Jan 1990 Advanced Calculus

Problem 1.1

So we are given that \( \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x| < 2 \). Then we know that the summation converges absolutely for \( |x| < 2 \). Then pick some \( x \) and \( \epsilon \) such that

\[
\left| x \right| < 2 \quad \text{and} \quad \left| (1 + \epsilon) x \right| < 2.
\]

Then notice

\[
\left| \sum_{n=1}^{\infty} b_n x^n \right| \leq \left| \sum_{n=1}^{\infty} b_n (1 + \epsilon)^n x^n \right| \leq \sum_{n=1}^{\infty} \frac{|b_n|(1 + \epsilon)^n x^n}{(1 + \epsilon)^n}
\]

\[
\leq \sum_{n=1}^{\infty} \frac{n^2 |a_n|}{(1 + \epsilon)^n} |(1 + \epsilon)x|^n \leq \sum_{n=1}^{N} \frac{n^2 |a_n|}{(1 + \epsilon)^n} |(1 + \epsilon)x|^n + \sum_{n=N}^{\infty} |a_n| |(1 + \epsilon)|^n
\]

Hence the summation \( \sum b_n \) converges.

Problem 1.2

We know that \( \sum_{n=3}^{\infty} \frac{1}{n \ln^2 n} \) converges if and only if

\[
\int_{3}^{\infty} \frac{1}{x \ln^2 x} \, dx
\]

and we know that the integral does indeed converge since

\[
\int_{3}^{\infty} \frac{1}{x \ln^2 x} \, dx = -\frac{1}{\ln \infty} + \frac{1}{\ln 3} = \frac{1}{\ln 3} < \infty
\]

Hence the summation \( \sum_{n=3}^{\infty} \frac{1}{n \ln^2 n} \) converges.

Problem 1.3

Recall that functions in \( C^k \) can take the \( k \)th derivative and still be continuous. Thus we can expand \( f \) by

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + E(x)
\]

where \( E(x) \) is the error function and

\[
\lim_{x \to 0} \frac{E(x)}{x^3} = 0
\]

Thus
\[
\sum_{n=1}^{\infty} \{n[f(1/n) - f(-1/n)] - 2f'(0)\}
\]

\[
= \sum_{n=1}^{\infty} n \left( f(0) + \frac{f'(0)}{n} + \frac{f''(0)}{3!n^3} + E(1/n) \right) - n \left( f(0) - \frac{f'(0)}{n} + \frac{f''(0)}{3!n^3} + E(-1/n) \right) - 2f'(0)
\]

\[
= \sum_{n=1}^{\infty} n \left( \frac{2f''(0)}{3!n^3} + E(1/n) - E(-1/n) \right) = \sum_{n=1}^{\infty} \frac{2f''(0)}{3n^3} + \sum_{n=1}^{\infty} (E(1/n) - E(-1/n))
\]

Notice from \(\lim_{x \to 0} E(x)/x^3 = 0\), let \(\epsilon = 1\). Then for \(|x| < \delta\), \(E(x) \leq x^3\). For \(n > 1/\delta\) we have \(E(1/n) \leq 1/n^3\) and \(|E(-1/n)| \leq 1/n^3\). Hence

\[
\sum_{n=1}^{\infty} (E(1/n) - E(-1/n)) \leq \sum_{n=1}^{1/\delta} n(E(1/n) - E(-1/n)) + \sum_{n=1/\delta}^{\infty} \frac{1}{n^2} < \infty
\]

Hence it converges.

**Problem 2.1**

Notice

\[
D_{\lambda \cdot \downarrow} f = \lim_{t \to 0} \frac{f(x_0 + t\lambda \cdot \downarrow) - f(x_0)}{t} = \lim_{t' \to 0} \frac{f(x_0 + t' \cdot \downarrow) - f(x_0)}{t'} = \lambda \lim_{t' \to 0} \frac{f(x_0 + t' \cdot \downarrow) - f(x_0)}{t'} = \lambda D_{\cdot \downarrow} f(x_0)
\]

**Problem 2.2**

By L’Hospitals rule, we have

\[
D_{\lambda \cdot \downarrow} f = \lim_{t \to 0} \frac{f(x_0 + t\lambda \cdot \downarrow) - f(x_0)}{t} = \lim_{t \to 0} \frac{f'(x_0 + t \cdot \downarrow)}{1} = \frac{\text{grad}(f) \cdot \downarrow}{1}
\]

**Problem 2.3**

\[
D_{\cdot \downarrow} f(0) = \lim_{t \to 0} \frac{f(t \cdot \downarrow) - 0}{t} = \lim_{t \to 0} \frac{(tv_x)^2(1 + 1/v_y)}{t} = \lim_{t \to 0} tv_x^2(1 + 1/v_y) = 0
\]
Problem 3.1

So we have
\[ \int \int F \cdot ndS = \int \int \int_{B'(0, \epsilon)} \nabla \cdot F dV \]

Where \( B'(0, \epsilon) \) is the ball centered at the origin with radius \( \epsilon \). Then
\[ \int \int_{|x|=1} F \cdot ndS + \int \int_{|x|=\epsilon} F \cdot ndS = \int \int \int_{\Omega} \frac{3}{r^3} dV \]

which implies
\[ \int \int_{|x|=1} F \cdot ndS = \int \int \int_{\Omega} \frac{3}{r^3} dV - \int \int_{|x|=\epsilon} F \cdot ndS = \frac{3}{r^3} \left( \frac{4}{3} \pi (1 - \epsilon^3) \right) - \int \int_{|x|=\epsilon} F \cdot ndS \]

