.5, \quad a_0 = m = 0.2, \quad \rho = -0.7$$



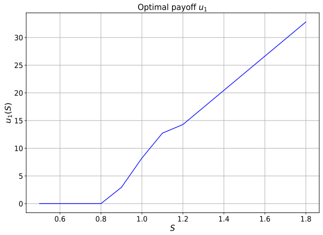


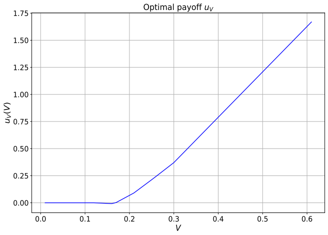


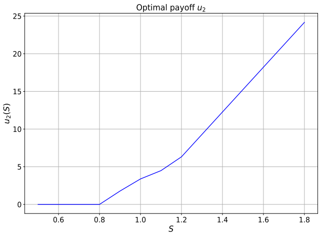
$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$



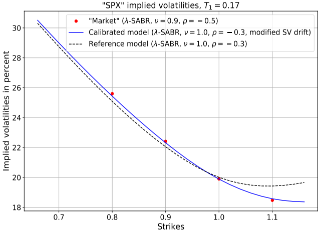
$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 1.2$, $\mathbb{P}_0 : \nu = 1$

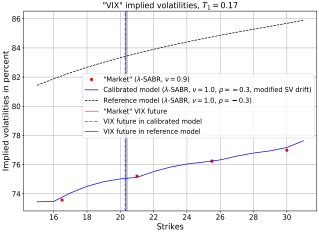


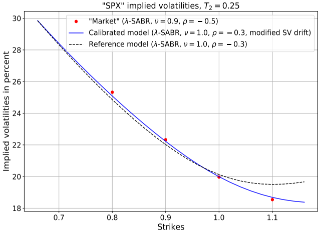




$da_t = -k(a_t - m)\,dt + \nu a_t\,dZ_t$. 'Market': $\nu = 0.9$, $\rho = -0.5$, $\mathbb{P}_0 : \nu = 1$, $\rho = -0.3$







Other approaches

- Guo-Loeper-Obłoj-Wang (2020): joint calibration via semimartingale optimal transport
  - More general cost function: volatilities and correlations are allowed to be modified from reference model
  - Model (S_t, Y_t) instead of (S_t, a_t) where Y is the price at t of the integrated variance over $[t, T_2]$
  - Terminal constraint on the semimartingale Y: $Y_{T_2} = 0$
- Cont-Kokholm (2013): Bergomi-like model with simultaneous jumps on SPX and VIX.
  - Best fit
  - An approximation of the VIX in the model is used
- Gatheral-Jusselin-Rosenbaum (2020): quadratic rough Heston volatility model.
  - Best fit
  - VIX smile well calibrated, not enough ATM SPX skew
- Fouque-Saporito (2018): Heston Stochastic Vol-of-Vol Model
- Pacati-Pompa-Renò (2018): displacement of multi-factor affine models with jumps (Heston++)
- Papanicolaou-Sircar (2014), Goutte-Amine-Pham (2017): regime-switching Heston model (with or without jumps)

Inversion of convex ordering in the VIX market

Quantitative Finance (September 2020)

Quantitative Finance, 2020
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Inversion of convex ordering in the VIX market

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(*Received 14 December 2019; accepted 3 April 2020*)

We investigate conditions for the existence of a *continuous* model on the S&P 500 index (SPX) that jointly calibrates to a *full surface* of SPX implied volatilities and to the VIX smiles. We present a novel approach based on the SPX smile calibration condition $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\text{lv}}^2(t, S_t)$. In the limiting case of instantaneous VIX, a novel application of martingale transport to finance shows that such model exists if and only if, for each time t, the local variance $\sigma_{\text{lv}}^2(t, S_t)$ is smaller than the instantaneous variance σ_t^2 in convex order. The real case of a 30-day VIX is more involved, as averaging over 30 days and projecting onto a filtration can undo convex ordering.

