A BDDC ALGORITHM FOR RAVIART-THOMAS VECTOR FIELDS

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Abstract. A BDDC preconditioner is defined by a coarse component, expressed in terms of primal constraints and a weighted average across the interface between the subdomains, and local components given in terms of Schur complements of local subdomain problems. A BDDC method for vector field problems discretized with Raviart-Thomas finite elements is introduced. Our method is based on a new type of weighted average developed to deal with more than one variable coefficient. A bound on the condition number of the preconditioned linear system is also provided which is independent of the values and jumps of the coefficients across the interface and has a polylogarithmic condition number bound in terms of the number of degrees of freedom of the individual subdomains. Numerical experiments for two and three dimensional problems are also presented, which support the theory and show the effectiveness of our algorithm even for certain problems not covered by our theory.

Key words. domain decomposition, BDDC preconditioner, Raviart-Thomas finite elements

AMS subject classifications. 65F08, 65F10, 65N30, 65N55

1. Introduction. Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$. We will work with the Hilbert space $H(\mathrm{div};\Omega)$, the subspace of vector valued functions $\boldsymbol{u} \in (L^2(\Omega))^3$ with $\mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)$. The space $H_0(\mathrm{div};\Omega)$ is the subspace of $H(\mathrm{div};\Omega)$ with a vanishing normal component on the boundary $\partial\Omega$.

We will consider the following problem: Find $\boldsymbol{u} \in H_0(\mathrm{div};\Omega)$, such that

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega (\alpha\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} + \beta\,\boldsymbol{u}\cdot\boldsymbol{v})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx, \qquad \boldsymbol{v} \in H_0(\mathrm{div};\Omega). \tag{1.1}$$

We will assume that the coefficient α is a nonnegative $L^\infty(\Omega)$-function, that β is a strictly positive $L^\infty(\Omega)$-function, and that the right hand side $\boldsymbol{f} \in (L^2(\Omega))^3$. We note that the norm of $\boldsymbol{u} \in H(\mathrm{div};\Omega)$ for a domain with unit diameter is given by $(a(\boldsymbol{u},\boldsymbol{u}))^{1/2}$ with $\alpha \equiv 1$ and $\beta \equiv 1$.

The bilinear form (1.1) arises from the following boundary value problem:

$$\begin{aligned} L\boldsymbol{u} := -\mathbf{grad}\,(\alpha\,\mathrm{div}\,\boldsymbol{u}) + \beta\,\boldsymbol{u} &= \boldsymbol{f} \text{ in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

Here, $\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$. The boundary value problem (1.2) is equivalent to a mixed formulation or a first order system least-squares problem as in [9]. There are also other applications of $H(\mathrm{div})$, e.g., in iterative solvers for the Reissner-Mindlin plate and the sequential regularization method for the Navier-Stokes equations. For more details, see [1, 28].

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Domain decomposition methods of iterative substructuring type for solving large linear algebraic systems originating from elliptic partial differential equations have been studied extensively. They are all preconditioned Krylov space methods; see [44]. Among them, the balancing Neumann-Neumann (BNN) and the finite element tearing and interconnecting (FETI) algorithms have proven quite successful; see, e.g., [14,16,17,24,29]. The balancing domain decomposition by constraints (BDDC) methods, introduced in [10], are modified BNN methods with the global component of the preconditioner determined by a set of primal continuity constraints between the substructures. For a pioneering analysis for scalar elliptic problems, see [30,31].

The BDDC methods are closely related to the dual-primal FETI (FETI-DP) methods; see [15,32] and the same can be said about the earlier BNN and one-level FETI methods. Thus the spectra relevant to the performance of a BDDC and a FETI-DP algorithm will be the same, except possibly for eigenvalues of 0 and 1, for the same set of primal constraints; see [7,27,31]. Hence, we can use results for BDDC methods to obtain results for FETI-DP methods and vice versa.

The main purpose of this paper is to construct and analyze a BDDC preconditioner for vector field problems discretized with Raviart-Thomas finite elements. Iterative substructuring methods for Raviart-Thomas problems were first considered in [52] and we will use several auxiliary results from that study in the analysis of our method. BNN, FETI, and FETI-DP methods for these types of problems have been developed in [40,42,43]. Overlapping Schwarz methods have also been introduced for vector field problems; see [2,19,35,36,41]. Other methods such as multigrid methods have been applied successfully in [3,18,23].

BDDC methods have also been widely extended to other problems such as flow in porous media in [45,46], incompressible Stokes equations in [26], Reissner-Mindlin plate models in [4,25], and advection-diffusion problems in [50]. Multilevel BDDC methods were introduced in [10,47–49] and other discretization methods, e.g., spectral element methods and discontinuous Galerkin methods, have been considered in [13,37, 38]. Recently, there has also been pioneering work on isogeometric element problems; see [5].

In the construction of a BDDC preconditioners, a set of primal constraints and a weighted averaging technique have to be chosen and these choices will very directly affect the rate of convergence. Effective primal constraints are very simple for the Raviart-Thomas elements; we need only choose the average value of the normal component over the subdomain faces as primal variables. However, the choice of averaging is much more intricate. We will use a new type of weighted averaging technique introduced in [12] for three dimensional $H(\mathbf{curl})$ problems.

In several previous studies on domain decomposition methods for vector field problems, see [40,42,43,52], the bound on the condition number of the preconditioned linear system depends on the ratio of the coefficients α and β and the diameters of the subdomains. This defect has been removed in several recent studies. Among them is a paper on an iterative substructuring method for two dimensional problems posed in $H(\mathbf{curl})$; see [11]. In addition, a BDDC algorithm for three-dimensional problems in $H(\mathbf{curl})$ has been considered in [12]. An overlapping Schwarz method for three-dimensional $H(\mathrm{div})$ problems has also been developed; see [36]. However, we know of no previous full analysis of BDDC or FETI-DP type methods for three-dimensional $H(\mathrm{div})$ problems. We will provide a BDDC method with an upper bound on the condition number which is indepedent of the values and jumps of the coefficients across the interface and provide a condition number bound which is polylogarithmic in the number of degrees of freedom of the individual subdomains.

The rest of this paper is organized as follows. In section 2, we introduce some standard Sobolev spaces, a finite element approximation based on Raviart-Thomas elements, and decompositions of the interface spaces. We introduce our BDDC algorithms for an interface problem and define various operators used to describe the algorithms in section 3. We next provide some auxiliary results and a proof of our main result in section 4. Finally, section 5 contains results of numerical experiments, which support our findings.

2. Function and finite element spaces.

2.1. Continuous spaces. We will use the Sobolev spaces $H^1(\Omega)$ and its trace space $H^{1/2}(\partial\Omega)$ and their norms and seminorms for bounded domains. Let H be the diameter of Ω. Then,

$$\|u\|_{1;\Omega}^2 := |u|_{1;\Omega}^2 + \frac{1}{H^2}\|u\|_{0;\Omega}^2, \qquad \|u\|_{1/2;\partial\Omega}^2 := |u|_{1/2;\partial\Omega}^2 + \frac{1}{H}\|u\|_{0;\partial\Omega}^2,$$

where the L^2-norm $\|\cdot\|_{0;\Omega}$ and the seminorms $|\cdot|_{1;\Omega}$ and $|\cdot|_{1/2;\partial\Omega}$ are defined by $\|u\|_{0;\Omega}^2 := \int_\Omega |u|^2\,dx$, $|u|_{1;\Omega}^2 := \int_\Omega |\nabla u|^2\,dx$ and

$$|u|_{1/2;\partial\Omega}^2 := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^3}\,dxdy,$$

respectively. The weights for the L^2-terms result from the standard definitions of the norms for a domain of diameter 1 and a dilation. We can also easily extend these definitions to vector-valued cases.

The space $H(\mathrm{div}\,;\Omega)$ is defined by

$$H(\mathrm{div}\,;\Omega) := \{\boldsymbol{u} \in (L^2(\Omega))^3 \,|\, \mathrm{div}\,\boldsymbol{u} \in L^2(\Omega)\}$$

with the scaled graph norm:

$$\|\boldsymbol{u}\|_{\mathrm{div};\Omega}^2 := \|\mathrm{div}\,\boldsymbol{u}\|_{0;\Omega}^2 + \frac{1}{H^2}\|\boldsymbol{u}\|_{0;\Omega}^2.$$

The normal component of any $\boldsymbol{u} \in H(\mathrm{div};\Omega)$ belongs to $H^{-1/2}(\partial\Omega)$; see [8,33]. The norm for the space $H^{-1/2}(\partial\Omega)$ is given by

$$\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{-1/2;\partial\Omega} := \sup_{\phi\in H^{1/2}(\partial\Omega),\phi\neq 0} \frac{\langle \boldsymbol{u}\cdot\boldsymbol{n}, \phi\rangle}{\|\phi\|_{1/2;\partial\Omega}}.$$

The angle brackets stand for the duality product of $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. We have the following trace theorem.

LEMMA 2.1. *There exists a constant C, which is independent of the diameter of Ω, such that, for all $\boldsymbol{u} \in H(\mathrm{div};\Omega)$,*

$$\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{-1/2;\partial\Omega}^2 \le C(H^2\|\mathrm{div}\,\boldsymbol{u}\|_{0;\Omega}^2 + \|\boldsymbol{u}\|_{0;\Omega}^2).$$

Proof. This follows directly from Green's identity on a domain with unit diameter and by applying a dilation; see [52, Lemma 2.1]. ∎

In developing our theory, we also need to work with the $H(\mathbf{curl}\,;\Omega)$ space defined by

$$H(\mathbf{curl}\,;\Omega) := \{\boldsymbol{u} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\boldsymbol{u} \in (L^2(\Omega))^3\}$$

with the scaled graph norm:

$$\|\boldsymbol{u}\|^2_{\mathbf{curl};\Omega} := \|\mathbf{curl}\,\boldsymbol{u}\|^2_{0;\Omega} + \frac{1}{H^2}\,\|\boldsymbol{u}\|^2_{0;\Omega}\,.$$

We finally introduce $H^1_0(\Omega)$, $H_0(\mathrm{div}\,;\Omega)$, and $H_0(\mathbf{curl}\,;\Omega)$ as the subspaces of $H^1(\Omega)$, $H(\mathrm{div}\,;\Omega)$, and $H(\mathbf{curl}\,;\Omega)$ with a vanishing boundary value, a vanishing normal component, and a vanishing tangential component on $\partial\Omega$, respectively.

REMARK 2.1. *The curl operator for two dimensions is just a simple rotation of the divergence operator. In two dimensions, we therefore can use results for* $H(\mathrm{div}\,;\Omega)$ *to obtain results for* $H(\mathbf{curl}\,;\Omega)$ *and vice versa.*

2.2. Finite element spaces. In this paper, we will only develop our theory for tetrahedral elements but we note that our results are equally valid for hexahedral elements. We first introduce a triangulation $\mathcal{T}_h$ of Ω of tetrahedral elements. We then decompose the domain Ω into N nonoverlapping subdomains Ω_i of diameter H_i. We assume that each subdomain Ω_i is a union of elements of the triangulation $\mathcal{T}_h$ and that each Ω_i is simply connected and has a connected boundary. Later, when we develop our theory, we will introduce additional assumptions on the subdomains. We also assume that the triangulation $\mathcal{T}_h$ is shape regular with nodes matching across the interface between the subdomains. The smallest diameter of the elements of Ω_i is denoted by h_i. We will use the fraction H/h in our estimates, to denote

$$H/h := \max_{1\le i\le N}\{H_i/h_i\}\,.$$

We also define the interface Γ by

$$\Gamma := \left(\bigcup_{i=0}^{N}\partial\Omega_i\right)\backslash\partial\Omega$$

and the local interfaces Γ_i by

$$\Gamma_i := \Gamma\cap\partial\Omega_i.$$

We will consider the lowest order Raviart-Thomas and Nédélec elements on the mesh $\mathcal{T}_h$; see [8, Chapter 3] and [34]. The Raviart-Thomas elements are conforming in $H(\mathrm{div}\,;\Omega)$ and those of lowest order are defined by

$$W := \{\boldsymbol{u} \mid \boldsymbol{u}_{|K} \in \mathcal{RT}(K), K\in\mathcal{T}_h \text{ and } \boldsymbol{u}\in H(\mathrm{div}\,;\Omega)\},$$

where the shape function $\mathcal{RT}(K)$ is given by four scalar parameters

$$\mathcal{RT}(K) := \begin{pmatrix} a_1\\ a_2\\ a_3\end{pmatrix} + b\begin{pmatrix} x\\ y\\ z\end{pmatrix}$$

for a tetrahedral element. The degrees of freedom for an element K in $\mathcal{T}_h$ are given by

$$\lambda_f\,(\boldsymbol{u}) := \frac{1}{|f|}\int_f \boldsymbol{u}\cdot\boldsymbol{n}\;ds, \qquad f\subset\partial K,$$

i.e., the average values of the normal components over the faces of the elements. These four values determine a_1, a_2, a_3, and b. We note that the direction of the normal of a subdomain face can be fixed arbitrarily but that we should choose the same direction for all element faces of any subdomain face. The basis functions of the lowest order Raviart-Thomas element space are supported in two elements of $\mathcal{T}_h$, and with the normal component equal to 1 for one of the two elements and -1 for the other on a specified face while vanishing on all others.