Now notice that
\[ \left| \int \int_{|x|=\epsilon} F \cdot ndS \right| \leq 4\pi \epsilon^2 \max |F| \leq 4\pi \epsilon^2 \frac{\sqrt{3\epsilon^3}}{r^3} = \frac{r \pi \sqrt{3\epsilon^3}}{r^3} \to 0 \]
as \( \epsilon \to 0 \). Hence
\[ \int \int_{|x|=1} F \cdot ndS = \frac{4\pi}{r^3} \]

Problem 3.1

Notice
\[ \int \int_{|x|=1} F \cdot ndS = \int \int_{|x|=1} \frac{1}{r^3} (x, y, z) \cdot \frac{(x, y, z)}{r} dS = \int \int_{|x|=1} \frac{x^2 + y^2 + z^2}{r^4} dS = \int \int_{|x|=1} dS = 4\pi \]

Problem 4.1

\[ \int_{\Omega} \int f(x, y) dx dy = \int_{\Gamma} f(x(u, v), y(u, v)) |J(u, v)| dudv \]

where
\[ J(u, v) = \left| \begin{array}{cc} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{array} \right| = \frac{dx}{du} \frac{dy}{dv} - \frac{dx}{dv} \frac{dy}{du} \]
Problem 4.2

Notice that \( x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \leq 2 \). Hence let \( u = x - y \) and \( v = y \). Thus \( x = u + v \) and \( y = v \) and \( |J(u, v)| = 1 \) and using the formula above we have

\[
(x^2 - 2xy + 2y^2)^3 = ((u + v)^2 - 2(u + v)v + 2v^2)^3 = (u^2 + v^2)^3
\]

Hence we have for \( R' = \{(u, v) : u^2 + v^2 \leq 2 \} \)

\[
\int_R \int (x^2 - 2xy + 2y^2)^3 \, dx \, dy = \int_{R'} \int (u^2 + v^2)^3 \, du \, dv = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} r^6 r \, dr \, d\theta = 4\pi
\]

Problem 5

Let \( g(x) = f^2(x) + f''(x) \). Then \( g'(x) = 2f'(x)(f(x) + f''(x)) \). Since \( f \in C^2 \), this implies that \( g \) is indeed continuous. Now notice that \( g(0) = 4 \) and \( |f'(0)| > 1 \) since \( f^2(0) + f''(0) = 4 \). Now consider the following cases **CASE 1:** \( f'(0) > 1 \). Now clearly \( f'(x) \neq 1 \) \( \forall x > 0 \) since \( f \) is bounded. Hence \( \exists \) a minimum \( a > 0 \) such that \( f'(a) = 1 \). This implies that \( g(a) < 2 \), and so by the Intermediate value theorem, \( \exists c_1 \) such that \( g(c_1) = 2 \).

Likewise **CASE 2:** Now notice \( f'(x) \neq 1 \) \( \forall x < 0 \) since \( f \) is bounded. Thus \( \exists \) a greatest \( b < 0 \) such that \( f'(b) = 1 \). Again we have \( g(b) < 2 \) and by the Intermediate value theorem, \( \exists c_2 \) such that \( f(c_2) = 2 \). Now by Rolle’s Theorem, \( \exists d \) such that \( g'(d) = 0 \), where \( d \in (c_1, c_2) \). And since we chose \( a \) and \( b \) as the smallest and largest, we know that \( f'(d) \neq 0 \). Hence

\[
f''(d) + f(d) = 0
\]

Jan 1990 Complex Variables

Problem 1

Let \( f(z) = z^5 + 3z + 1 \). Using the argument principle, let \( C \) be the quarter circle in the first quadrant with radius \( R \). Then along the real axis \( \arg f \) does not change. Then along the arc, we have \( z = Re^{i\theta} \) for \( \theta : 0 \to \pi/2 \), we have

\[
f(z) = R^5 e^{5i\theta} \left( 1 + \frac{3}{R^4 e^{4i\theta}} + \frac{1}{R^5 e^{5i\theta}} \right)
\]

As \( R \to \infty \) notice \( f(z) \approx R^5 e^{5i\theta} \). Hence the charge in \( \arg f \) along the arc in \( 5\pi/2 \). Then for \( z = iy \) where \( y : R \to 0 \), we have

\[
f(iy) = 1 + i(y^5 + y)
\]

Thus the \( \arg g \) goes back to \( \pi/2 \). Hence total change is \( 2\pi \). By the argument principle the number of zero’s in the first quadrant is 1.
Problem 2
Let $\gamma$ be the contour around the wedge $\{|z| \leq R, 0 \leq \arg z \leq 2\pi/n\}$ and $f(z) = 1/(1 + z^n)$. Then by the Residue Theorem we have

$$\int_{\gamma} \frac{dz}{1 + z^n} = 2\pi i \text{Res} \left( \frac{1}{1 + z^n}, e^{i\pi/n} \right) = -2\pi i \frac{e^{i\pi/n}}{n}$$

where $n \geq 2$. Now along the arc $C_R$, we have $z = Re^{i\theta}$ and notice

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{2\pi}{n} \frac{R}{R^2 - 1} \to 0$$
as $R \to \infty$. Now let $\gamma_1$ be the part of the contour on the real axis and $\gamma_2$ be the rest of the contour.

$$\int_{\gamma} \frac{dz}{1 + z^n} = \int_{\gamma_1} \frac{dz}{1 + z^n} + \int_{\gamma_2} \frac{dz}{1 + z^n} = -2\pi i \frac{e^{i\pi/n}}{n}$$

and notice on $\gamma_2$, we have $z = xe^{2\pi i/n}$, and $dz = e^{2\pi i/n}dx$ for $0 \leq x \leq R$. Then

$$\int_{\gamma_2} f(z)dz = \int_{0}^{R} \frac{-e^{2\pi i/n}dx}{1 + x^n} = -e^{2\pi i/n} \int_{\gamma_1} f(z)dz$$