We show that in usual market conditions, and for reasonable smile extrapolations, the distribution of VIX_T^2 in the market local volatility model is *larger* than the market-implied distribution of VIX_T^2 in convex order for short maturities T, and that the two distributions are not rankable in convex order for intermediate maturities. In particular, a *necessary* condition for continuous models to jointly calibrate to the SPX and VIX markets is the *inversion of convex ordering* property: the fact that, even though associated local variances are smaller than instantaneous variances in convex order, the VIX

Continuous model on SPX calibrated to SPX options

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t, \qquad S_0 = x. \tag{4.1}$$

- Corresponding local volatility function σ_{loc}: $\sigma^2_{\text{loc}}(t, S_t) := \mathbb{E}[\sigma_t^2 | S_t]$.
- Corresponding local volatility model:

$$\frac{dS_t^{\text{loc}}}{S_t^{\text{loc}}} = \sigma_{\text{loc}}(t, S_t^{\text{loc}}) \, dW_t, \qquad S_0^{\text{loc}} = x.$$

- From Gyöngy (1986): $\forall t \geq 0, \quad S_t^{\text{loc}} \stackrel{(d)}{=} S_t$.
- Using Dupire (1994), we conclude that Model (4.1) is calibrated to the full SPX smile if and only if $\sigma_{\text{loc}} = \sigma_{\text{lv}}$ (market local volatility computed using Dupire's formula).
- Market local volatility model:

$$\frac{dS_t^{\text{lv}}}{S_t^{\text{lv}}} = \sigma_{\text{lv}}(t, S_t^{\text{lv}}) \, dW_t, \qquad S_0^{\text{lv}} = x.$$

VIX

- By definition, the (idealized) VIX at time $T \geq 0$ is the implied volatility of a 30 day log-contract on the SPX index starting at T. For continuous models (4.1), this translates into

$$\mathrm{VIX}_T^2 = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2 \, dt \middle| \mathcal{F}_T\right] = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\sigma_t^2 \middle| \mathcal{F}_T\right] dt.$$

- Since $\mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|\mathcal{F}_T] = \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$, $\mathrm{VIX}_{\mathrm{loc},T}$ satisfies

$$\mathrm{VIX}_{\mathrm{loc},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}] \, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}}) \, dt \middle| S_T^{\mathrm{loc}}\right].$$

- Similarly,

$$\mathrm{VIX}_{\mathrm{lv},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}})|S_T^{\mathrm{lv}}] \, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}}) \, dt \middle| S_T^{\mathrm{lv}}\right].$$

Reminder on convex order

- (The distributions of) two random variables X and Y are said to be in convex order if and only if, for any convex function f, $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$.
- Denoted by $X \leq_c Y$.
- Both distributions have same mean, but distribution of Y is more "spread" than that of X.
- **In financial terms: X and Y have the same forward value, but calls (puts) on Y are more expensive than calls (puts) on X** (dimension 1).

The case of instantaneous VIX: $\tau \to 0$

- Assume SV model is calibrated to the SPX smile: $\mathbb{E}[\sigma_t^2 | S_t] = \sigma_{\mathrm{lv}}^2(t, S_t)$.
- As observed by Dupire (2005), by conditional Jensen, $\sigma_{\mathrm{lv}}^2(t, S_t) \leq_c \sigma_t^2$, i.e.,

$$\text{mkt local var}_t \leq_c \text{instVIX}_t^2.$$

- Conversely, if mkt local var$_t \leq_c$ instVIX$_t^2$, there exists a jointly calibrating SPX/instVIX model (G., 2017).
- $\Longrightarrow$ **Convex order condition is necessary and sufficient for instVIX**.
- Proof uses a **new type of application of martingale transport to finance**: martingality constraint applies to (mkt local var$_t$, instVIX$_t^2$) at a single date, instead of (S_1, S_2).

The real VIX: $\tau = 30$ days

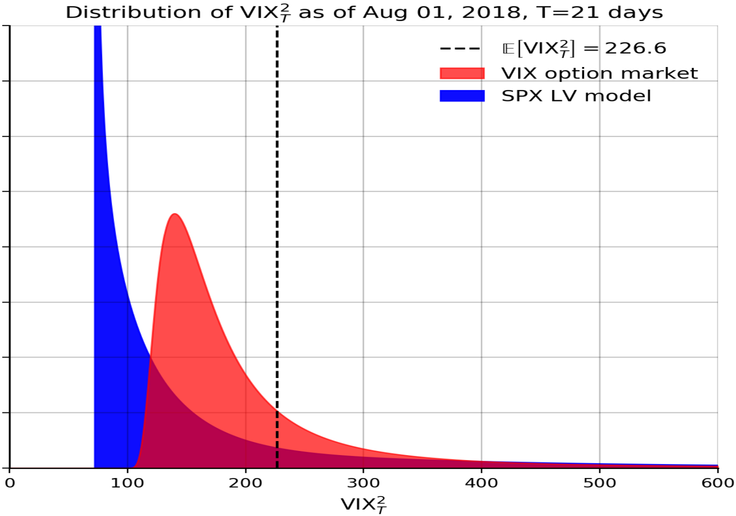
- In reality, squared VIX are not instantaneous variances but the **fair strikes** of **30-day** realized variances.
- Let us look at market data (August 1, 2018). We compare the market distributions of