The l^2-norm of the vector of the coefficients $\lambda_f(\boldsymbol{u})$ can be used to estimate the L^2-norm of $\boldsymbol{u}$; the proof of the following lemma is elementary and a simple modification of [39, Proposition 6.3.1].

LEMMA 2.2. *Let $K \in \mathcal{T}_h$. Then, there exist strictly positive constants, c and C, depending only on the aspect ratio of K, such that for all $\boldsymbol{u} \in W$,*

$$c \sum_{f \subset \partial K} h_f^3 \lambda_f\left(\boldsymbol{u}\right)^2 \leq \|\boldsymbol{u}\|_{0;K}^2 \leq C \sum_{f \subset \partial K} h_f^3 \lambda_f\left(\boldsymbol{u}\right)^2 \tag{2.1}$$

and

$$\|\operatorname{div} \boldsymbol{u}\|_{0;K}^2 \leq C \sum_{f \subset \partial K} h_f \lambda_f\left(\boldsymbol{u}\right)^2, \tag{2.2}$$

where h_f is the diameter of f.

The following lemma follows directly from Lemma 2.2.

LEMMA 2.3 (inverse inequality). *Let $K \in \mathcal{T}_h$. Then, there exist a constant C, depending only on the aspect ratio of K, such that for all $\boldsymbol{u} \in W$,*

$$h_K \left\|\operatorname{div} \boldsymbol{u}\right\|_{0;K} \leq C \left\|\boldsymbol{u}\right\|_{0;K}, \tag{2.3}$$

where h_K is the diameter of K.

We also need $\widehat{W}_0$, the finite element counterpart of $H_0(\operatorname{div};\Omega)$:

$$\widehat{W}_0(\Omega) := W(\Omega) \cap H_0(\operatorname{div};\Omega).$$

We will now consider the variational problem (1.1). We obtain the stiffness matrix A by restricting this problem to $\widehat{W}_0$; A is symmetric and positive definite.

When developing our theory, we will need several additional spaces. Let S be the space of continuous, piecewise linear functions on the tetrahedral elements, and let S_0 be the subspace of elements of S which vanish on $\partial\Omega$. Let Q be the space of piecewise constant functions on the same elements. Finally, let X be the space of the lowest order Nédélec elements. We recall that the Lagrange P_1, Raviart-Thomas, and Nédélec spaces are conforming finite element spaces in H^1, $H(\mathrm{div})$, and $H(\mathbf{curl})$, respectively.

Let $\mathcal{V}_h$ and $\mathcal{F}_h$ be the set of vertices and faces of $\mathcal{T}_h$, respectively. The interpolation operators I_h, and Π_h^{RT} for sufficiently smooth functions $u \in H^1$ and $\boldsymbol{v} \in H(\mathrm{div})$ onto S and W, respectively, are defined as follows:

$$I_h u := \sum_{p \in \mathcal{V}_h} u(p) \phi_p^P \text{ and } \Pi_h^{RT} \boldsymbol{v} := \sum_{f \in \mathcal{F}_h} \lambda_f(\boldsymbol{v}) \boldsymbol{\phi}_f^{RT},$$

where ϕ_p^P and $\boldsymbol{\phi}_f^{RT}$ are the basis functions of P_1 and Raviart-Thomas finite elements associated with the node p and the element face f, respectively. We also denote by Π_h the projection operator from L^2 onto Q.

We finally recall the following error estimate for the Raviart-Thomas interpolation operator and a commuting property.

LEMMA 2.4. *For any* $\boldsymbol{u} \in (H^1(\Omega)^3)$, *we have*

$$\left\|\boldsymbol{u} - \Pi_h^{RT}\boldsymbol{u}\right\|_{0;\Omega} \leq Ch\left|\boldsymbol{u}\right|_{1;\Omega}. \tag{2.4}$$

Proof. See [6, Lemma 5.5]. □

LEMMA 2.5. *Let* $\boldsymbol{u}$ *be sufficiently regular. Then, the following commuting property holds:*

$$\operatorname{div}\left(\Pi_h^{RT}\boldsymbol{u}\right) = \Pi_h\left(\operatorname{div}\boldsymbol{u}\right). \tag{2.5}$$

Proof. See [6, Property 5.3]. □

We note that the commuting property (2.5) is a part of the discrete de Rham diagram described, e.g., in [33, section 5.7].

2.3. The discrete problem. The description of the BDDC algorithm and its analysis require the introduction of a number of spaces. Let $W^{(i)}$ be the space of the lowest order Raviart-Thomas finite elements on Ω_i with a zero normal component on $\partial\Omega \cap \partial\Omega_i$. We decompose $W^{(i)}$ into two subspaces, an interior space $W_I^{(i)}$ and an interface space $W_\Gamma^{(i)}$. The interface space $W_\Gamma^{(i)}$ is then decomposed into a primal space $W_\Pi^{(i)}$ and a dual space $W_\Delta^{(i)}$. Hence, we have the following decompositions:

$$W^{(i)} := W_I^{(i)} \oplus W_\Gamma^{(i)} := W_I^{(i)} \oplus W_\Delta^{(i)} \oplus W_\Pi^{(i)}.$$

We will also use the following product spaces:

$$W_0 := \prod_{i=1}^{N} W^{(i)},\ W_I := \prod_{i=1}^{N} W_I^{(i)},\ W_\Gamma := \prod_{i=1}^{N} W_\Gamma^{(i)},$$

and

$$W_\Delta := \prod_{i=1}^{N} W_\Delta^{(i)},\ W_\Pi := \prod_{i=1}^{N} W_\Pi^{(i)}.$$

We then have

$$W_0 = W_I \oplus W_\Gamma = W_I \oplus W_\Delta \oplus W_\Pi.$$

In general, the functions in W_Γ have discontinuous normal components across the interface while those of the finite element solutions are continuous. We denote the subspace with continuous normal components by $\widehat{W}_\Gamma$. We also consider a space $\widetilde{W}_\Gamma$, for which all the primal constraints are enforced. We can then decompose $\widehat{W}_\Gamma$ and $\widetilde{W}_\Gamma$ into $\widehat{W}_\Delta \oplus \widehat{W}_\Pi$ and $W_\Delta \oplus \widehat{W}_\Pi$, respectively, where $\widehat{W}_\Delta$ is the continuous dual variable subspace and $\widehat{W}_\Pi$ is the continuous primal variable subspace.

We can now obtain the local stiffness matrix $A^{(i)}$ by restricting the bilinear form to Ω_i and replacing $H(\operatorname{div};\Omega_i)$ by the finite element space $W^{(i)}$. But before we do so, it is convenient to make a change of variables by introducing a basis for the primal degrees of freedom and a complementary basis for the dual subspace $W_\Delta^{(i)}$. Here we can follow the recipes of [27, subsection 3.3] closely. For our problem, the only primal variables will be the averages of the normal component over the subdomain faces, and after our change of variables, the complementary dual subspace will be represented by elements for which the same averages vanish. We note that there is also evidence that such a change of variables enhances the numerical stability of BDDC and FETI-DP algorithms; see [22].

After this change of variable, the restriction of our problem to the subdomain Ω_i can be written in terms of a local stiffness matrix $A^{(i)}$ as

$$A^{(i)} \begin{bmatrix} \boldsymbol{u}_I^{(i)} \\ \boldsymbol{u}_\Delta^{(i)} \\ \boldsymbol{u}_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_I^{(i)} \\ \boldsymbol{u}_\Delta^{(i)} \\ \boldsymbol{u}_\Pi^{(i)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_I^{(i)} \\ \boldsymbol{f}_\Delta^{(i)} \\ \boldsymbol{f}_\Pi^{(i)} \end{bmatrix}, \tag{2.6}$$

where $\boldsymbol{u}_I^{(i)} \in W_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)} \in W_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)} \in W_\Pi^{(i)}$.

We can then obtain the global linear system of algebraic equations by assembling the local subdomain problems:

$$A \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \boldsymbol{u}_\Pi \end{bmatrix} = \begin{bmatrix} A_{II} & A_{I\Delta} & A_{I\Pi} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta\Pi} \\ A_{\Pi I} & A_{\Pi\Delta} & A_{\Pi\Pi} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \boldsymbol{u}_\Pi \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \end{bmatrix}, \tag{2.7}$$

where $\boldsymbol{u}_I \in W_I$, $\boldsymbol{u}_\Delta \in \widehat{W}_\Delta$, and $\boldsymbol{u}_\Pi \in \widehat{W}_\Pi$.

3. The BDDC methods.

3.1. Some useful operators. We will now define several operators which perform restrictions, extensions, scalings, and averaging between different spaces. They will be used to present our algorithm and in our proofs. We first consider the restriction operators. $R_\Gamma^{(i)}$ maps the space $\widehat{W}_\Gamma$ to the subdomain subspace $W_\Gamma^{(i)}$. Similarly, we can define $\overline{R}_\Gamma^{(i)} : \widetilde{W}_\Gamma \to W_\Gamma^{(i)}$. Moreover, $R_\Delta^{(i)} : W_\Delta \to W_\Delta^{(i)}$ and $R_\Pi^{(i)} : \widehat{W}_\Pi \to W_\Pi^{(i)}$ map global interface vectors defined on Γ to their components on Γ_i. $\widetilde{R}_{\Gamma\Delta}$ and $\widetilde{R}_{\Gamma\Pi}$ are the restriction operators from the intermediate space $\widetilde{W}_\Gamma$ to W_Δ and $\widehat{W}_\Pi$, respectively. Similarly, we can define the restriction operator $R_{\Gamma\Delta}^{(i)}$ from $W_\Gamma^{(i)}$ to $W_\Delta^{(i)}$. R_Γ and $\overline{R}_\Gamma$ are the direct sums of the $R_\Gamma^{(i)}$ and $\overline{R}_\Gamma^{(i)}$, respectively. Furthermore, $\widetilde{R}_\Gamma : \widehat{W}_\Gamma \to \widetilde{W}_\Gamma$ is the direct sum of $\widehat{R}_\Pi$ and the $\widehat{R}_\Delta^{(i)}$, where $\widehat{R}_\Pi$ represents the restriction from $\widehat{W}_\Gamma$ to $\widehat{W}_\Pi$ and $\widehat{R}_\Delta^{(i)}$ maps the space $\widehat{W}_\Gamma$ into $W_\Delta^{(i)}$.

