Geometry Seminar
Tuesday, Nov 1, 2011
Room 512 WWH at 6:00 P.M.

The computation of multiple roots of polynomials whose coefficients are inexact

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This lecture will show by example some of the problems that occur when the roots of a polynomial are computed using a standard polynomial root solver. In particular, polynomials of high degree with a large number of multiple roots will be considered, and it will be shown that even roundoff error due to floating point arithmetic, in the absence of data errors, is sufficient to cause totally incorrect results to be obtained. Since data errors are usually larger than roundoff errors (and fundamentally different in character), the errors encountered with real world data are significant and emphasise the need for a computationally robust polynomial root solver.

The inability of commonly used polynomial root solvers to compute high degree multiple roots correctly requires investigation. A method developed by Gauss for computing the roots of a polynomial will be discussed, and it will be shown that it has an elegant geometric interpretation in terms of pejorative manifolds, which were introduced by William Kahan (Berkeley). Polynomials defined by points on these manifolds satisfy properties that are fundamentally different from the properties of polynomials defined by points that are not on these manifolds. The numerical interpretation of this difference provides the motivation for the method of Gauss, and the geometric properties of pejorative manifolds will therefore be emphasised and considered in detail. Furthermore, these properties explain why multiple roots are preserved in a floating point environment when the coefficients of the polynomial are corrupted by noise.

This numerical interpretation leads naturally to a discussion of a structured condition number of a root of a polynomial, where structure refers to the form of the perturbations that are applied to the coefficients. It will be shown that this structured condition number, where the perturbations are such that the multiplicities of the roots are preserved, differs significantly from the standard componentwise and normwise condition numbers, which refer to random (unstructured) perturbations of the coefficients. Several examples will be given

and it will be shown that the condition number of a multiple root of a polynomial due to a random perturbation in the coefficients is large, but the structured condition number of the same root is small. This large difference is typically several orders of magnitude.

The computational implementation of the method of Gauss raises some non-trivial issues the determination of the rank of a matrix in a floating point environment and the quotient of two inexact polynomials and they will be discussed because they are ill-posed operations. They must be implemented with care because simple methods will necessarily lead to incorrect results. Furthermore, problems occur when the coefficients of the polynomial span several orders of magnitude, in which case the polynomial must be processed before its roots are computed in order to guarantee computationally reliable arithmetic operations.

I will finish the talk by demonstrating Matlab code that implements the method on several high degree polynomials whose coefficients have been corrupted by noise and whose theoretically exact forms have multiple roots of high degree.

For more information please visit the seminar website at:
`http://www.math.nyu.edu/seminars/geometry_seminar.html`.