Hence as $R \to \infty$, we have

$$\int_{\gamma} \frac{dz}{1 + z^n} = \int_{0}^{\infty} \frac{dx}{1 + x^n} (1 - e^{2\pi i/n}) = -2\pi i \frac{e^{i\pi/n}}{n}$$

Hence

$$\int_{0}^{\infty} \frac{(-2\pi i/n)e^{i\pi/n}}{1 - e^{2\pi i/n}} = \frac{(-2\pi i/n)}{-2i \sin(\pi/n)} = \frac{(\pi/n)}{\sin(\pi/n)}$$

Problem 3
Since $f$ is entire, it has a Taylor series about $z = 0$, and hence

$$f(z) = \sum_{i=0}^{\infty} a_i z^i \quad \text{where} \quad a_i = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta^{i+1}}d\zeta$$

Since $M(2R) \leq 2^N M(R)$ $\forall R > 0$, we have $M(R) \leq 2^N \log R M(1)$. Hence let

$$|a_i| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta^{i+1}}d\zeta \right| \leq \frac{1}{2\pi} 2\pi R \max \{|f(\zeta)|\} \frac{\max \{|f(\zeta)|\}}{R^{i+1}} = \frac{M(1)R^N}{R^i}$$

Hence if $i > N$, and as $R \to \infty$, $a_i = 0$. Thus $f$ is a polynomial of degree at most $N$. 


Problem 4
We first map the wedge to the upper half plane \( U \) by \( z^{1/\alpha} \). Then we map \( U \) to the unit disk by \( \frac{z-i}{z+i} \). Hence our conformal map is

\[
w = \frac{z^{1/\alpha} - i}{z^{1/\alpha} + i}
\]

Jan 1990 Linear Algebra

Problem 1.1
\( A \) is invertible if and only if \( \det(A) = ad - bc \neq 0 \).

Problem 1.2
We must have \( ad - bc = 0 \) and \( \max(|a|, |b|, |c|, |d|) > 0 \).

Problem 1.3
We must have \( b = c \) and \( a \geq 0 \) and \( ad - bc \geq 0 \). Recall that this also implies that the eigenvalues are positive.

Problem 1.4
We need \( A^2 = A \) so

\[
a^2 + bc = a
\]
\[
ab + bd = b
\]
\[
ac + dc = c
\]
\[
bc + d^2 = d
\]

Problem 1.5
\( \det(A) = ad - bc = \pm 1 \)

Problem 1.6
If \( a \) and \( d \) are real, and \( b = \overline{c} \).
Problem 1.7
We must have $A^H A = AA^H = I$. Thus

\[
\begin{align*}
    a\bar{a} + c\bar{c} &= 1 \\
    \bar{a}b + \bar{c}d &= 0 \\
    a\bar{b} + c\bar{d} &= 0 \\
    b\bar{b} + d\bar{d} &= 1
\end{align*}
\]

Problem 1.8
For $A$ to be diagonalizable, we must have $A^H A = AA^H$. Hence

\[
\begin{align*}
    \bar{a}\bar{c} &= \bar{b}\bar{d} \\
    \bar{a}b + \bar{c}d &= a\bar{c} + b\bar{d} \\
    a\bar{b} + d\bar{c} &= ac + db \\
    b\bar{b} + d\bar{d} &= c\bar{c} + d\bar{d}
\end{align*}
\]

Problem 2.1
Let $A$ be an $m \times n$ matrix. Then \exists a $n \times m$ matrix $B$ such that $AB = I_m$ if and only if

\[
A \left( \begin{array}{c|c|c|c}
    b_1 & b_2 & \cdots & b_m \\
\end{array} \right) = \left( \begin{array}{c|c|c|c}
    \bar{e}_1 & \bar{e}_2 & \cdots & \bar{e}_m \\
\end{array} \right)
\]

if and only if for $i \leftarrow 1$ to $m$ we have

\[
Ab_i = \bar{e}_i
\]

if and only if the column space of $A$ equals to $\mathbb{R}^m$, if and only if $\text{Rank}(A) = m$. Thus $r = m \leq n$
Problem 2.2
Let $A$ be an $m \times n$ matrix. Then $\exists$ a $m \times n$ matrix $C$ such that $CA = I_n$ if and only if
$$
\begin{pmatrix}
\leftarrow c_1 \rightarrow \\
\leftarrow c_2 \rightarrow \\
\vdots \\
\leftarrow c_n \rightarrow 
\end{pmatrix} A =
\begin{pmatrix}
\leftarrow e_1 \rightarrow \\
\leftarrow e_2 \rightarrow \\
\vdots \\
\leftarrow e_n \rightarrow 
\end{pmatrix}
$$
if and only if for $i \leftarrow 1$ to $n$,
$$c_i A = e_i$$
if and only if the row space of $A$ is the $\mathbb{R}^n$, if and only if $\text{Rank}(A) = n$. Thus $r = n \geq m$. Now if both inverses exists, then notice
$$C = C(AB) + (CA)B = B$$

Problem 3.1
We will apply Gram Schmidt to $u_0, u_1, u_2$ and get the orthonormal basis $w_0, w_1, w_2$ where
$$w_0 = \frac{u_0}{\|u_0\|} = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$$
$$w_1 = u_1' - \frac{(u_1, w_0)}{(w_0, w_0)} w_0^* = (0, -1/2 - i/2, 1/2 + i/2)^T$$
$$w_2 = u_2^* - \frac{(u_2, w_0)}{(w_0, w_0)} w_0^* - \frac{(u_2, w_1)}{(w_1, w_1)} w_1^* = (1, 0, 0)^T$$

Problem 3.2
Thus by part (a) we have the decomposition $A = UR$ where
$$A = UR = \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ q_2^T b & q_2^T c \\ q_3^T c \end{bmatrix}$$
which implies
$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/2 - i/2 & 0 \\ 1/\sqrt{2} & 1/2 + i/2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} + i/\sqrt{2} & 1/\sqrt{2} + i/\sqrt{2} \\ 1/\sqrt{2} + i/\sqrt{2} & 1 & -1 \\ 1/\sqrt{2} + i/\sqrt{2} & 1 & 1 \end{pmatrix}$$
and $\det(U) = 1/\sqrt{2} + i/\sqrt{2}$.