$$\mathrm{VIX}^2_{\mathrm{lv},T} := \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S^{\mathrm{lv}}_t)\, dt \middle| S^{\mathrm{lv}}_T\right]$$

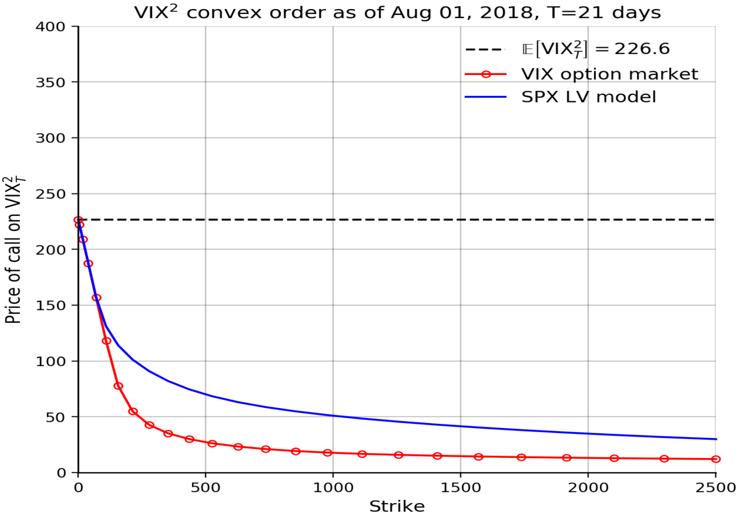
and

$$\mathrm{VIX}^2_{\mathrm{mkt},T} \quad \left(\longleftrightarrow \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\, dt \middle| \mathcal{F}_T\right]\right)$$

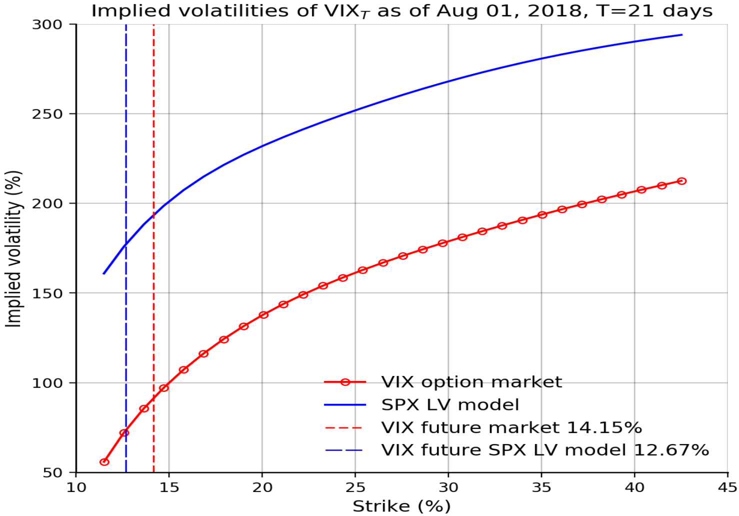
$T = 21$ days


Distribution of VIX$_T^2$ as of Aug 01, 2018, T=21 days
$\mathbb{E}[\text{VIX}_T^2] = 226.6$
VIX option market
SPX LV model
0
100
200
300
400
500
600
VIX$_T^2$