We next introduce scaling matrices, $D^{(i)}$, acting on the degrees of freedom associated with the Γ_i. They are combined into a block diagonal matrix and should provide a discrete partition of unity, i.e.,

$$R_\Gamma^T \begin{bmatrix} D^{(1)} & & & \\ & D^{(2)} & & \\ & & \ddots & \\ & & & D^{(N)} \end{bmatrix} R_\Gamma = I. \tag{3.1}$$

We can now define a scaled operator $R_{D,\Gamma}^{(i)} := D^{(i)} R_\Gamma^{(i)}$ by pre-multiplying $R_\Gamma^{(i)}$ by the scaling matrix $D^{(i)}$. Another locally scaled operator $\widetilde{R}_{D,\Delta}^{(i)}$ is defined by $R_{\Gamma\Delta}^{(i)} R_{D,\Gamma}^{(i)}$. We next consider a globally scaled operator $\widetilde{R}_{D,\Gamma}$ defined by the direct sum of $\widehat{R}_\Pi$

and the $\widetilde{R}^{(i)}_{D,\Delta}$. We note that

$$\widetilde{R}_\Gamma^T \widetilde{R}_{D,\Gamma} = \widetilde{R}_{D,\Gamma}^T \widetilde{R}_\Gamma = I. \tag{3.2}$$

Finally, we introduce an averaging operator $E_D : \widetilde{W}_\Gamma \to \widehat{W}_\Gamma$ by

$$E_D := \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^T. \tag{3.3}$$

This operator, which is a projection, provides a weighted average across the interface Γ. We will provide details on our choice of scaling matrices in subsection 3.3.

3.2. Schur complements and a reduced interface problem. Before we introduce the BDDC algorithm, we eliminate all interior unknowns locally by using direct solvers. After this step, we obtain the local Schur complements:

$$S_\Gamma^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{I\Gamma}^{(i)}, \tag{3.4}$$

where

$$A_{\Gamma\Gamma}^{(i)} := \begin{bmatrix} A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}.$$

We will also consider the global Schur complement S_Γ, the direct sum of the local Schur complements $S_\Gamma^{(i)}$.

By using the local Schur complements, we can build a reduced global interface problem. The global problem is given by

$$\widehat{S}_\Gamma \boldsymbol{u}_\Gamma = \boldsymbol{g}_\Gamma, \tag{3.5}$$

where

$$\widehat{S}_\Gamma := \sum_{i=1}^N R_\Gamma^{(i)T} S_\Gamma^{(i)} R_\Gamma^{(i)} := R_\Gamma^T S_\Gamma R_\Gamma. \tag{3.6}$$

and

$$\boldsymbol{g}_\Gamma := \sum_{i=1}^N R_\Gamma^{(i)T} \left\{ \begin{bmatrix} \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \end{bmatrix} - \begin{bmatrix} A_{\Delta I}^{(i)} \\ A_{\Pi I}^{(i)} \end{bmatrix} A_{II}^{(i)-1} \boldsymbol{f}_I^{(i)} \right\}. \tag{3.7}$$

We note that once $\boldsymbol{u}_\Gamma^{(i)}$ has been computed, we can find the interior values $\boldsymbol{u}_I^{(i)}$ by solving the following equation:

$$A_{II}^{(i)} \boldsymbol{u}_I^{(i)} = \boldsymbol{f}_I^{(i)} - \left[A_{I\Delta}^{(i)} A_{I\Pi}^{(i)} \right] \boldsymbol{u}_\Gamma^{(i)}.$$

We will construct a preconditioner for the interface problem (3.5).

3.3. The BDDC algorithm. We now define a different Schur complement

$$\widetilde{S}_\Gamma := \overline{R}_\Gamma^T S_\Gamma \overline{R}_\Gamma.$$

After eliminating the interior residuals, we obtain the following linear system which also defines $\widetilde{S}_\Gamma$:

$$\widetilde{A}\begin{bmatrix} \boldsymbol{u}_I^{(1)} \\ \boldsymbol{u}_\Delta^{(1)} \\ \vdots \\ \boldsymbol{u}_I^{(N)} \\ \boldsymbol{u}_\Delta^{(N)} \\ \boldsymbol{u}_\Pi \end{bmatrix} = \begin{bmatrix} 0 \\ R_\Delta^{(1)}\widetilde{R}_{\Gamma\Delta}\widetilde{S}_\Gamma\boldsymbol{u}_\Gamma \\ \vdots \\ 0 \\ R_\Delta^{(N)}\widetilde{R}_{\Gamma\Delta}\widetilde{S}_\Gamma\boldsymbol{u}_\Gamma \\ \widetilde{R}_{\Gamma\Pi}\widetilde{S}_\Gamma\boldsymbol{u}_\Gamma \end{bmatrix}.$$

We note that $\widetilde{A}$ is the partially subassembled stiffness matrix and $\widetilde{S}_\Gamma$ is the partially assembled Schur complement. Hence, we need to further assemble it to obtain the fully assembled Schur complement $\widehat{S}_\Gamma$. By using restriction and extension operators, we find that $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma$. We can then rewrite the interface problem (3.5) as

$$\widetilde{R}_\Gamma^T\widetilde{S}_\Gamma\widetilde{R}_\Gamma\boldsymbol{u}_\Gamma = \boldsymbol{g}_\Gamma. \tag{3.8}$$

The BDDC preconditioner has the following form:

$$M^{-1} = \widetilde{R}_{D,\Gamma}^T\widetilde{S}_\Gamma^{-1}\widetilde{R}_{D,\Gamma}. \tag{3.9}$$

Here, $\widetilde{S}_\Gamma^{-1}$ can be obtained by using a block Cholesky factorization of $\widetilde{A}$ as in [27, section 4]:

$$\widetilde{S}_\Gamma^{-1} := \widetilde{R}_{\Gamma\Delta}^T\left(\sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ R_\Delta^{(i)} \end{bmatrix}\right)\widetilde{R}_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1}\Phi^T \tag{3.10}$$

with

$$\Phi := \widetilde{R}_{\Gamma\Pi}^T - \widetilde{R}_{\Gamma\Delta}^T\sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)T} \end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1}\begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix} R_\Pi^{(i)}$$

and where

$$S_{\Pi\Pi} := \sum_{i=1}^N R_\Pi^{(i)T}\left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1}\begin{bmatrix} A_{\Pi I}^{(i)T} \\ A_{\Pi\Delta}^{(i)T} \end{bmatrix}\right) R_\Pi^{(i)}.$$

The first term of (3.10) is related to the local Schur complements and the second term to the coarse-level problem associated with the primal space.

In order to specify the algorithm completely, we need to define the weighted averaging operator $D^{(i)}$. Conventional weighted averaging techniques, known as stiffness and ρ scalings, are described in [10,31]. However, these methods are designed for constant coefficients or for one variable coefficient. For more than one variable coefficient, we need a different approach and we will use the new weighted averaging technique introduced in [12] for $H(\mathbf{curl})$ problems.

We first consider the Schur complements related to F_{ij}, the common face of two adjacent subdomains Ω_i and Ω_j. Two local stiffness matrices associated with F_{ij} are given, for $k = i$ and j, by

$$A_{F_{ij}}^{(k)} := \begin{bmatrix} A_{II}^{(k)} & A_{IF_{ij}}^{(k)} \\ A_{F_{ij}I}^{(k)} & A_{F_{ij}F_{ij}}^{(k)} \end{bmatrix}. \tag{3.11}$$

The two Schur complements associated with F_{ij} are given by

$$S_{F_{ij}}^{(k)} := A_{F_{ij}F_{ij}}^{(k)} - A_{F_{ij}I}^{(k)} {A_{II}^{(k)}}^{-1} A_{IF_{ij}}^{(k)} \tag{3.12}$$

for $k = i$ and j. We will use the scaling matrices $D_j^{(i)} := \left(S_{F_{ij}}^{(i)} + S_{F_{ij}}^{(j)}\right)^{-1} S_{F_{ij}}^{(i)}$. We note that we can apply the operator $\left(S_{F_{ij}}^{(i)} + S_{F_{ij}}^{(j)}\right)^{-1}$ by solving a Dirichlet problem on $\Omega_i \cup F_{ij} \cup \Omega_j$ with zero Dirichlet boundary conditions. The scaling operator $D^{(i)}$ is then given by

$$D^{(i)} := \begin{bmatrix} D_{j_1}^{(i)} & & & \\ & D_{j_2}^{(i)} & & \\ & & \ddots & \\ & & & D_{j_k}^{(i)} \end{bmatrix}, \tag{3.13}$$

where $j_1, j_2, \ldots, j_k \in \mathcal{N}_i$ and $\mathcal{N}_i$ is the set of indices of the Ω_j's $(i \neq j)$ which share a subdomain face with Ω_i.

With this scaling operator $D^{(i)}$ and the operators of section 3.1, the BDDC preconditioner (3.9) is completely determined. We remark that there are two scaling matrices for each subdomain face and that it is easy to show that the partition of unity condition (3.1) is satisfied, i.e., that

$$\sum_{i=1}^{N} {R_\Gamma^{(i)}}^T D^{(i)} R_\Gamma^{(i)} = I. \tag{3.14}$$

We obtain the following preconditioned linear system:

$$M^{-1}\widehat{S}_\Gamma \boldsymbol{u}_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma \boldsymbol{u}_\Gamma = \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \boldsymbol{g}_\Gamma = M^{-1}\boldsymbol{g}_\Gamma \tag{3.15}$$

We use the preconditioned conjugate gradient method to solve the linear system (3.15).

We next define norms related to the Schur complements. The S_Γ-norm is given by $\|\boldsymbol{u}_\Gamma\|_{S_\Gamma}^2 := \boldsymbol{u}_\Gamma^T S_\Gamma \boldsymbol{u}_\Gamma$ for all $\boldsymbol{u}_\Gamma \in W_\Gamma$. Similarly, the local norms associated with $S_\Gamma^{(i)}$ and $S_{F_{ij}}^{(i)}$ are defined by $\left\|\boldsymbol{u}_\Gamma^{(i)}\right\|_{S_\Gamma^{(i)}}^2 := {\boldsymbol{u}_\Gamma^{(i)}}^T S_\Gamma^{(i)} \boldsymbol{u}_\Gamma^{(i)}$ and $\left\|\boldsymbol{u}_{F_{ij}}^{(i)}\right\|_{S_{F_{ij}}^{(i)}}^2 := {\boldsymbol{u}_{F_{ij}}^{(i)}}^T S_{F_{ij}}^{(i)} \boldsymbol{u}_{F_{ij}}^{(i)}$. The norms $\|\cdot\|_{\widehat{S}_\Gamma}$ and $\|\cdot\|_{\widetilde{S}_\Gamma}$ are defined by $\|\widehat{\boldsymbol{u}}_\Gamma\|_{\widehat{S}_\Gamma}^2 := \widehat{\boldsymbol{u}}_\Gamma^T \widehat{S}_\Gamma \widehat{\boldsymbol{u}}_\Gamma$ and $\|\widetilde{\boldsymbol{u}}_\Gamma\|_{\widetilde{S}_\Gamma}^2 := \widetilde{\boldsymbol{u}}_\Gamma^T \widetilde{S}_\Gamma \widetilde{\boldsymbol{u}}_\Gamma$ for $\widehat{\boldsymbol{u}}_\Gamma \in \widehat{W}_\Gamma$ and $\widetilde{\boldsymbol{u}}_\Gamma \in \widetilde{W}_\Gamma$, respectively. Using the definition of R_Γ and $\overline{R}_\Gamma$, we can easily see that $\|\widehat{\boldsymbol{u}}_\Gamma\|_{\widehat{S}_\Gamma}^2 = \|R_\Gamma \widehat{\boldsymbol{u}}_\Gamma\|_{S_\Gamma}^2$ and $\|\widetilde{\boldsymbol{u}}_\Gamma\|_{\widetilde{S}_\Gamma}^2 = \left\|\overline{R}_\Gamma \widetilde{\boldsymbol{u}}_\Gamma\right\|_{S_\Gamma}^2$.

In developing FETI-DP and BDDC theory, we need to estimate jump and averaging operators and the traditional variants require bounds for each subdomain face and involve norms associated with the both subdomains that share a face, e.g., [45, Lemma 5.3]. Such results usually require relatively strong assumptions such as a shape regular coarse triangulation and a quasi-uniform fine triangulation. The following lemma for our averaging technique, introduced earlier in this subsection, will greatly simplify the analysis and will reduce our work to estimates for individual subdomains. We also no longer need to use of a finite element extension theorem.