Problem 3.3
Notice that $R$ can be modified if we reorder $u_i$’s. But notice $u_0'^T a > 0$ and $u_0'^T b$ and $u_0'^T c$ is not $> 0$. Therefore we must have it in the order above.
Problem 4.1
So we know there is an eigenvalue of 1 corresponding to an eigenvector $v$ since it’s a rotation matrix. Therefore

$$R - I = \begin{pmatrix} -1 - \sqrt{2} & -1 & 0 \\ \sqrt{2} & \sqrt{2} & 1 \\ -1 & 1 & 1 - \sqrt{2} \end{pmatrix}$$

By performing gaussian elimination we get

$$v = \begin{pmatrix} -\sqrt{7} \\ 3 + 2\sqrt{2} \\ 2 + \sqrt{2} \end{pmatrix}$$

Problem 4.2
We can see that $Tr(R) = -1/2$ and $det(R) = 1$. Also we know that

$$\lambda_1 + \lambda_2 + 1 = \frac{-1}{2} \quad \text{and} \quad \lambda_1 \lambda_2 = 1$$

hence we have two equations and two unknowns. Also since all entries of $R$ are real, the eigenvalues $\lambda_1$ and $\lambda_2$ are complex conjugates of each other. Therefore

$$\lambda_1 = \frac{-3 + i\sqrt{7}}{4} \quad \lambda_2 = \frac{-3 - i\sqrt{7}}{4}$$

Problem 4.2
By definition of a rotation matrix, it preserves length. Therefore for eigenvector $h$ of $\lambda_i$

$$||Ah|| = ||\lambda_i h|| = ||\lambda_i|| ||h||$$

implies that $|\lambda_i| = 1$.

Sept 1990 Advanced Calculus

Problem 2.a
Notice

$$\lim_{n \to \infty} n^2 \left[ \left( \frac{1 + \frac{p}{n}}{1 - \frac{p}{n}} \right)^{n/(2p)} - 1 \right] = \lim_{n \to \infty} \left( \frac{2n + 2p}{2n - 2p} \right)^{n/(2p)} - e$$

and by L’Hopital’s rule we have
\[
\left(\frac{2n+2p}{2n-2p}\right)^{n/(2p)} \left(\frac{1}{2p} \ln \left(\frac{2n+2p}{2n-2p}\right) + \frac{n}{2p} \left(\frac{2n-2p}{2n+2p}\right) \left(\frac{2(2n-2p) - 2(2n+2p)}{(2n-2p)^2}\right)\right)_{-2n^{-3}}
\]

which approaches \(\infty\) as \(n \to \infty\).

**Problem 2.b**

Notice this is just a Riemann sum

\[
\lim_{n \to \infty} \sum_{m=1}^{n} \frac{m^{p-1}}{n^p + m^p} = \lim_{n \to \infty} \sum_{m=1}^{n} \frac{1}{n} \left(\frac{m}{n}\right)^{p-1} = \int_{0}^{1} \frac{x^{p-1}}{1 + x^p} dx = \frac{\ln(2)}{p}
\]

**Problem 5**

So we want to solve

\[a_n = Aa_{n-1} + Ba_{n-2}\]

and by what we are given

\[4 = 2A + B \quad 9 = 3A + B\]

Thus \(A = 5\) and \(B = -6\). So we have

\[a_n = 5a_{n-1} - 5a_{n-2}\]

Hence by solving \(y^2 - 5y + 6 = 0\), we have \(y_1 = 3\) and \(y_2 = 2\). Therefore our general solution is

\[a_n = C3^n + D2^n\]

and since \(a_0 = a_1 = 1\) we have

\[a_n = 2^{n+1} - 3^n\]

**Sept 1990 Complex Variables**

**Problem 1.a**

\(|z - i| < |z|\) is simply \(\{x + iy : y > 1/2\}\). Then in polar coordinates we have \(z = re^{i\theta}\) where \(0 < \theta < \pi\) and \(\frac{1}{2} \csc \theta < r < \infty\).
Problem 1.b

We have
\[2|x - 1 + iy| < |x - 2 + iy| \Rightarrow 4((x - 1)^2 + y^2) < (x - 2)^2 + y^2\]
which implies
\[3x^2 - 4x + 3y^2 < 0 \Rightarrow x^2 - \frac{4}{3}x + y^2 < 0\]
Finally we have \((x - 2/3)^2 + y^2 < 4/9\). Hence in polar coordinates we have
\[z = \frac{2}{3} + re^{i\theta}\]
where \(0 < r < 2/3\) and \(0 < \theta < 2\pi\).

Problem 1.b

For \(|z + 1||z - 1| < 1\) the region is the two unit circle centered at 1 and -1.