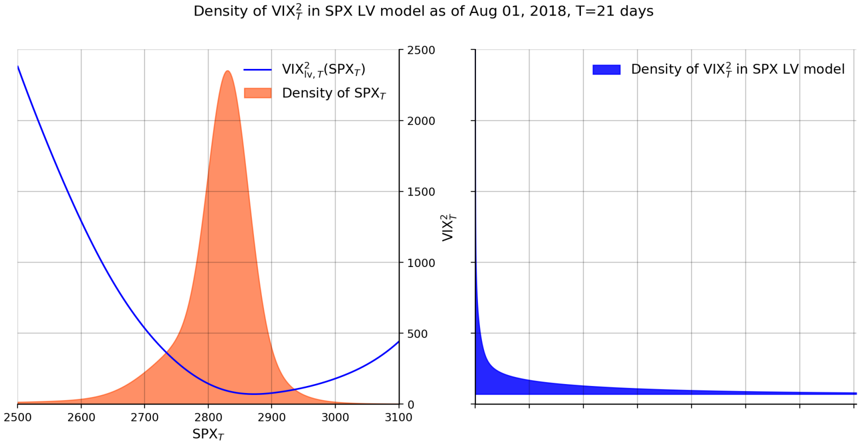
$T = 21$ days


VIX^2 convex order as of Aug 01, 2018, T=21 days
$\mathbb{E}[\text{VIX}_T^2] = 226.6$
VIX option market
SPX LV model
Price of call on VIX$_T^2$
Strike
400
350
300
250
200
150
100
50
0
0
500
1000
1500
2000
2500

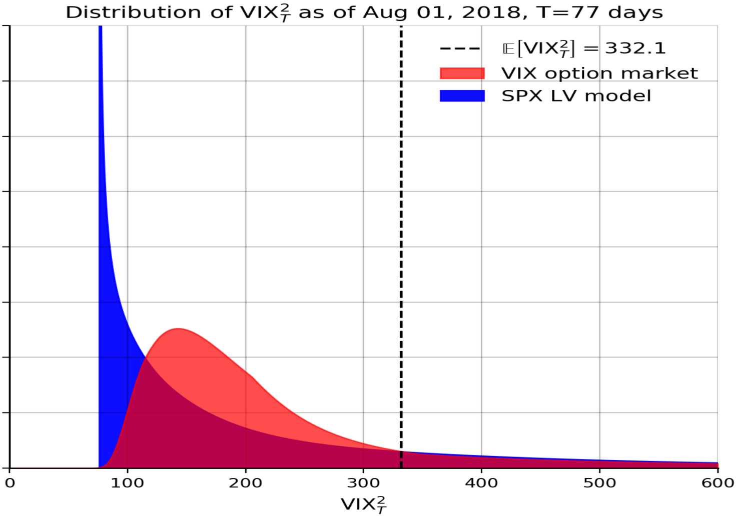
$T = 21$ days


Implied volatilities of VIX$_T$ as of Aug 01, 2018, T=21 days
Implied volatility (%)
Strike (%)
300
250
200
150
100
50
10
15
20
25
30
35
40
45
VIX option market
SPX LV model
VIX future market 14.15%
VIX future SPX LV model 12.67%

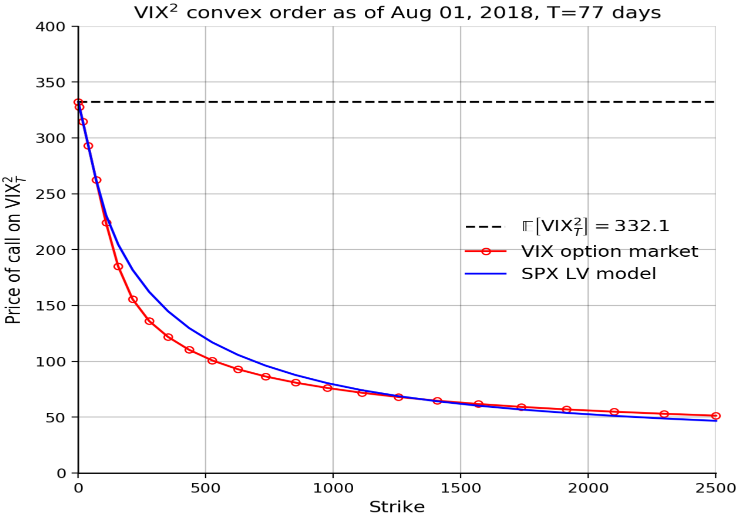
$\mathrm{VIX}^2_{\mathrm{lv},T}(S^{\mathrm{lv}}_T)$