Lemma 3.1. *For any $\boldsymbol{u}_i \in W_\Gamma^{(i)}$, let $\boldsymbol{u}_{ij}$ be the restriction of $\boldsymbol{u}_i$ to a subdomain* *face $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$. We then have the following estimates:*

$$\left\| \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(i)}_{F_{ij}} \boldsymbol{u}_{ij} \right\|_{S^{(i)}_{F_{ij}}} \leq \|\boldsymbol{u}_{ij}\|_{S^{(i)}_{F_{ij}}} \tag{3.16}$$

$$\left\| \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(i)}_{F_{ij}} \boldsymbol{u}_{ij} \right\|_{S^{(j)}_{F_{ij}}} \leq \|\boldsymbol{u}_{ij}\|_{S^{(i)}_{F_{ij}}} . \tag{3.17}$$

Proof. These bounds are equivalent to the inequalities

$$S^{(i)}_{F_{ij}} \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(i)}_{F_{ij}} \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(i)}_{F_{ij}} \leq S^{(i)}_{F_{ij}}$$

and

$$S^{(i)}_{F_{ij}} \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(j)}_{F_{ij}} \left(S^{(i)}_{F_{ij}} + S^{(j)}_{F_{ij}} \right)^{-1} S^{(i)}_{F_{ij}} \leq S^{(i)}_{F_{ij}}.$$

These inequalities follow easily by expanding any function $\boldsymbol{u}_{ij}$ in the eigenvectors of the generalized eigenvalue problem defined by the two Schur complements and using the fact that all its eigenvalues are positive.
□

4. Technical tools and the main result. From now on, we will assume that the coefficients α and β are constant in each subdomain and that they thus only can have jumps across the interface Γ. We will also assume that all subdomains are convex polyhedra. We can then write the bilinear form of (1.1) in the following way:

$$a(\boldsymbol{u}, \boldsymbol{v}) := \sum_{i=1}^{N} a_i(\boldsymbol{u}, \boldsymbol{v}),$$

where the local energy bilinear form for each subdomain Ω_i are defined as follows:

$$a_i(\boldsymbol{u}, \boldsymbol{u}) := \alpha_i \int_{\Omega_i} \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{u} \, dx + \beta_i \int_{\Omega_i} \boldsymbol{u} \cdot \boldsymbol{u} \, dx.$$

4.1. Technical tools. To have access to all the technical tools that we need, we from now on have to assume that our subdomains are polyhedra and that each of them is the union of a few shape-regular large tetrahedra, which define a coarse finite element mesh $\mathcal{T}_H$.

In our proofs, we also need some standard tools for the space $S(\Omega_i)$, which we can borrow from [44, subsection 4.6]. They are related to the subdomain faces and the wire basket $\mathcal{W}^i$, which is the union of the edges and vertices of $\partial\Omega_i$.

LEMMA 4.1. *There are functions $\vartheta_{\mathcal{W}^i} \in S(\Omega_i)$ and $\vartheta_{\mathcal{F}^j} \in S(\Omega_i)$ such that*

$$\vartheta_{\mathcal{W}^i} + \sum_{F_{ij} \subset \partial\Omega_i} \vartheta_{F_{ij}} = 1, \tag{4.1}$$

and where $\vartheta_{F_{ij}} \equiv 0$ on $\partial\Omega_i \backslash F_{ij}$. Moreover, for any $u \in S(\Omega_i)$, there exists a constant independent of h_i and H_i, such that

$$|I_h(\vartheta_{\mathcal{W}^i} u)|^2_{1;\Omega_i} \leq C\,(1 + \log H_i/h_i)\, \|u\|^2_{1;\Omega_i} \tag{4.2}$$

and

$$\left|I_h\left(\vartheta_{F_{ij}} u\right)\right|^2_{1;\Omega_i} \le C\left(1+\log H_i/h_i\right)^2 \|u\|^2_{1;\Omega_i}. \tag{4.3}$$

In addition, the following estimate holds:

$$\|u\|^2_{0;\partial F_{ij}} \le C\left(1+\log H_i/h_i\right) \|u\|^2_{1;\Omega_i}. \tag{4.4}$$

Proof. See [44, subsection 4.6]. □

Unlike for the gradient operator, it is quite complicated to classify the kernel and the range of the curl and divergence operators. The discrete regular decompositions given in [21] provide useful tools to analyze problems posed in $H(\mathbf{curl})$ and $H(\mathrm{div})$. We can then apply techniques developed for H^1-functions by using

LEMMA 4.2 (Hiptmair-Xu decomposition). *Let Ω_i be a convex polyhedron. Then, for all $\boldsymbol{v}_h \in W(\Omega_i)$, there exist $\boldsymbol{\Psi}_h \in S(\Omega_i)$, $\boldsymbol{q}_h \in X(\Omega_i)$, and $\widetilde{\boldsymbol{v}}_h \in W(\Omega_i)$ such that*

$$\boldsymbol{v}_h = \widetilde{\boldsymbol{v}}_h + \Pi_h^{RT}\left(\boldsymbol{\Psi}_h\right) + \mathbf{curl}\,\boldsymbol{q}_h \tag{4.5}$$

and

$$\left\|h_i^{-1}\widetilde{\boldsymbol{v}}_h\right\|^2_{0;\Omega_i} + \|\boldsymbol{\Psi}_h\|^2_{1;\Omega_i} \le C\,\|\mathrm{div}\,\boldsymbol{v}_h\|^2_{0;\Omega_i}, \tag{4.6}$$

$$\|\mathbf{curl}\,\boldsymbol{q}_h\|^2_{0;\Omega_i} + \|\boldsymbol{\Psi}_h\|^2_{0;\Omega_i} \le C\,\|\boldsymbol{v}_h\|^2_{0;\Omega_i}. \tag{4.7}$$

Proof. See [21, Lemma 5.1 and 5.2]. □

We note that this important paper was preceeded by [20], which concerns another application of the same decomposition.

We next introduce a stable operator which provides a divergence-free extension.

LEMMA 4.3 (divergence free extension). *There exists an extension operator $\widetilde{\mathcal{H}}_i$ from the normal trace space of $W_\Delta^{(i)}$, such that, for all $\boldsymbol{u} \in W_\Delta^{(i)}$,*

$$\left(\widetilde{\mathcal{H}}_i\mu\right)\cdot\boldsymbol{n} = \mu,\ \mathrm{div}\widetilde{\mathcal{H}}_i\mu = 0,$$

where $\mu := \boldsymbol{u}\cdot\boldsymbol{n}$. Moreover,

$$\left\|\widetilde{\mathcal{H}}_i\mu\right\|_{0;\Omega_i} \le C\,\|\mu\|_{-1/2;\partial\Omega_i}.$$

Proof. See [52, Lemma 4.3] or [51, Lemma 2.6]. □

We then have the following estimate for the discrete harmonic extensions which have the minimal energy property for a given normal trace. For more detail, see [44, section 10.2] and [36, section 3.1].

COROLLARY 4.4 (discrete harmonic extension). *Let $\mathcal{H}_i$ be the energy minimizing discrete harmonic extension. For all $\boldsymbol{u} \in W_\Delta^{(i)}$, we have*

$$\left(\mathcal{H}_i\mu\right)\cdot\boldsymbol{n} = \mu := \boldsymbol{u}\cdot\boldsymbol{n}.$$

Furthermore,

$$\alpha_i\,\|\mathrm{div}\,\mathcal{H}_i\mu\|^2_{0;\Omega_i} + \beta_i\,\|\mathcal{H}_i\mu\|^2_{0;\Omega_i} \le C\beta_i\,\|\mu\|^2_{-1/2;\partial\Omega_i}.$$

Proof. $\mathcal{H}_i$ is the minimal-energy extension operator for a given subdomain interface. Therefore, we have

$$\alpha_i \left\|\operatorname{div} \mathcal{H}_i \mu\right\|_{0;\Omega_i}^2 + \beta_i \left\|\mathcal{H}_i \mu\right\|_{0;\Omega_i}^2 \leq \alpha_i \left\|\operatorname{div} \widetilde{\mathcal{H}}_i \mu\right\|_{0;\Omega_i}^2 + \beta_i \left\|\widetilde{\mathcal{H}}_i \mu\right\|_{0;\Omega_i}^2 .$$

But $\operatorname{div}\widetilde{\mathcal{H}}_i \mu = 0$ and thus by using Lemma 4.3,

$$\alpha_i \left\|\operatorname{div} \mathcal{H}_i \mu\right\|_{0;\Omega_i}^2 + \beta_i \left\|\mathcal{H}_i \mu\right\|_{0;\Omega_i}^2 \leq C\beta_i \left\|\mu\right\|_{-1/2;\partial\Omega_i}^2 .$$

□

We next consider the coarse interpolation operator Π_H^{RT} onto the Raviart-Thomas space of the coarse mesh $\mathcal{T}_H$.

LEMMA 4.5 (stability of the coarse interpolant). *For all $\boldsymbol{u}_i \in W^{(i)}$, we have the following estimates:*

$$\left\|\operatorname{div}\left(\Pi_H^{RT} \boldsymbol{u}_i\right)\right\|_{0;\Omega_i}^2 \leq \left\|\operatorname{div} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2 \tag{4.8}$$

and

$$\left\|\Pi_H^{RT} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2 \leq C\left(\left(1 + \log H_i/h_i\right)\left\|\boldsymbol{u}_i\right\|_{0;\Omega_i}^2 + H_i^2 \left\|\operatorname{div} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2\right), \tag{4.9}$$

where the constant C depends neither on h_i nor on H_i.

Proof. See [51, Lemma 2.4] or [52, Lemma 4.1]. □

LEMMA 4.6. *Let $\boldsymbol{u}_{F_{ij}} \in W^{(i)}$ with $\lambda_f\left(\boldsymbol{u}_{F_{ij}}\right) = 0, \forall f \subset \partial\Omega_i \backslash F_{ij}$. Let $\boldsymbol{u}_{F_{ij}}^H := \mathcal{H}_i\left(\Pi_H^{RT} \boldsymbol{u}_{F_{ij}} \cdot \boldsymbol{n}\right)$ and assume that $\alpha_i \leq \beta_i H_i^2$. We then have*

$$a_i\left(\boldsymbol{u}_{F_{ij}}^H, \boldsymbol{u}_{F_{ij}}^H\right) \leq C a_i\left(\boldsymbol{u}_{F_{ij}}, \boldsymbol{u}_{F_{ij}}\right), \tag{4.10}$$

where C is indepedent of H_i, h_i, α_i, and β_i.

Proof. We will modify the proof of [36, Lemma 5.6].

Let $\Omega_{i,F_{ij}}^{d_i} \subset \Omega_i$ be the set of all points which are within a distance d_i of F_{ij}. We first introduce a piecewise linear scalar cut-off function $\chi_{F_{ij}}$ which has the value 1 on F_{ij} and vanishes in $\Omega_i \backslash \Omega_{i,F_{ij}}^{d_i}$ for some $h_i \leq d_i \leq H_i$. Moreover, $\left\|\chi_{F_{ij}}\right\|_\infty \leq 1$ and $\left\|\nabla \chi_{F_{ij}}\right\|_\infty \leq C/d_i$.

We next consider the coarse basis function related to the discrete harmonic extension. The basis function $\widetilde{\boldsymbol{\phi}}_{F_{ij}}^{RT}$ is obtained from the standard basis function $\boldsymbol{\phi}_{F_{ij}}^{RT}$, i.e., $\widetilde{\boldsymbol{\phi}}_{F_{ij}}^{RT} = \mathcal{H}_i\left(\boldsymbol{\phi}_{F_{ij}}^{RT} \cdot \boldsymbol{n}\right)$. We note that $\left\|\boldsymbol{\phi}_{F_{ij}}^{RT}\right\|_{0;\Omega_i}^2 \leq C H_i^3$ and $\left\|\operatorname{div} \boldsymbol{\phi}_{F_{ij}}^{RT}\right\|_{0;\Omega_i}^2 \leq C H_i$.