Problem 2

Let \(\gamma_1\) be the circle \(|z| = 1 - \epsilon\) where \(\epsilon > 0\) is arbitrarily small. Then by Rouche notice
\[\lim_{n \to \infty} |z^n + z^3 + z + 2 - 2| = \lim_{n \to \infty} |z^n + z^3 + z| \leq \lim_{n \to \infty} |z^n| + |z^3| + |z| = 0 + (1 - \epsilon)^3 + (1 - \epsilon) < 2 = |2|\]
Hence there are no zeros inside \(|z| = 1 - \epsilon\) for really small \(\epsilon\). Now let \(\gamma_2\) be the circle \(|z| = 1 + \epsilon\) where \(\epsilon\) is arbitrarily small. Then notice \(\exists \ N\) such that
\[\lim_{n \to \infty} |z^n + z^3 + z + 2 - z^n| = \lim_{n \to \infty} |z^3 + z + 2| \leq \lim_{n \to \infty} |z^3| + |z| + 2\]
\[= (1 + \epsilon)^3 + 1 + \epsilon + 2 < 10 \leq (1 + \epsilon)^N = |z|^N \leq \lim_{n \to \infty} |z^n|\]
Hence all the zeros are inside \(\gamma_2\). Since \(\epsilon\) is arbitrarily small, all zeros converge to \(|z| = 1\) as \(n \to \infty\).

Problem 4

Let \(\gamma\) be the right-half semi-circle with radius \(R\). Then let \(f(z) = \frac{1}{z^3 + z^2 + 2z + 2} = \frac{1}{(z-1)(z+\sqrt{2})(z-\sqrt{2})}\).

Then by the Residue Theorem, we have
\[\int_\gamma f(z)\,dz = 2\pi i \left(-1 + \frac{1}{2\sqrt{2}(\sqrt{2} - 1)}\right)\]
Now notice on the arc, \(z = Re^{i\theta}\) and
\[\left|\int_{CR} f(z)\,dz\right| \leq \pi R \frac{1}{(R - 1)(R^2 - 2)} \to 0\]
as \( R \to \infty \). Hence
\[
\int_{-\infty}^{\infty} f(z) \, dz = 2\pi i \left( -1 + \frac{1}{2\sqrt{2}(\sqrt{2} - 1)} \right)
\]

**Problem 5**

By simply figuring out the inverse, we get \( z = \frac{\omega + i}{1 + \omega} \). Which is a linear transformation. So it maps lines/circles to lines/circles. Now if the line in the image is the imaginary axis, then clearly the pre-image is imaginary axis, all other pre-images of lines through the origin will be circles. They will be circles going through points \( i \) and \( -i \) since \( 0 \mapsto i \) and \( \infty \mapsto -i \). So let \( w = e^{i\theta} \). Then
\[
z = \frac{e^{i\theta} + e^{i\pi/2}}{1 + e^{i(\theta + \pi/2)}} = \frac{\cos \theta}{1 + \cos(\theta + \pi/2)}
\]

So given a line through the image with an angle \( \theta \), the pre-image is a circle going through the points \( i, -i \), and \( \frac{\cos \theta}{1 + \cos(\theta + \pi/2)} \).

**Sept 1990 Linear Algebra**

**Problem 1.1**

Notice
\[
\det \begin{pmatrix} -4 - \lambda & -5 \\ 10 & 11 - \lambda \end{pmatrix} = \lambda^2 - 7\lambda + 6 = 0
\]

Then we have \( \lambda_1 = 6 \) and \( \lambda_2 = 1 \). Then \( v_1 = (1, -1)^T \) and \( v_2 = (-1, 2)^T \). Hence
\[
A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]

Hence
\[
A^N = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6^N \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 - 6^N & 1 - 6^N \\ -2 + 2 \cdot 6^N & -1 + 2 \cdot 6^N \end{pmatrix}
\]

**Problem 1.2**

Notice
\[
I + \sum_{n=1}^{N} \epsilon^n A^n = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1 - \epsilon^{N+1}}{1 - \epsilon} & 0 \\ 0 & \frac{1 - (6\epsilon)^{N+1}}{1 - 6\epsilon} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}
\]

Hence it clearly converges for \( \epsilon < 1/6 \).
Problem 1.3

For $\epsilon < 1/6$, notice the limit is

$$\lim_{N \to \infty} I + \sum_{n=1}^{N} \epsilon^n A^n = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \epsilon & 0 & \frac{1}{1-6\epsilon} \\ \frac{1}{1-\epsilon} & 1 & \frac{2}{1-6\epsilon} \end{pmatrix} = (I-\epsilon)^{-1}$$

Indeed since $\lim A^N \to 0$ and

$$(I + \epsilon A + \epsilon^2 A^2 + \cdots + \epsilon^{N-1} A^{N-1})(I - \epsilon A) = I + \epsilon N A^N$$

Problem 2

Let $A = A_1$ be an $n \times n$ matrix. Then let $\lambda_1$ be any eigenvalue of $A$ with unit eigenvector $v_1$. Then let $B_1$ be the orthogonal basis $\{u_1, u_2, \ldots, u_n\}$ such that $u_1 = v_1$. This is possible by Gram Schmidt. Hence for

$$S_1 = \begin{pmatrix} | & | & \cdots & | \\ u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

we have $S^H S = S S^H = I$ and

$$S_1^H A S_1 = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \vdots & & \ \ \\ 0 & \vdots & A_2 & \\ 0 & \vdots & & \ \ \\ & \vdots & \vdots & \vdots \end{pmatrix}$$

now let $v_2$ be a unit eigenvector of $A_2$ with eigenvalues $\lambda_2$. Then let $u_2, \ldots, u_n$ be the new orthonormal basis of $S(A)$ such that $u_2 = v_2$. Then

$$S_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & | & \cdots & | \\ \vdots & u_2 & \cdots & u_n \\ 0 & | & \cdots & | \end{pmatrix}$$

Then we have $S_2^H S_2 = S_2 S_2^H = I$. Therefore

$$S_2^H S_1^H A S_1 S_2 = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \lambda_2 & \times & \times \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \vdots & A_2 \end{pmatrix}$$

As we continue this argument we have
\[
S^H_{n-1} \cdots S^H_2 S^H_1 A S_2 \cdots S_{n-1} = \begin{pmatrix}
\lambda_1 & \times & \times \\
\cdot & \times \\
\lambda_n
\end{pmatrix}
\]

Where \( U = S_1 S_2 \cdots S_n \). Then \( U^H U = UU^H = I \). Hence \( U^H A H = T \).