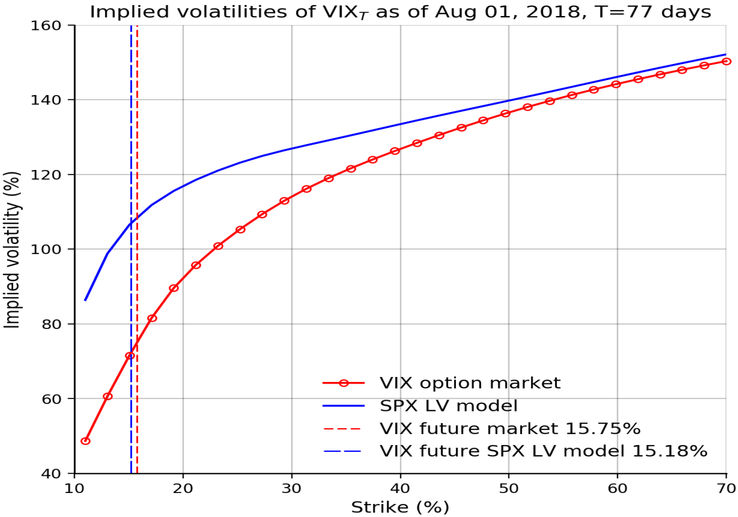
$T = 77$ days


Distribution of VIX_T^2 as of Aug 01, 2018, T=77 days
$\mathbb{E}[VIX_T^2] = 332.1$
VIX option market
SPX LV model
0
100
200
300
400
500
600
VIX_T^2

$T = 77$ days


VIX² convex order as of Aug 01, 2018, T=77 days
Price of call on VIX_T^2
Strike
0
50
100
150
200
250
300
350
400
0
500
1000
1500
2000
2500
$\mathbb{E}[VIX_T^2] = 332.1$
VIX option market
SPX LV model

$T = 77$ days


Implied volatilities of VIX$_T$ as of Aug 01, 2018, T=77 days
Implied volatility (%)
Strike (%)
160
140
120
100
80
60
40
10
20
30
40
50
60
70
VIX option market
SPX LV model
VIX future market 15.75%
VIX future SPX LV model 15.18%

Inversion of convex ordering

- **Inversion of convex ordering**: the fact that, for small T, $\mathrm{VIX}^2_{\mathrm{loc},T} \geq_c \mathrm{VIX}^2_T$ **despite the fact that for all** t**,** $\sigma^2_{\mathrm{loc}}(t, S_t) \leq_c \sigma^2_t$.
- A **necessary** condition for continuous models to jointly calibrate to the SPX and VIX markets.
- In the paper, we numerically show that when the spot-vol correlation is large enough in absolute value,
  - (a) traditional SV models with **large mean reversion**, and
  - (b) rough volatility models with **small Hurst exponent**

  satisfy the inversion of convex ordering property, and more generally can reproduce the market term-structure of convex ordering of the local and stochastic squared VIX.
- Not a sufficient condition though.
- Actually we have proved that **inversion of convex ordering can be produced by a continuous SV model**.
- In such models, for small T, $\mathrm{VIX}^2_{\mathrm{loc},T} >_c \mathrm{VIX}^2_T$ so ($x \mapsto \sqrt{x}$ concave)

$$\mathbb{E}[\mathrm{VIX}_T] > \mathbb{E}\left[\mathrm{VIX}_{\mathrm{loc},T}\right]:$$

Local volatility does NOT maximize the price of VIX futures.

SIAM J. Financial Math. 11(1):SC1–SC13, 2020 (with B. Acciaio)

SIAM J. FINANCIAL MATH.
Vol. 11, No. 1, pp. SC1–SC13

Short Communication: Inversion of Convex Ordering: Local Volatility Does Not Maximize the Price of VIX Futures*

Beatrice Acciaio† and Julien Guyon‡

Abstract. It has often been stated that, within the class of continuous stochastic volatility models calibrated to vanillas, the price of a VIX future is maximized by the Dupire local volatility model. In this article we prove that this statement is incorrect: we build a continuous stochastic volatility model in which a VIX future is *strictly more expensive* than in its associated local volatility model. More generally, in our model, strictly convex payoffs on a squared VIX are strictly cheaper than in the associated local volatility model. This corresponds to an *inversion of convex ordering* between local and stochastic variances, when moving from instantaneous variances to squared VIX, as convex payoffs on instantaneous variances are always cheaper in the local volatility model. We thus prove that this inversion of convex ordering, which is observed in the S&P 500 market for short VIX maturities, can be produced by a continuous stochastic volatility model. We also prove that the model can be extended so that, as suggested by market data, the convex ordering is preserved for long maturities.