We can then write the function $\boldsymbol{u}_{F_{ij}}^H$ as follows:

$$\boldsymbol{u}_{F_{ij}}^H = \lambda_{F_{ij}}\left(\boldsymbol{u}_{F_{ij}}\right) \widetilde{\boldsymbol{\phi}}_{F_{ij}}^{RT}. \tag{4.11}$$

In order to estimate the energy of $\boldsymbol{u}_{F_{ij}}^H$, we estimate the degree of freedom $\lambda_{F_{ij}}\left(\boldsymbol{u}_{F_{ij}}\right)$ and the energy of $\widetilde{\boldsymbol{\phi}}_{F_{ij}}^{RT}$ separately.

We first estimate the degree of freedom. By the divergence theorem, and the fact that $\left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) \cdot \boldsymbol{n}$ vanishes on $\partial\Omega^{d_i}_{i,F_{ij}} \backslash F_{ij}$,

$$
\begin{aligned}
|F_{ij}| \, \lambda_{F_{ij}} \left(\boldsymbol{u}_{F_{ij}}\right) = |F_{ij}| \, \lambda_{F_{ij}} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) &= \int_{F_{ij}} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) \cdot \boldsymbol{n} \, ds \\
&= \int_{\Omega^{d_i}_{i,F_{ij}}} \operatorname{div} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) \, dx - \int_{\partial\Omega^{d_i}_{i,F_{ij}} \backslash F_{ij}} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) \cdot \boldsymbol{n} \, ds \\
&= \int_{\Omega^{d_i}_{i,F_{ij}}} \operatorname{div} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right) \, dx. \qquad (4.12)
\end{aligned}
$$

By using the Cauchy-Schwarz inequality and the shape-regularity, we obtain

$$
\begin{aligned}
\left|\lambda_{F_{ij}} \left(\boldsymbol{u}_{F_{ij}}\right)\right|^2 &\leq C \frac{d_i}{H_i^2} \left\|\operatorname{div} \left(\chi_{F_{ij}} \boldsymbol{u}_{F_{ij}}\right)\right\|^2_{0;\Omega^{d_i}_{i,F_{ij}}} \\
&\leq C \frac{d_i}{H_i^2} \left(\left\|\chi_{F_{ij}}\right\|^2_\infty \left\|\operatorname{div} \boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i} + \left\|\nabla \chi_{F_{ij}}\right\|^2_\infty \left\|\boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i} \right) \\
&\leq C \frac{d_i}{H_i^2} \left\|\operatorname{div} \boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i} + C \frac{1}{H_i^2 d_i} \left\|\boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i}. \qquad (4.13)
\end{aligned}
$$

We now estimate the basis function. From the minimal energy property, we find

$$
\begin{aligned}
&\alpha_i \left\|\operatorname{div} \widetilde{\boldsymbol{\phi}}^{RT}_{F_{ij}}\right\|^2_{0;\Omega_i} + \beta_i \left\|\widetilde{\boldsymbol{\phi}}^{RT}_{F_{ij}}\right\|^2_{0;\Omega_i} \\
\leq\ & \alpha_i \left\|\operatorname{div} \left(\Pi^{RT}_h \left(\chi_{F_{ij}} \boldsymbol{\phi}^{RT}_{F_{ij}}\right)\right)\right\|^2_{0;\Omega_i} + \beta_i \left\|\left(\Pi^{RT}_h \left(\chi_{F_{ij}} \boldsymbol{\phi}^{RT}_{F_{ij}}\right)\right)\right\|^2_{0;\Omega_i} \\
\leq\ & C\alpha_i \left(d_i + H_i^2/d_i\right) + C\beta_i H_i^2 d_i \leq C\alpha_i H_i^2/d_i + C\beta_i H_i^2 d_i. \qquad (4.14)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
a_i \left(\boldsymbol{u}^H_{F_{ij}}, \boldsymbol{u}^H_{F_{ij}}\right) &= \alpha_i \left\|\lambda_{F_{ij}} \left(\boldsymbol{u}_{F_{ij}}\right) \left(\operatorname{div} \widetilde{\boldsymbol{\phi}}^{RT}_{F_{ij}}\right)\right\|^2_{0;\Omega_i} + \beta_i \left\|\lambda_{F_{ij}} \left(\boldsymbol{u}_{F_{ij}}\right) \left(\widetilde{\boldsymbol{\phi}}^{RT}_{F_{ij}}\right)\right\|^2_{0;\Omega_i} \\
&\leq C \left(\alpha_i + \beta_i d_i^2\right) \left\|\operatorname{div} \boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i} + C \left(\alpha_i/d_i^2 + \beta_i\right) \left\|\boldsymbol{u}_{F_{ij}}\right\|^2_{0;\Omega_i}. \qquad (4.15)
\end{aligned}
$$

Let $d_i = \max\{\sqrt{\alpha_i/\beta_i}, h_i\}$. We note that $h_i \leq d_i \leq H_i$. By using (4.15) and Lemma 2.3, we obtain

$$
a_i \left(\boldsymbol{u}^H_{F_{ij}}, \boldsymbol{u}^H_{F_{ij}}\right) \leq C a_i \left(\boldsymbol{u}_{F_{ij}}, \boldsymbol{u}_{F_{ij}}\right). \qquad (4.16)
$$

□

We next introduce a partition of unity associated with the faces of an individual subdomain Ω_i as in [44, Chapter 10.2.1],

$$
\sum_{F \subset \partial\Omega_i} \zeta_F \equiv 1, \text{a.e. on } \partial\Omega_i \backslash \partial\Omega,
$$

where

$$
\zeta_F(x) = \begin{cases} 1, & x \in F \\ 0, & x \in \partial\Omega_i \backslash F. \end{cases} \qquad (4.17)
$$

We then have the following estimates for the subdomain face components; we recall that for $\boldsymbol{u}_i \in W_{\Delta}^{(i)}$, $\int_{F_{ij}} \boldsymbol{u}_i \cdot \boldsymbol{n}\, ds = 0$ for each subdomain face $F_{ij} \subset \partial\Omega_i$.

LEMMA 4.7. *For any $\boldsymbol{u}_i \in W_{\Delta}^{(i)}$ and for any $\boldsymbol{u}_i^H \in W_{\Pi}^{(i)}$, let $\mu_i := \boldsymbol{u}_i \cdot \boldsymbol{n}$, $\mu_{F_{ij}} = \zeta_{F_{ij}} \mu_i$, and $\mu_i^H := \boldsymbol{u}_i^H \cdot \boldsymbol{n}$. Then,*

$$\left\|\mu_{F_{ij}}\right\|_{-1/2;\partial\Omega_i}^2 \le C\left(1+\log H_i/h_i\right)\left(\left(1+\log H_i/h_i\right)\left\|\mu_i+\mu_i^H\right\|_{-1/2;\partial\Omega_i}^2+\left\|\mu_i\right\|_{-1/2;\partial\Omega_i}^2\right)$$

where C is independent of μ_i^H, H_i, and h_i.

Proof. See [52, Lemma 4.4] or [51, Lemma 2.7]. ◻

LEMMA 4.8. *For any $\boldsymbol{u}_i \in W^{(i)}$, there exist $\boldsymbol{v}_{i,F_{ij}}$ and $\boldsymbol{v}_{i,F_{ij}}^H \in W^{(i)}$ such that*

$$\begin{cases} \lambda_f\left(\boldsymbol{v}_{i,F_{ij}}\right) = \lambda_f\left(\boldsymbol{u}_i\right) & \text{if } f \subset F_{ij}; \\ \lambda_f\left(\boldsymbol{v}_{i,F_{ij}}\right) = 0 & \text{if } f \subset \partial\Omega_i \backslash F_{ij} \end{cases} \tag{4.18}$$

and

$$\begin{cases} \lambda_f\left(\boldsymbol{v}_{i,F_{ij}}^H\right) = \lambda_f\left(\Pi_H^{RT} \boldsymbol{u}_i\right) & \text{if } f \subset F_{ij}; \\ \lambda_f\left(\boldsymbol{v}_{i,F_{ij}}^H\right) = 0 & \text{if } f \subset \partial\Omega_i \backslash F_{ij}. \end{cases} \tag{4.19}$$

Furthermore,

$$a_i\left(\boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}_{i,F_{ij}}^H, \boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}_{i,F_{ij}}^H\right) \le C\left(1+\log H_i/h_i\right)^2 a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right), \tag{4.20}$$

where C is independent of α_i, β_i, H_i, and h_i.

Proof. We will only consider the case where $\alpha_i \le \beta_i H_i^2$ since the proof is straightforward by using Corollary 4.4 and Lemmas 4.7, 2.1, and 4.5 if $\beta_i H_i^2 \le \alpha_i$; for more details, see [51, 52].

By using Lemma 4.2, we can find $\widetilde{\boldsymbol{u}}_i$, $\boldsymbol{\Psi}_i$, and $\boldsymbol{q}_i$ such that

$$\boldsymbol{u}_i = \widetilde{\boldsymbol{u}}_i + \Pi_h^{RT}\left(\boldsymbol{\Psi}_i\right) + \mathbf{curl}\, \boldsymbol{q}_i. \tag{4.21}$$

We note that $\operatorname{div}\left(\mathbf{curl}\, \boldsymbol{q}_i\right) = 0$ and that (4.6) and (4.7) provide bounds for the different terms.

We first consider $\widetilde{\boldsymbol{u}}_i$. We define $\widetilde{\boldsymbol{u}}_{i,F_{ij}} = \sum_{f \subset F_{ij}} \lambda_f\left(\widetilde{\boldsymbol{u}}_i\right) \boldsymbol{\phi}_f^{RT}$, where $\boldsymbol{\phi}_f^{RT}$ is the Raviart-Thomas basis function associated with the face f. By using Lemmas 2.2, 2.3, and 4.2, we have

$$\begin{aligned} \left\|\widetilde{\boldsymbol{u}}_{i,F_{ij}}\right\|_{0;\Omega_i}^2 &\le C \sum_{f \subset F_{ij}} h_i^3 \lambda_f\left(\widetilde{\boldsymbol{u}}_i\right)^2 \le C\left\|\widetilde{\boldsymbol{u}}_i\right\|_{0;\Omega_i}^2 \\ &\le C h_i^2 \left\|\operatorname{div} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2 \le C\left\|\boldsymbol{u}_i\right\|_{0;\Omega_i}^2 \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \left\|\operatorname{div} \widetilde{\boldsymbol{u}}_{i,F_{ij}}\right\|_{0;\Omega_i}^2 &\le C \sum_{f \subset F_{ij}} h_i \lambda_f\left(\widetilde{\boldsymbol{u}}_i\right)^2 \\ &\le C\left\|h_i^{-1} \widetilde{\boldsymbol{u}}_i\right\|_{0;\Omega_i}^2 \le C\left\|\operatorname{div} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2. \end{aligned} \tag{4.23}$$

Hence, from (4.22) and (4.23),

$$a_i\left(\widetilde{\boldsymbol{u}}_{i,F_{ij}}, \widetilde{\boldsymbol{u}}_{i,F_{ij}}\right) \le C a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \tag{4.24}$$

We also define $\widetilde{\boldsymbol{u}}^H_{i,F_{ij}} = \mathcal{H}_i\left(\Pi_H^{RT}\widetilde{\boldsymbol{u}}_{i,F_{ij}}\cdot\boldsymbol{n}\right)$. By using Lemma 4.6 and (4.24), we obtain

$$
\begin{aligned}
a_i\left(\widetilde{\boldsymbol{u}}^H_{i,F_{ij}}, \widetilde{\boldsymbol{u}}^H_{i,F_{ij}}\right) &\le C a_i(\widetilde{\boldsymbol{u}}_{i,F_{ij}}, \widetilde{\boldsymbol{u}}_{i,F_{ij}}) \\
&= C a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.25)
\end{aligned}
$$