**Problem 3**

First I claim that

\[
\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \det(C)
\]

Indeed since recall the big formula for determinant

\[
\det(M) = \sum (\det P)a_{1\alpha}a_{2\beta} \cdots a_{n\omega}
\]

where \( P = (\alpha, \beta, \cdots, \omega) \). Basically the determinant is one entry from every row and column. So \( \det(A) \) is the sum over all \( n! \) column permutations \( P \). Then by this formula, the permutations picked in the lower left part of the matrix is 0. Likewise, every term picked in the upper right will force a choice in the lower left, which will multiply to 0. Hence

\[
\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \det(C)
\]

Since \( A^{-1} \) exists, let \( E = \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \). Then notice \( \det(E) = 1 \). Hence

\[
\begin{align*}
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(E) \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \left( \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \\
&= \det \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} = \det \left( A(D - CA^{-1}B) \right)
\end{align*}
\]

Since \( AC = CA \), we have

\[
ACA^{-1} = C \Rightarrow CA^{-1}A^{-1}C
\]

Hence we have

\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \left( A(A^{-1}AD - A^{-1}CB) \right) = \det \left( (AD - CB) \right)
\]
Problem 4.1

So we know we going to have

\[ A \begin{pmatrix} w_1 & w_2 & 2_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 & w_2 & 2_3 \end{pmatrix} \]

So

\[ Aw_1 = w_2 \]

By performing Gaussian elimination, we can see that \( w_1 = (1, -1, 1)^T \). Thus

\[ Aw_2 = w_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \]

solving this equation, we have \( w_2 = (0, -1, 0)^T \).

Problem 4.2

Solving \( Aw_3 = w_2 = (0, -1, 0)^T \) we get \( w_3 = (1/3, 1, 1)^T \).

Problem 4.3

So we have

\[ M = \begin{pmatrix} 1 & 0 & 1/3 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \]

Then \( M^{-1}AM = J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \).

Jan 1991 Advanced Calculus

Problem 1.1

We know that \( \exists N \) such that for all \( n > N \), we have \( \log^3 n < n \). Hence for \( n > N \),

\[ \frac{1}{n} < \frac{1}{\log^3 n} \Rightarrow \sum_{n=N}^{\infty} \frac{1}{n} < \sum_{n=N}^{\infty} \frac{1}{\log^3 n} \leq \sum_{n=2}^{\infty} \frac{1}{\log^3 n} \]

Since \( \sum_{n=N}^{\infty} \frac{1}{n} \to \infty \), the summation \( \sum_{n=2}^{\infty} \frac{1}{\log^3 n} \) converges.
Problem 1.2

Here I claim that \( \exists N \) such that for all \( n > N \), we have

\[
x^2 < (\log x)^{\log x}
\]

Indeed since for large enough \( x \) we have

\[
\log x^2 = 2\log x < \log x \log \log x
\]

Thus

\[
\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty
\]

Hence the summation converges.

Problem 2

We are given that

\[
\lim_{x \to \infty} f(x) = M < \infty
\]

Thus notice

\[
\lim_{x \to \infty} \frac{df}{dx}(x) = \lim_{x \to \infty} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \lim_{x \to \infty} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 0 = 0
\]

Hence

\[
\lim_{x \to \infty} \left( f(x) + \frac{df}{dx}(x) \right) = 0 \Rightarrow \lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} \frac{df}{dx}(x)
\]

Problem 3.a

By taking the derivative of \( f_n \) we have

\[
f'_n(x) = n(1-x)^n + n(1-x)^{n-1}(-1)n.x = n(1-x)^n - n^2x(1-x)^{n-1} = 0
\]

which implies

\[
(1-x)^n = nx(1-x)^{n-1} = 0 \Rightarrow 1 - x - nx = 0
\]

thus \( x = 1/(n+1) \). So the function is maximized at \( x_n = 1/(n+1) \), and we have

\[
M_n = \frac{n}{n+1} \left( 1 - \frac{1}{n+1} \right)^n = \frac{n}{n+1} \left( \frac{n}{n+1} \right)^n = \left( \frac{n}{n+1} \right)^{n+1}
\]
Problem 3.b
It suffices to show that $|f_n(x_n)| \leq M$ to prove that $f_n$ is uniformly bounded in $[0, 1]$. Notice
\[ f_n(x_n) = n \left( \frac{1}{n+1} \right) \left( 1 - \frac{1}{n+1} \right)^n \rightarrow e^{-1} < \infty \]

Notice pointwise, the function $f_n(x) \rightarrow 0$. Indeed since
\[ \lim_{n \to \infty} n^{x_n} = \lim_{n \to \infty} \frac{n x_n}{e^{-n \log(1-x)}} = \lim_{n \to \infty} \frac{x}{(1-x)^n(-\log(1-x))} = \lim_{n \to \infty} \frac{-x}{n \log(1-x)} (1-x)^n \rightarrow 0 \]
since $1-x < 0$. If $x = 1$, then it’s trivial. Now assume that it does converge uniformly. Then for all $\epsilon > 0$, $\exists N$ such that for all $n > N$, we have
\[ |f_n(x)| < \epsilon \]
for all $x \in [0, 1]$. But for some $n > N$, notice
\[ f_n(x_n) = n \left( \frac{1}{n+1} \right) \left( 1 - \frac{1}{n+1} \right)^n > e^{-1}/2 \]
hence we have a contradiction, and so the function does not converge uniformly.