Key words. VIX, VIX futures, stochastic volatility, local volatility, convex order, inversion of convex ordering

AMS subject classifications. 91G20, 91G80, 60H30

DOI. 10.1137/19M129303X

1. Introduction. For simplicity, let us assume zero interest rates, repos, and dividends. Let $\mathcal{F}_t$ denote the market information available up to time t. We consider continuous stochastic volatility models on the S&P 500 index (SPX) of the form

Thanks!

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Risk, April 2020

The joint S&P 500/Vix smile calibration puzzle solved

Since Vix options started trading in 2006, many researchers have tried to build a model that jointly and exactly calibrates to the prices of Standard & Poor's 500 options, Vix futures and Vix options. In this article, Julien Guyon solves this long-standing puzzle by casting it as a discrete-time dispersion-constrained martingale transport problem, which he solves in a non-parametric way using Sinkhorn's algorithm

Volatility indexes, such as the Vix index, do not just serve as market-implied indicators of volatility. Futures and options on these indexes are also widely used as risk management tools to hedge the volatility exposure of options portfolios. The existence of a liquid market for these futures and options has led to the need for models that jointly calibrate to the prices of options on the underlying asset and the prices of volatility derivatives. Without such models, financial institutions could possibly arbitrage each other: even market-making desks within the same institution could do so, eg, the Vix desk could arbitrage the S&P 500 (SPX) desk. By using models that fail to correctly incorporate the prices of the hedging instruments, such as SPX options, Vix futures and Vix options, exotic desks may misprice options, especially (but not only) those with payoffs that involve both the underlying and its volatility index.

For this reason, since Vix options began trading in 2006, many researchers and practitioners have tried to build a model that jointly and exactly calibrates to the prices of SPX futures, SPX options, Vix futures and Vix options. This is known to be a very challenging problem, especially for short maturities. In particular, the very large negative skew of short-term SPX options,

and Vix smiles: that the distribution of the Dupire market local variance be smaller than the distribution of the (instantaneous) Vix squared in the convex order, at all times. He also reported that for short maturities the distribution of the true Vix squared in the market local volatility model is actually larger than the market-implied distribution of the true Vix squared in the convex order. Guyon showed numerically that when the (typically negative) spot-vol correlation is large enough in absolute value, both (a) traditional stochastic volatility models with large mean reversion and (b) rough volatility models with a small Hurst exponent can reproduce this inversion of convex ordering. Acciaio & Guyon (2020) provide a mathematical proof that the inversion of convex ordering can be produced by continuous models. However, the inversion of convex ordering is only a necessary condition for the joint SPX/Vix calibration of continuous models; it is not sufficient.

Since it looks to be very difficult to jointly calibrate the SPX and Vix smiles with continuous models, many authors have incorporated jumps in the dynamics of the SPX: see references in Guyon (2019a). Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM Vix implied volatility. However, the attempts so far have only produced

Equations for $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$

$$
\begin{aligned}
\frac{\partial \Psi_{\bar\mu}}{\partial u_1(s_1)} = 0 : \quad & \forall s_1 > 0, & u_1(s_1) &= \Phi_1(s_1; u_V, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_V(v)} = 0 : \quad & \forall v \geq 0, & u_V(v) &= \Phi_V(v; u_1, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_2(s_2)} = 0 : \quad & \forall s_2 > 0, & u_2(s_2) &= \Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_S(s_1, v)} = 0 : \quad & \forall s_1 > 0,\ \forall v \geq 0, & 0 &= \Phi_{\Delta_S}(s_1, v; \Delta_S(s_1, v), \Delta_L(s_1, v)) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_L(s_1, v)} = 0 : \quad & \forall s_1 > 0,\ \forall v \geq 0, & 0 &= \Phi_{\Delta_L}(s_1, v; \Delta_S(s_1, v), \Delta_L(s_1, v))
\end{aligned}
$$