We next consider the second term $\Pi_h^{RT}\left(\boldsymbol{\Psi}_i\right)$ of (4.21). Let $\boldsymbol{\Psi}_{i,F_{ij}} := I_h\left(\vartheta_{F_{ij}}\boldsymbol{\Psi}_i\right)$. By using Lemmas 4.1 and 4.2, we obtain

$$
\left\|\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i} \le C\left\|\boldsymbol{\Psi}_i\right\|^2_{0;\Omega_i} \le C\left\|\boldsymbol{u}_i\right\|^2_{0;\Omega_i}
$$

and

$$
\begin{aligned}
\left\|\operatorname{div}\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i} &\le C\left|\boldsymbol{\Psi}_{i,F_{ij}}\right|^2_{1;\Omega_i} \le C\left(1+\log H_i/h_i\right)^2\left\|\boldsymbol{\Psi}_i\right\|^2_{1;\Omega_i} \\
&\le C\left(1+\log H_i/h_i\right)^2\left\|\operatorname{div}\boldsymbol{u}_i\right\|^2_{0;\Omega_i}.
\end{aligned}
$$

Moreover, by using Lemma 2.4, an inverse estimate, and Lemma 2.5, we obtain

$$
\begin{aligned}
\left\|\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} &\le 2\left\|\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i} + 2\left\|\boldsymbol{\Psi}_{i,F_{ij}} - \Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} \\
&\le 2\left\|\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i} + Ch_i^2\left|\boldsymbol{\Psi}_{i,F_{ij}}\right|^2_{1;\Omega_i} \le C\left\|\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i},
\end{aligned}
$$

and

$$
\left\|\operatorname{div}\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} = \left\|\Pi_h\left(\operatorname{div}\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} \le \left\|\operatorname{div}\boldsymbol{\Psi}_{i,F_{ij}}\right\|^2_{0;\Omega_i}.
$$

Therefore,

$$
\left\|\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} \le C\left\|\boldsymbol{u}_i\right\|^2_{0;\Omega_i} \qquad (4.26)
$$

and

$$
\left\|\operatorname{div}\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right\|^2_{0;\Omega_i} \le C\left(1+\log H_i/h_i\right)^2\left\|\operatorname{div}\boldsymbol{u}_i\right\|^2_{0;\Omega_i}. \qquad (4.27)
$$

Hence, from (4.26) and (4.27), we obtain

$$
a_i\left(\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right), \Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right) \le C\left(1+\log H_i/h_i\right)^2 a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.28)
$$

Let $\boldsymbol{\Psi}^H_{i,F_{ij}} = \mathcal{H}_i\left(\Pi_H^{RT}\left(\Pi_h^{RT}\boldsymbol{\Psi}_{i,F_{ij}}\right)\cdot\boldsymbol{n}\right)$. By using Lemma 4.6 and (4.28), we have

$$
\begin{aligned}
a_i\left(\boldsymbol{\Psi}^H_{i,F_{ij}}, \boldsymbol{\Psi}^H_{i,F_{ij}}\right) &\le C a_i\left(\Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right), \Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right)\right) \\
&\le C\left(1+\log H_i/h_i\right)^2 a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.29)
\end{aligned}
$$

Let $\boldsymbol{\Psi}_{i,\mathcal{W}_i} = \Pi_h^{RT}\left(I_h\left(\vartheta_{\mathcal{W}^i}\boldsymbol{\Psi}_i\right)\right)$ and $\boldsymbol{\Psi}_{i,\partial F_{ij}} = \sum_{f\subset F_{ij}}\lambda_f\left(\boldsymbol{\Psi}_{i,\mathcal{W}_i}\right)\boldsymbol{\phi}_f^{RT}$. By using Lemmas 2.2, 2.4, 4.1, and 4.2, an inverse estimate, and an estimate for the P_1 basis functions of $S(\Omega_i)$, we obtain

$$
\begin{aligned}
\left\|\boldsymbol{\Psi}_{i,\partial F_{ij}}\right\|^2_{0;\Omega_i} &\le C\sum_{f\subset F_{ij}} h_i^3\lambda_f\left(\boldsymbol{\Psi}_{i,\mathcal{W}_i}\right)^2 \le C\left\|\boldsymbol{\Psi}_{i,\mathcal{W}_i}\right\|^2_{0;\Omega_i} \\
&\le C\left\|I_h\left(\vartheta_{\mathcal{W}^i}\boldsymbol{\Psi}_i\right)\right\|^2_{0;\Omega_i} \le C\left\|\boldsymbol{\Psi}_i\right\|^2_{0;\Omega_i} \le C\left\|\boldsymbol{u}_i\right\|^2_{0;\Omega_i} \qquad (4.30)
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\operatorname{div} \boldsymbol{\Psi}_{i,\partial F_{ij}}\right\|_{0;\Omega_i}^2 &\leq C \sum_{f \subset F_{ij}} h_i \lambda_f \left(\boldsymbol{\Psi}_{i,\mathcal{W}_i}\right)^2 \leq C \frac{1}{h_i^2} \left\|\boldsymbol{\Psi}_{i,\mathcal{W}_i}\right\|_{0;\Omega_i}^2 \\
&\leq C \frac{1}{h_i^2} \left\| I_h \left(\vartheta_{\mathcal{W}^i} \boldsymbol{\Psi}_i\right)\right\|_{0;\Omega_i}^2 \leq C \left\|\boldsymbol{\Psi}_i\right\|_{0;\partial F_{ij}}^2 \\
&\leq C \left(1 + \log H_i/h_i\right) \left\|\boldsymbol{\Psi}_i\right\|_{1;\Omega_i}^2 \\
&\leq C \left(1 + \log H_i/h_i\right) \left\|\operatorname{div} \boldsymbol{u}_i\right\|_{0;\Omega_i}^2 . \qquad (4.31)
\end{aligned}
$$

Hence, combining (4.30) and (4.31), we obtain

$$
a_i \left(\boldsymbol{\Psi}_{i,\partial F_{ij}}, \boldsymbol{\Psi}_{i,\partial F_{ij}}\right) \leq C \left(1 + \log H_i/h_i\right) a_i \left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.32)
$$

Let $\boldsymbol{\Psi}_{i,\partial F_{ij}}^H := \mathcal{H}_i \left(\Pi_H^{RT} \boldsymbol{\Psi}_{i,\partial F_{ij}} \cdot \boldsymbol{n}\right)$. From Lemma 4.6 and (4.32), we obtain

$$
\begin{aligned}
a_i \left(\boldsymbol{\Psi}_{i,\partial F_{ij}}^H, \boldsymbol{\Psi}_{i,\partial F_{ij}}^H\right) &\leq C a_i \left(\boldsymbol{\Psi}_{i,\partial F_{ij}}, \boldsymbol{\Psi}_{i,\partial F_{ij}}\right) \\
&\leq C \left(1 + \log H_i/h_i\right) a_i \left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.33)
\end{aligned}
$$

We finally consider the $\mathbf{curl}\,\boldsymbol{q}_i$-term of (4.21). Let $\boldsymbol{q}_i^H := \Pi_H^{RT} \left(\mathbf{curl}\,\boldsymbol{q}_i\right)$, $q_i := \left(\mathbf{curl}\,\boldsymbol{q}_i\right) \cdot \boldsymbol{n}$, and $q_i^H := \boldsymbol{q}_i^H \cdot \boldsymbol{n}$. Moreover, let $\boldsymbol{q}_{i,F_{ij}}^H := \mathcal{H}_i \left(\zeta_{F_{ij}} q_i^H\right)$ and $\boldsymbol{q}_{i,F_{ij}} = \mathcal{H}_i \left(\zeta_{F_{ij}} \left(q_i - q_i^H\right)\right)$. From Corollary 4.4 and Lemma 4.7, we obtain

$$
\begin{aligned}
& a_i \left(\boldsymbol{q}_{i,F_{ij}}, \boldsymbol{q}_{i,F_{ij}}\right) \\
&\leq C\beta_i \left\|\zeta_{F_{ij}} \left(q_i - q_i^H\right)\right\|_{-1/2;\partial\Omega_i}^2 \\
&\leq C\beta_i \left(1 + \log H_i/h_i\right) \left(\left(1 + \log H_i/h_i\right) \left\|q_i\right\|_{-1/2;\partial\Omega_i}^2 + \left\|q_i - q_i^H\right\|_{-1/2;\partial\Omega_i}^2\right) \\
&\leq C\beta_i \left(1 + \log H_i/h_i\right) \left(\left(1 + \log H_i/h_i\right) \left\|q_i\right\|_{-1/2;\partial\Omega_i}^2 + \left\|q_i^H\right\|_{-1/2;\partial\Omega_i}^2\right) \\
&\leq C\beta_i \left(1 + \log H_i/h_i\right)^2 \left\|q_i\right\|_{-1/2;\partial\Omega_i}^2 + C\beta_i \left(1 + \log H_i/h_i\right) \left\|q_i^H\right\|_{-1/2;\partial\Omega_i}^2 . \qquad (4.34)
\end{aligned}
$$

We note that $\left\|\operatorname{div} \boldsymbol{q}_i^H\right\|_{0;\Omega_i} \leq \left\|\operatorname{div} \left(\mathbf{curl}\,\boldsymbol{q}_i\right)\right\|_{0;\Omega_i} = 0$ from Lemma 4.5. Hence, by using Lemmas 2.1 and 4.5, we obtain

$$
\begin{aligned}
\left\|q_i^H\right\|_{-1/2;\partial\Omega_i}^2 &\leq C \left(H_i^2 \left\|\operatorname{div} \boldsymbol{q}_i^H\right\|_{0;\Omega}^2 + \left\|\boldsymbol{q}_i^H\right\|_{0;\Omega}^2\right) = C \left\|\boldsymbol{q}_i^H\right\|_{0;\Omega}^2 \\
&\leq C \left(\left(1 + \log H_i/h_i\right) \left\|\mathbf{curl}\,\boldsymbol{q}_i\right\|_{0;\Omega}^2 + H_i^2 \left\|\operatorname{div} \left(\mathbf{curl}\,\boldsymbol{q}_i\right)\right\|_{0;\Omega}^2\right) \\
&= C \left(1 + \log H_i/h_i\right) \left\|\mathbf{curl}\,\boldsymbol{q}_i\right\|_{0;\Omega}^2 \qquad (4.35)
\end{aligned}
$$

and

$$
\left\|q_i\right\|_{-1/2;\partial\Omega_i}^2 \leq C \left(H_i^2 \left\|\operatorname{div} \left(\mathbf{curl}\,\boldsymbol{q}_i\right)\right\|_{0;\Omega}^2 + \left\|\mathbf{curl}\,\boldsymbol{q}_i\right\|_{0;\Omega}^2\right) = C \left\|\mathbf{curl}\,\boldsymbol{q}_i\right\|_{0;\Omega}^2 . \qquad (4.36)
$$

Therefore, by combining (4.34), (4.35), (4.36), and Lemma 4.2, we obtain

$$
\begin{aligned}
a_i \left(\boldsymbol{q}_{i,F_{ij}}, \boldsymbol{q}_{i,F_{ij}}\right) &\leq C \left(1 + \log H_i/h_i\right)^2 \beta_i \left\|\mathbf{curl}\,\boldsymbol{q}_i\right\|_{0;\Omega}^2 \\
&\leq C \left(1 + \log H_i/h_i\right)^2 \beta_i \left\|\boldsymbol{u}_i\right\|_{0;\Omega}^2 \\
&\leq C \left(1 + \log H_i/h_i\right)^2 a_i \left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \qquad (4.37)
\end{aligned}
$$

We can now define $\boldsymbol{v}_{i,F_{ij}}$ as follows:

$$\boldsymbol{v}_{i,F_{ij}} := \widetilde{\boldsymbol{u}}_{i,F_{ij}} + \Pi_h^{RT}\left(\boldsymbol{\Psi}_{i,F_{ij}}\right) + \boldsymbol{\Psi}_{i,\partial F_{ij}} + \boldsymbol{q}^H_{i,F_{ij}} + \boldsymbol{q}_{i,F_{ij}} \tag{4.38}$$

and $\boldsymbol{v}^H_{i,F_{ij}}$ by

$$\boldsymbol{v}^H_{i,F_{ij}} := \widetilde{\boldsymbol{u}}^H_{i,F_{ij}} + \boldsymbol{\Psi}^H_{i,F_{ij}} + \boldsymbol{\Psi}^H_{i,\partial F_{ij}} + \boldsymbol{q}^H_{i,F_{ij}}. \tag{4.39}$$

Then, $\boldsymbol{v}_{i,F_{ij}}$ and $\boldsymbol{v}^H_{i,F_{ij}}$ satisfy the conditions (4.18) and (4.19), respectively. Furthermore, we obtain the following estimate by using (4.24), (4.25), (4.28), (4.32), (4.29), (4.33), and (4.37):

$$a_i\left(\boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}^H_{i,F_{ij}}, \boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}^H_{i,F_{ij}}\right) \le C\left(1+\log H_i/h_i\right)^2 a_i\left(\boldsymbol{u}_i, \boldsymbol{u}_i\right). \tag{4.40}$$

□

We conclude this section by introducing an extension lemma which extend the face components to the entire interface.