Problem 3.c
Notice that $f_n$ is measurable for all $n$, and $|f_n| \leq 1$ for all $n$ and $x \in [0, 1]$. Also by (b), we see that $f_n \rightarrow f = 0$, pointwise almost everywhere as $n \to \infty$. Thus $f$ is integrable and
\[ \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \to \infty} f_n(x) \, dx \]

Problem 4.a
*Implicit Function Theorem:* Suppose $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ continuously and differently on an open set $A \subset \mathbb{R}^n \times \mathbb{R}$. Also suppose $\exists (a, b) \in \mathbb{R}^n \times \mathbb{R}$ such that $F(a, b) = 0$ and
\[ \frac{d}{du} F(a, b) = 0 \]
Then $\exists$ a neighborhood $W \subset \mathbb{R}^n$ of $a$ and a unique differential function $g : W \rightarrow \mathbb{R}$ such that $g(a) = b$ and
\[ F(t, g(t)) = 0 \quad \forall t \in W \]

Problem 4.b
Let $F(x, y, u) = x + y + u + x y u^3$. Then $F(0, 0, 0) = 0$ and $\frac{dF}{du}(0, 0, 0) = 1 \neq 0$. Thus by the the implicit function theorem, $\exists$ a continuous function $f(x, y)$ such that $f(x, y) = u$ in the neighborhood of $(0,0)$. -17
Problem 4.c
Thus \( f(x, y) \) in part (b) has a Taylor Expansion,
\[
f(x, y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j
\]
where
\[
a_i = \frac{1}{i!} \frac{d^i f}{dx^i}(0, 0) \quad \text{and} \quad b_j = \frac{1}{j!} \frac{d^j f}{dy^j}(0, 0)
\]
Therefore we have \( x + y + f(x, y) + xyf(x, y) = 0 \), which implies
\[
1 + \frac{df}{dx}(x, y) + yf(x, y) + xy \frac{df}{dx}(x, y) = 0
\]
Hence \( \frac{df}{dx}(0, 0) = -1 \). Likewise
\[
1 + \frac{df}{dy}(x, y) + xf(x, y) + xy \frac{df}{dy}(x, y) = 0
\]
Hence \( \frac{df}{dy}(0, 0) = -1 \). By following the hint, we can see that
\[
x + y + (-x - y + \cdots) + xy(-x - y + \cdots)^3 = 0
\]
implies that
\[
h(x, y) = -x - y + x^4y + xy^4 + \cdots
\]
Problem 5.a
We first make a change of basis so that the coordinates \( u, v, w \) leaves the circle in the \( u, v \)-plane. So we know that \((1, 1, 1)\) is the perpendicular vector of the plane \( x + y + z = 0 \). So let \( w = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \), then let \( u = (1/\sqrt{2}, 1/\sqrt{2}, 0) \) and \( v = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}) \). This implies
\[
x = \frac{1}{\sqrt{3}} w + \frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{6}} v
\]
\[
y = \frac{1}{\sqrt{3}} w - \frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{6}} v
\]
\[
z = \frac{1}{\sqrt{3}} w - \frac{2}{\sqrt{6}} v
\]
Then we have
\[
dx = \frac{1}{\sqrt{3}} dw + \frac{1}{\sqrt{2}} du + \frac{1}{\sqrt{6}} dv
\]
\[
dy = \frac{1}{\sqrt{3}} dw - \frac{1}{\sqrt{2}} du + \frac{1}{\sqrt{6}} dv
\]
\[ dz = \frac{1}{\sqrt{3}} dw - \frac{2}{\sqrt{6}} dv \]

Then on \( C \) we have \( w = 0 \) and \( dw = 0 \). Therefore

\[
\int_C zdx + xdy + ydz = \int_C \frac{-2}{\sqrt{6}} \left( \frac{1}{\sqrt{2}} du + \frac{1}{\sqrt{6}} dv \right) + \left( \frac{-1}{\sqrt{2}} u + \frac{1}{\sqrt{6}} v \right) \left( \frac{-1}{\sqrt{2}} u + \frac{1}{\sqrt{6}} v \right) \left( -\frac{2}{\sqrt{6}} dv \right)
\]

which simplifies to

\[
= \int_C \frac{-3}{2\sqrt{3}} vdu - \frac{1}{2} vdv - \frac{1}{2} udu + \frac{2}{2\sqrt{3}} udu
\]

Then let \( u = \cos \theta \) and \( v = \sin \theta \) we have

\[
= \int_0^{2\pi} \frac{3}{2\sqrt{3}} \sin^2 \theta - \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta + \frac{2}{2\sqrt{2}} \cos^2 \theta d\theta = \int_0^{2\pi} \frac{3}{2\sqrt{3}} d\theta = \frac{3\pi}{\sqrt{3}}
\]

**Problem 5.b**

Recall Stoke’s Theorem

\[ \oint F(r) \cdot dr = \int \int_S \nabla \times F(r) \cdot ndS \]

So we have

\[
\int_C zdx + xdy + ydz = \int_C (z, x, y) \cdot dr = \int \int_S \nabla (z, x, y) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) dS
\]

\[
= \int \int_S (1, 1, 1) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) dS = \frac{3}{\sqrt{3}} \int \int_S dS = \frac{3\pi}{\sqrt{3}}
\]

**Jan 1991 Complex Variables**

**Problem 1.a**

Now we know that if \( \sum |a_n| \) converges, then \( \sum a_n \) converges. Notice if \( |z| < 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{|z|^n}{1 - z^{2n}} \leq \sum_{n=1}^{\infty} \frac{|z|^n}{1 - |z|^2} \leq \frac{1}{1 - |z|^2} \sum_{n=1}^{\infty} |z|^n < \infty
\]

So the summation converges for all \( z \) such that \( |z| < 1 \). Now for \( |z| > 1 \) notice

\[
\sum_{n=1}^{\infty} \frac{|z|^n}{1 - z^{2n}} = \sum_{n=1}^{\infty} \left| \frac{1/2}{1 - z^{-n}} - \frac{1/2}{1 + z^{-n}} \right| \leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{1 - z^{-n}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{1 + z^{-n}} < \infty
\]

Hence the summation converges for all \( z \) with \( |z| > 1 \).
Problem 1.b

Now let \( D = \{ z \in \mathbb{C} : |z| = 1 \} \). Then notice the summation is defined as an analytic function on \( \mathbb{C} \setminus D \). Notice for all \( z \in D \)

\[
\lim_{n \to \infty} \frac{z^n}{1 - z^{2n}} \not\to 0
\]

which implies that the summation does not converge.