$$
\begin{aligned}
\Phi_1(s_1; u_V, \Delta_S, \Delta_L) &:= \ln \mu_1(s_1) - \ln\left(\int \bar\mu(s_1, dv, ds_2) e^{u_V(v) + u_2(s_2) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2)} \right) \\
\Phi_V(v; u_1, \Delta_S, \Delta_L) &:= \ln \mu_V(v) - \ln\left(\int \bar\mu(ds_1, v, ds_2) e^{u_1(s_1) + u_2(s_2) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2)} \right) \\
\Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) &:= \ln \mu_2(s_2) - \ln\left(\int \bar\mu(ds_1, dv, s_2) e^{u_1(s_1) + u_V(v) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2)} \right) \\
\Phi_{\Delta_S}(s_1, v; u_2, \delta_S, \delta_L) &:= \int \bar\mu(s_1, v, ds_2)(s_2 - s_1) e^{u_2(s_2) + \delta_S(s_2 - s_1) + \delta_L\left(L\left(\frac{s_2}{s_1}\right) - v^2\right)} \\
\Phi_{\Delta_L}(s_1, v; u_2, \delta_S, \delta_L) &:= \int \bar\mu(s_1, v, ds_2) \left(L\left(\frac{s_2}{s_1}\right) - v^2\right) e^{u_2(s_2) + \delta_S(s_2 - s_1) + \delta_L\left(L\left(\frac{s_2}{s_1}\right) - v^2\right)}.
\end{aligned}
$$

Implementation details

Practically, we consider market strikes $\mathcal{K} := (\mathcal{K}_1, \mathcal{K}_V, \mathcal{K}_2)$ and market prices (C^1_K, C^V_K, C^2_K) of vanilla options on S_1, V, and S_2, and we build the model

$$\mu^*_{\mathcal{K}}(ds_1, dv, ds_2) = \bar{\mu}(ds_1, dv, ds_2) e^{c^* + \Delta^{0*}_S s_1 + \Delta^{0*}_V v + \sum_{K\in\mathcal{K}_1} a^{1*}_K (s_1 - K)_+} \\ e^{\sum_{K\in\mathcal{K}_V} a^{V*}_K (v-K)_+ + \sum_{K\in\mathcal{K}_2} a^{2*}_K (s_2-K)_+ + \Delta^{*(S)}_S (s_1,v,s_2) + \Delta^{*(L)}_L (s_1,v,s_2)}$$

where $\theta^* := (c^*, \Delta^{0*}_S, \Delta^{0*}_V, a^{1*}, a^{V*}, a^{2*}, \Delta^*_S, \Delta^*_L)$ maximizes

$$\bar{\Psi}_{\bar{\mu},\mathcal{K}}(\theta) := c + \Delta^0_S S_0 + \Delta^0_V F_V + \sum_{K\in\mathcal{K}_1} a^1_K C^1_K + \sum_{K\in\mathcal{K}_V} a^V_K C^V_K + \sum_{K\in\mathcal{K}_2} a^2_K C^2_K$$

$$-\mathbb{E}^{\bar{\mu}}\left[e^{c + \Delta^0_S S_1 + \Delta^0_V V + \sum_{\mathcal{K}_1} a^1_K (S_1-K)_+ + \sum_{\mathcal{K}_V} a^V_K (V-K)_+ + \sum_{\mathcal{K}_2} a^2_K (S_2-K)_+ + \Delta^{(S)}_S (\dots) + \Delta^{(L)}_L (\dots)} \right]$$

over the set Θ of portfolios $\theta := (c, \Delta^0_S, \Delta^0_V, a^1, a^V, a^2, \Delta_S, \Delta_L)$ such that $c, \Delta^0_S, \Delta^0_V \in \mathbb{R}$, $a^1 \in \mathbb{R}^{\mathcal{K}_1}$, $a^V \in \mathbb{R}^{\mathcal{K}_V}$, $a^2 \in \mathbb{R}^{\mathcal{K}_2}$, and $\Delta_S, \Delta_L : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are measurable functions of (s_1, v).

Implementation details

- This corresponds to solving the entropy minimization problem

$$P_{\bar{\mu},\mathcal{K}} := \inf_{\mu \in \mathcal{P}(\mathcal{K})} H(\mu, \bar{\mu}) = \sup_{\theta \in \Theta} \bar{\Psi}_{\bar{\mu},\mathcal{K}}(\theta) =: D_{\bar{\mu},\mathcal{K}}$$

where $\mathcal{P}(\mathcal{K})$ denotes the set of probability measures μ on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ such that

$$\mathbb{E}^{\mu}[S_1] = S_0, \quad \mathbb{E}^{\mu}[V] = F_V, \quad \forall K \in \mathcal{K}_1,\ \mathbb{E}^{\mu}\left[(S_1 - K)_+\right] = C^1_K,$$
$$\forall K \in \mathcal{K}_V,\ \mathbb{E}^{\mu}\left[(V - K)_+\right] = C^V_K, \quad \forall K \in \mathcal{K}_2,\ \mathbb{E}^{\mu}\left[(S_2 - K)_+\right] = C^2_K,$$
$$\mathbb{E}^{\mu}\left[S_2 | S_1, V\right] = S_1, \quad \mathbb{E}^{\mu}\left[L\left(\frac{S_2}{S_1}\right) \middle| S_1, V\right] = V^2.$$