Lemma 4.9 (face extension lemma). *For all $\boldsymbol{v}_i \in W_\Gamma^{(i)}$, we can find $\boldsymbol{v}_i^H \in W_\Pi^{(i)}$ which satisfies $\int_F \left(\boldsymbol{v}_i - \boldsymbol{v}_i^H\right)\cdot \boldsymbol{n}\, ds = 0$ for each subdomain face F of Ω_i. Let $\boldsymbol{v}_{ij}$ and $\boldsymbol{v}_{ij}^H$ be the restriction of $\boldsymbol{v}_i$ and $\boldsymbol{v}_i^H$ to the subdomain face F_{ij}, respectively. We then have the following estimate:*

$$\left\|\boldsymbol{v}_{ij} - \boldsymbol{v}_{ij}^H\right\|^2_{S^{(i)}_{F_{ij}}} \le C\left(1+\log H_i/h_i\right)^2 \left\|\boldsymbol{v}_i\right\|^2_{S^{(i)}_\Gamma}, \tag{4.41}$$

where C is independent of h_i and H_i.

Proof. Let $\boldsymbol{u}_i := \mathcal{H}_i\left(\boldsymbol{v}_i\cdot\boldsymbol{n}\right)$. Then, by Lemma 4.8, there exist $\boldsymbol{v}_{i,F_{ij}}$ and $\boldsymbol{v}^H_{i,F_{ij}}$ such that

$$\begin{aligned}\left\|\boldsymbol{v}_{ij} - \boldsymbol{v}_{ij}^H\right\|^2_{S^{(i)}_{F_{ij}}} &= a_i\left(\mathcal{H}_i\left(\zeta_{F_{ij}}\left(\left(\boldsymbol{v}_i - \boldsymbol{v}_i^H\right)\cdot\boldsymbol{n}\right)\right), \mathcal{H}_i\left(\zeta_{F_{ij}}\left(\left(\boldsymbol{v}_i - \boldsymbol{v}_i^H\right)\cdot\boldsymbol{n}\right)\right)\right)\\ &\le a_i\left(\boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}^H_{i,F_{ij}}, \boldsymbol{v}_{i,F_{ij}} - \boldsymbol{v}^H_{i,F_{ij}}\right) \le C\left(1+\log H_i/h_i\right)^2 a_i\left(\boldsymbol{u}_i,\boldsymbol{u}_i\right)\\ &= C\left(1+\log H_i/h_i\right)^2 \left\|\boldsymbol{v}_i\right\|^2_{S^{(i)}_\Gamma}.\end{aligned}$$

□

4.2. A stability estimate. The averaging operator E_D, defined in (3.3), satisfies the following estimate:

Lemma 4.10. *There is a constant C, which is independent of H_i, h_i, α_i, and β_i, such that, for all $\boldsymbol{u}_\Gamma \in \widetilde{W}_\Gamma$,*

$$\left\|E_D\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma} \le C\left(1+\log H/h\right)^2 \left\|\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma}. \tag{4.42}$$

Proof. We have that

$$\begin{aligned}\left\|E_D\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma} &\le 2\left(\left\|\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma} + \left\|\boldsymbol{u}_\Gamma - E_D\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma}\right)\\ &= 2\left(\left\|\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma} + \left\|\overline{R}_\Gamma\left(\boldsymbol{u}_\Gamma - E_D\boldsymbol{u}_\Gamma\right)\right\|^2_{S_\Gamma}\right)\\ &= 2\left(\left\|\boldsymbol{u}_\Gamma\right\|^2_{\widetilde{S}_\Gamma} + \sum_{i=1}^N \left\|\overline{R}^{(i)}_\Gamma\left(\boldsymbol{u}_\Gamma - E_D\boldsymbol{u}_\Gamma\right)\right\|^2_{S^{(i)}_\Gamma}\right).\end{aligned} \tag{4.43}$$

We will use the scaling operators introduced in subsection 3.3. Let $\boldsymbol{u}_i := \overline{R}_\Gamma^{(i)} \boldsymbol{u}_\Gamma$. We then have $\boldsymbol{u}_\Gamma - E_D \boldsymbol{u}_\Gamma = D_i^{(j)} (\boldsymbol{u}_i - \boldsymbol{u}_j)$ on $F_{ij} := \partial\Omega_i \cap \partial\Omega_j$. Let $\boldsymbol{u}_{ij}$ and $\boldsymbol{u}_{ji}$ be the restrictions of $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ to F_{ij}, respectively. We then obtain

$$\left\| \overline{R}_\Gamma^{(i)} (\boldsymbol{u}_\Gamma - E_D \boldsymbol{u}_\Gamma) \right\|^2_{S_\Gamma^{(i)}} = \sum_{i \neq j, F_{ij} \subset \partial\Omega_i} \left\| D_j^{(i)} (\boldsymbol{u}_{ij} - \boldsymbol{u}_{ji}) \right\|^2_{S_{F_{ij}}^{(i)}} . \tag{4.44}$$

Let $\boldsymbol{u}_i^H \in W_\Pi^{(i)}$ defined by $\mathcal{H}_i \left(\left(\Pi_H^{RT} \mathcal{H}_i (\boldsymbol{u}_i \cdot \boldsymbol{n}) \right) \cdot \boldsymbol{n} \right)$. We then have $\int_F (\boldsymbol{u}_i - \boldsymbol{u}_i^H) \cdot \boldsymbol{n} \, ds = 0$ for each subdomain face F of Ω_i. We define $\boldsymbol{u}_j^H$ similarly. Let $\boldsymbol{u}_{ij}^H$ and $\boldsymbol{u}_{ji}^H$ be the restrictions of $\boldsymbol{u}_i^H$ and $\boldsymbol{u}_j^H$ to F_{ij}. By our choice of primal constraints, $\boldsymbol{u}_{ij}^H = \boldsymbol{u}_{ji}^H$ on F_{ij}. From Lemmas 3.1 and 4.9, we obtain

$$\begin{aligned}
\left\| D_j^{(i)} (\boldsymbol{u}_{ij} - \boldsymbol{u}_{ji}) \right\|^2_{S_{F_{ij}}^{(i)}} &= \left\| D_j^{(i)} \left((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ij}^H) - (\boldsymbol{u}_{ji} - \boldsymbol{u}_{ji}^H) \right) \right\|^2_{S_{F_{ij}}^{(i)}} \\
&\leq 2 \left\| D_j^{(i)} \left(\boldsymbol{u}_{ij} - \boldsymbol{u}_{ij}^H \right) \right\|^2_{S_{F_{ij}}^{(i)}} + 2 \left\| D_j^{(i)} \left(\boldsymbol{u}_{ji} - \boldsymbol{u}_{ji}^H \right) \right\|^2_{S_{F_{ij}}^{(i)}} \\
&\leq 2 \left\| \boldsymbol{u}_{ij} - \boldsymbol{u}_{ij}^H \right\|^2_{S_{F_{ij}}^{(i)}} + 2 \left\| \boldsymbol{u}_{ji} - \boldsymbol{u}_{ji}^H \right\|^2_{S_{F_{ij}}^{(j)}} \\
&\leq C \left((1 + \log H_i/h_i)^2 \left\| \boldsymbol{u}_i \right\|^2_{S_\Gamma^{(i)}} + (1 + \log H_j/h_j)^2 \left\| \boldsymbol{u}_j \right\|^2_{S_\Gamma^{(j)}} \right) .
\end{aligned} \tag{4.45}$$

By summing over all faces and subdomains, we obtain

$$\left\| E_D \boldsymbol{u}_\Gamma \right\|^2_{\widetilde{S}_\Gamma} \leq C \left(1 + \log H/h \right)^2 \left\| \boldsymbol{u}_\Gamma \right\|^2_{\widetilde{S}_\Gamma} . \tag{4.46}$$

□

4.3. Main result. We consider the preconditioned linear system $M^{-1} \widehat{S}_\Gamma \boldsymbol{u}_\Gamma = M^{-1} \boldsymbol{g}_\Gamma$.

THEOREM 4.11 (condition number estimate). *The condition number of the preconditioned linear system $M^{-1} \widehat{S}_\Gamma \boldsymbol{u}_\Gamma = M^{-1} \boldsymbol{g}_\Gamma$ satisfies*

$$\kappa \left(M^{-1} \widehat{S}_\Gamma \right) \leq C \left(1 + \log H/h \right)^2 . \tag{4.47}$$

Proof. We need upper and lower bounds on the eigenvalues of the generalized eigenvalue problem $\widehat{S}_\Gamma \boldsymbol{u}_\Gamma = \lambda M \boldsymbol{u}_\Gamma$. We note that the lower bound is always greater than or equal to 1 and the upper bound is given by the norm of the averaging operator (3.3); see [31, Theorem 25], [26, Theorem 1], and [46, Theorem 6.1]. Therefore, (4.47) follows from Lemma 4.10. □

5. Numerical results.

5.1. The two-dimensional case. We have applied the BDDC algorithm to our model problem (1.2). For algorithmic details, we follow [27] and section 3.3. We set $\Omega = (0, 1)^2$ and decompose the unit square into N^2 subdomains. Each subdomain has a side length $H = 1/N$. Moreover, we assume that the coefficients α and β have jumps across the interface between the subdomains with a checkerboard pattern as in Fig. 5.1. We discretize the model problem (1.2) by using the lowest order Raviart-Thomas finite elements for triangles and use the preconditioned conjugate gradient

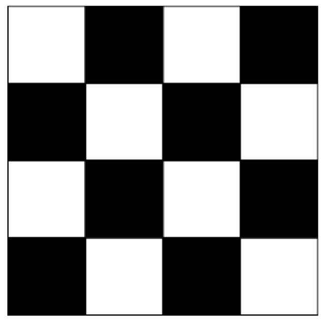

FIG. 5.1. *Checkerboard distribution of the coefficients (2D case)*

method to solve the discretized problem. The iteration is stopped when the l^2-norm of the residual has been reduced by a factor of 10^{-6}.

We have three different sets of experiments. We first fix the value of β and vary α. Second, we fix the value of α and vary β. Tables 5.1 and 5.2 show the first two sets of results. For the final set of experiments, we use a different distribution, instead of the checkerboard distribution, We first generate $2N^2$ random numbers $\{r_{\alpha_i}\}_{i=1,\ldots,N^2}$ and $\{r_{\beta_i}\}_{i=1,\ldots,N^2}$ in $[-3,3]$ with a uniform distribution. We then assign $10^{r_{\alpha_i}}$ and $10^{r_{\beta_i}}$ for α_i and β_i, respectively. The last results can be found in Table 5.3. In Fig. 5.2, we see that the condition number grows quadratically with the logarithm of H/h; it is insensitive to the jumps of coefficients.

TABLE 5.1

Condition numbers (Cond) and iteration counts (Iters): 2D case. $\alpha_i = \alpha_w = 1$ for the white subregions and $\alpha_i = \alpha_b$ for the black subregions as indicated in a checkerboard pattern as in Fig. 5.1, $\beta_i \equiv 1$, and $N = 4$.

	$H/h=4$		$H/h=8$		$H/h=16$		$H/h=32$		$H/h=64$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
$\alpha_i=10^{-2}$	1.49	6	2.03	8	2.72	9	3.54	11	4.51	12
$\alpha_i=10^{-1}$	1.61	7	2.19	8	2.92	10	3.79	11	4.80	12
$\alpha_i=10^{0}$	1.62	6	2.21	8	2.95	9	3.82	10	4.84	11
$\alpha_i=10^{1}$	1.62	7	2.21	8	2.95	9	3.83	11	4.84	12
$\alpha_i=10^{2}$	1.63	7	2.21	8	2.95	9	3.83	11	4.84	12

TABLE 5.2

Condition numbers (Cond) and iteration counts (Iters): 2D case. $\beta_i = \beta_w = 1$ for the white subregions and $\beta_i = \beta_b$ for the black subregions as indicated in a checkerboard pattern as in Fig. 5.1, $\alpha_i \equiv 1$, and $N = 4$.

	$H/h=4$		$H/h=8$		$H/h=16$		$H/h=32$		$H/h=64$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
$\beta_i=10^{-2}$	1.03	3	1.05	4	1.08	4	1.12	4	1.17	5
$\beta_i=10^{-1}$	1.22	5	1.43	6	1.69	7	2.00	8	2.37	9
$\beta_i=10^{0}$	1.62	6	2.21	8	2.95	9	3.82	10	4.84	11
$\beta_i=10^{1}$	1.21	5	1.42	6	1.68	7	2.00	7	2.36	9
$\beta_i=10^{2}$	1.02	3	1.05	4	1.08	4	1.12	4	1.16	5

TABLE 5.3
Condition numbers (Cond) and iteration counts (Iters) with random coefficients: 2D case. $N = 4$.

	$H/h = 4$		$H/h = 8$		$H/h = 16$		$H/h = 32$		$H/h = 64$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
Set 1	1.39	6	1.82	7	2.36	9	3.01	9	3.77	10
Set 2	1.32	5	1.79	7	2.31	8	2.94	9	3.67	10
Set 3	1.26	5	1.67	6	2.11	7	2.64	8	3.26	9
Set 4	1.64	7	2.21	8	2.95	9	3.84	11	4.87	12
Set 5	1.46	6	1.95	7	2.58	8	3.33	9	4.20	11

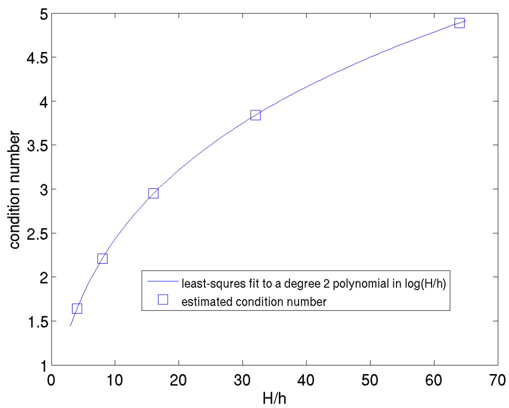

FIG. 5.2. *Estimated condition numbers in Table 5.3 and least-squares fit to a degree 2 polynomial in $\log H/h$, versus H/h (2D case).*

5.2. The three-dimensional case. For the three-dimensional case, we use the unit cube $(0,1)^3$ for Ω. In a way similar to the two-dimensional case, we decompose the domain into N^3 subdomains with the side length $H = 1/N$. We use the lowest order hexahedral Raviart-Thomas elements for this case and a similar checkerboard distribution of the coefficients as in the two-dimensional case; see Fig. 5.3. We use the stopping criteria of reducing the l^2-norm of the residual by a factor of 10^{-6} for the preconditioned conjugate gradient method. Other general settings are also similar to those of the two-dimensional case.
We find that the results for the three-dimensional case are quite similar to those of the two-dimensional case; see Tables 5.4, 5.5, and 5.6, and Fig. 5.4. The condition numbers depend quadratically on the value of $\log H/h$ and are independent of the jumps of coefficients across the interface.

5.3. Jumps inside subdomains. We report on numerical experiments for the case where coefficients have jumps inside the subdomains. We follow the general settings in section 5.2 for these experiments but use different coefficient distributions. For each subdomain Ω_i, we let $\Omega_i^o = \{(x,y,z)\,|\,1/4 \leq x^o, y^o, z^o \leq 1/2, \text{where}\, x^o = x/H - \lfloor x/H \rfloor, y^o = y/H - \lfloor y/H \rfloor, \text{and}\, z^o = z/H - \lfloor z/H \rfloor.\}$. Here, $\lfloor x \rfloor = \max\{m \in \mathbb{Z}\,|\,m \leq x\}$, where $\mathbb{Z}$ is the set of integers. We use the α_i and β_i specified in section 5.2 as coefficients for $\Omega_i \backslash \Omega_i^o$. For Ω_i^o, we assign $100\alpha_i$ and $100\beta_i$ as coefficients in the black subregions and α_i and β_i as coefficients in the white subregions. Table 5.7 and 5.8 show the results. We note that our theory does not cover these cases. However, we see that our method works well even though we have discontinuities inside the

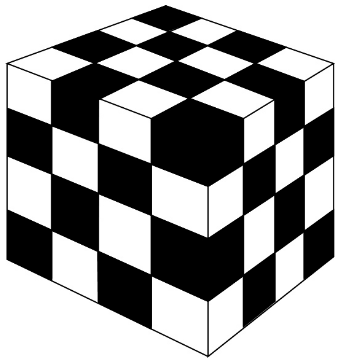

Fig. 5.3. *Checkerboard distribution of the coefficients (3D case)*

Table 5.4

Condition numbers (Cond) and iteration counts (Iters): 3D case. $\alpha_i = \alpha_w = 1$ for the white subregions and $\alpha_i = \alpha_b$ for the black subregions as indicated in a checkerboard pattern as in Fig. 5.3, $\beta_i \equiv 1$, and $N = 4$.

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
$\alpha_b = 10^{-2}$	1.64	7	2.32	9	3.26	11	4.37	13
$\alpha_b = 10^{-1}$	1.80	7	2.64	9	3.70	12	4.94	13
$\alpha_b = 10^{0}$	1.83	7	2.69	10	3.75	11	5.01	14
$\alpha_b = 10^{1}$	1.83	7	2.69	10	3.76	11	5.02	14
$\alpha_b = 10^{2}$	1.83	7	2.69	10	3.76	11	5.02	14

Table 5.5

Condition numbers (Cond) and iteration counts (Iters): 3D case. $\beta_i = \beta_w = 1$ for the white subregions and $\beta_i = \beta_b$ for the black subregions as indicated in a checkerboard pattern as in Fig. 5.3, $\alpha_i \equiv 1$, and $N = 4$.

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
$\beta_b = 10^{-2}$	1.03	3	1.06	4	1.09	4	1.12	4
$\beta_b = 10^{-1}$	1.28	5	1.53	6	1.89	8	2.31	9
$\beta_b = 10^{0}$	1.83	7	2.69	10	3.75	11	5.01	14
$\beta_b = 10^{1}$	1.27	5	1.51	6	1.85	7	2.27	9
$\beta_b = 10^{2}$	1.02	3	1.05	4	1.08	4	1.12	4

Table 5.6

Condition numbers (Cond) and iteration counts (Iters) with random coefficients: 3D case. $N = 4$.

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Cond	Iters	Cond	Iters	Cond	Iters	Cond	Iters
Set 1	1.80	8	2.69	11	3.76	13	5.01	16
Set 2	1.65	8	2.37	9	3.39	11	4.61	14
Set 3	1.78	8	2.50	10	3.49	12	4.82	14
Set 4	1.67	8	2.50	10	3.50	12	4.68	14
Set 5	1.74	8	2.49	10	3.45	13	4.54	15

subdomains.

5.4. The effect of using a conventional weighted averaging technique. In this section, for a comparison, we report on some numerical experiments using conventional techniques. We have performed three different types of experiments with

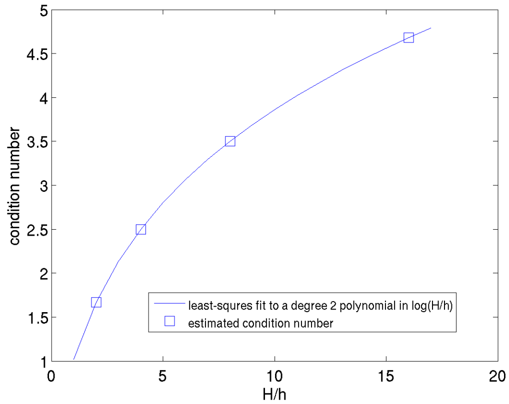

Fig. 5.4. *Estimated condition numbers in Table 5.6 and least-squares fit to a degree 2 polynomial in* $\log H/h$*, versus* H/h *(3D case).*

Table 5.7

Condition numbers (Cond) and iteration counts (Iters). Specified values as indicated in Table 5.4 with jumps inside subdomains and $N = 4$*.*

	$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Cond	Iters	Cond	Iters	Cond	Iters
$\alpha_i = 10^{-2}$	2.32	9	3.34	11	4.41	13
$\alpha_i = 10^{-1}$	2.64	9	3.83	12	5.05	14
$\alpha_i = 10^{0}$	2.69	10	3.90	12	5.16	14
$\alpha_i = 10^{1}$	2.69	10	3.91	12	5.17	14
$\alpha_i = 10^{2}$	2.69	10	3.91	12	5.17	14

Table 5.8

Condition numbers (Cond) and iteration counts (Iters). Specified values as indicated in Table 5.5 with jumps inside subdomains and $N = 4$*.*

	$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Cond	Iters	Cond	Iters	Cond	Iters
$\beta_i = 10^{-2}$	1.05	4	1.09	4	1.13	4
$\beta_i = 10^{-1}$	1.51	6	1.90	8	2.34	9
$\beta_i = 10^{0}$	2.69	10	3.90	12	5.16	14
$\beta_i = 10^{1}$	1.53	6	1.95	8	2.39	9
$\beta_i = 10^{2}$	1.06	4	1.09	4	1.13	4

the same set of coefficients distribution. The first set of experiments, named "sc", is based on the weighted averaging techniques described in (3.13). In the second, "diag", we use the conventional methods described in [10,31]. In this case, the scaling is based on the diagonal entries of each subdomain matrix. We use the cardinality in the last set, "card". For Raviart-Thomas elements, only two subdomains share a subdomain face in common. Hence, we use $1/2$ as scaling factors. For other general settings, we follow section 5.2. As we see in Table 5.9, our weighted averaging technique works well while the others are sensitive to the discontinuities across the interface.

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TABLE 5.9

Condition numbers (Cond) and iteration counts(Iters). $\alpha_i = \alpha_w = 1$ and $\beta_i = \beta_w = 1$ for the white subregions and $\alpha_i = \alpha_b$ and $\beta_i = \beta_b$ for the black subregions as indicated in a checkerboard pattern as in Fig. 5.3, $N = 4$, and $H/h = 8$.

	sc		diag		card	
	Cond	Iters	Cond	Iters	Cond	Iters
$\alpha_b = 10^{-2}$, $\beta_b = 10^2$	1.17	4	1.88e2	36	5.13e1	31
$\alpha_b = 10^{-1}$, $\beta_b = 10^1$	1.82	7	7.22e1	43	2.19e1	30
$\alpha_b = 10^0$, $\beta_b = 10^0$	3.75	11	3.75	11	3.75	11
$\alpha_b = 10^1$, $\beta_b = 10^{-1}$	1.89	8	8.63e1	48	2.61e1	32
$\alpha_b = 10^2$, $\beta_b = 10^{-2}$	1.09	4	1.01e3	74	2.58e2	66

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