- One can directly check that model $\mu^*_{\mathcal{K}}$ is an arbitrage-free model that jointly calibrates the prices of SPX futures, options, VIX future, and VIX options. Indeed, if $\bar{\Psi}_{\bar{\mu},\mathcal{K}}$ reaches its maximum at θ^*, then θ^* is solution to $\frac{\partial \bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial \theta_i}(\theta) = 0$:

Implementation details

$$\bar{\Psi}_{\bar{\mu},\mathcal{K}}(\theta) := c + \Delta_S^0 S_0 + \Delta_V^0 F_V + \sum_{K\in\mathcal{K}_1} a_K^1 C_K^1 + \sum_{K\in\mathcal{K}_V} a_K^V C_K^V + \sum_{K\in\mathcal{K}_2} a_K^2 C_K^2$$

$$-\mathbb{E}^{\bar{\mu}}\left[e^{c+\Delta_S^0 S_1+\Delta_V^0 V+\sum_{\mathcal{K}_1} a_K^1(S_1-K)_+ +\sum_{\mathcal{K}_V} a_K^V(V-K)_+ +\sum_{\mathcal{K}_2} a_K^2(S_2-K)_+ +\Delta_S^{(S)}(\dots)+\Delta_L^{(L)}(\dots)}\right]$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial c} = 0 : \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = 1 \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_S^0} = 0 : \mathbb{E}^{\bar{\mu}}\left[S_1\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = S_0$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_V^0} = 0 : \mathbb{E}^{\bar{\mu}}\left[V\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = F_V \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^1} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_1-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^1$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^V} = 0 : \mathbb{E}^{\bar{\mu}}\left[(V-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^V \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^2} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^2$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_S(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-S_1)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_L(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$

SLV calibrated to SPX: VIX smile (Aug 1, 2018)

- All continuous models on SPX that are calibrated to full SPX smile are of the form:

$$\frac{dS_t}{S_t} = \frac{a_t}{\sqrt{\mathbb{E}[a_t^2|S_t]}}\sigma_{\text{loc}}(t, S_t)\, dW_t.$$

- They are stochastic local volatility (SLV) models

$$\frac{dS_t}{S_t} = a_t \ell(t, S_t)\, dW_t$$

with stochastic volatility (SV) (a_t) and leverage function

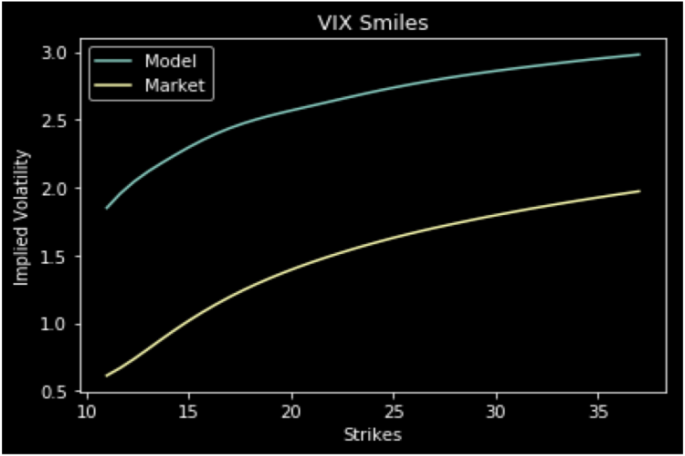
$$\ell(t, S_t) = \frac{\sigma_{\text{loc}}(t, S_t)}{\sqrt{\mathbb{E}[a_t^2|S_t]}}.$$

- In those models ($\tau := 30$ days)

$$\text{VIX}_T^2 \quad = \quad \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\frac{a_t^2}{\mathbb{E}[a_t^2|S_t]}\sigma_{\text{loc}}^2(t, S_t)\middle|\mathcal{F}_T\right] dt.$$

- Optimize SV parameters to fit VIX options.

SLV calibrated to SPX: VIX smile, $T = 21$ days (Aug 1, 2018)



SLV model, SV $=$ skewed 2-factor Bergomi model
SV params optimized to fit VIX smile