Problem 2

\( f(z) \) has poles at \( z_1, \ldots, z_n \) where \( |z_i| = 1 \) for all \( i \). Notice for

\[
g(z) = f(z) - \sum_{i=1}^{k} \frac{A_i}{z - z_i}
\]

which is analytic on \( |z| \leq 1 \). Since \( g \) is continuous on \( |z| \leq 1 \), which is compact, then

\[
|g(z)| = \left| f(z) - \sum_{i=1}^{l} \frac{A_i}{z - z_i} \right| = |c_0 + c_1 z + \cdots| \leq M
\]

Thus \( |c_i| \leq K \) for some \( K > 0 \) since \( g(z) \) is analytic at \( z = 1 \). Now

\[
\frac{A_i}{z - z_i} = \frac{-A_i}{1 - \frac{z}{z_i}} = \frac{-A_i}{z_i} \left( 1 + \left( \frac{z}{z_i} \right) + \left( \frac{z}{z_i} \right)^2 + \cdots \right)
\]

So

\[
g(z) = a_0 + a_1 z + \cdots - \frac{A_1}{z_1} \left( 1 + \frac{z}{z_1} + \cdots \right) - \cdots - \frac{A_k}{z_k} \left( 1 + \frac{z}{z_k} + \cdots \right)
\]

so \( |a_i| \leq M \) for some \( M > 0 \).

Problem 3.a

\[
T(z) = \frac{az + 1}{z + a}
\]

Problem 3.b

Now we want to map \( \infty \) and 2 to opposite ends of a circle. Hence

\[
\frac{2a + 1}{2 + a} = -a \Rightarrow a = -2 \pm \sqrt{3}
\]

So let’s look at

\[
T(z) = \frac{(-2 - \sqrt{3})z + 1}{z - 2 - \sqrt{3}}
\]

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Then \( T(1) = 1, T(-1) = -1 \), and \( T(i) = \frac{\sqrt{3}}{2}i - \frac{1}{2} \). Hence The unit circle is mapped to the unit circle. Now \( T(2) = 2 + \sqrt{3} \) and \( T(\infty) = -2 - \sqrt{3} \) and

\[
T(2 + i) = \frac{(2 + \sqrt{3}) + i(3 + 2\sqrt{3})}{2}
\]

Which implies that the line \( Re(z) = 2 \) is mapped to the circle \( |z| = 2 + \sqrt{3} \). Finally notice that

\[
T(-2) = \frac{-14 - 3\sqrt{3}}{13}
\]

Which is inside the annulus. Hence the linear transformation

\[
T(z) = \frac{(-2 - \sqrt{3})z + 1}{z - 2 - \sqrt{3}}
\]

**Problem 4**

Let \( C \) be the contour of two semicircles with radius \( R \) and \( \epsilon \) and \( f(z) = \frac{(\log z)^2}{z^2 + 1} \). By the Residue Theorem, we have

\[
\int_C f(z)dz = 2\pi i \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}
\]

Now notice on \( C_R \) we have \( z = Re^{i\theta} \). Hence

\[
\left| \int_C f(z)dz \right| \leq R\pi \frac{(\log R)^2}{R^2 - 1} \to 0
\]
as \( R \to \infty \). Also notice on \( C_{\epsilon} \), we have

\[
\left| \int_C f(z)dz \right| \leq \epsilon\pi \frac{(\log(\epsilon))^2}{\epsilon^2 - 1} \to 0
\]
as \( \epsilon \to 0 \). Thus we have

\[
\int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}dz + \int_{-R}^{-\epsilon} \frac{(\log z)^2}{z^2 + 1}dz = \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}dz + \int_{\epsilon}^{R} \frac{(\log z + i\pi)^2}{z^2 + 1}dz
\]

\[
= \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}dz + \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}dz + 2\pi i \int_{\epsilon}^{R} \frac{\log z}{z^2 + 1}dz + \int_{\epsilon}^{R} \frac{-\pi^2}{z^2 + 1}dz
\]

Thus as \( R \to \infty \) and \( \epsilon \to 0 \), we have

\[
\int_{0}^{\infty} \frac{\log z}{z^2 + 1}dz = 0
\]

Now Recall that

\[
\int_{0}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}
\]
Indeed since we can take the integral around the semicircle of radius $R$ and apply the Residue Theorem. Thus by above

\[
\int_0^\infty \frac{(\log z)^2}{z^2 + 1} \, dz = \frac{-\pi^3}{6} + \frac{\pi^3}{2} = \frac{\pi^3}{8}
\]

**Problem 5.a**

Let $\gamma$ be the contour of a semi-circle with radius $R$ in the right-half plane. Then for arbitrary large $R$, notice

\[
|(z - 1)^ne^z| = |(x - 1 + iy)^ne^{x+iy}| = |e^x||(x - 1)^2 + y^2|^{n/2} \geq 1
\]

since $x \geq 0$. Hence

\[
|(z - 1)^ne^z - a - (z - 1)^ne^z| = |a| < 1 \leq |(z - 1)^ne^z|
\]

So by Rouche we have $n$ zeros in the RHP.

**Problem 5.b**

Notice for any solutions $z = x + iy$ for

\[
(x - 1 + iy)^ne^{x+iy} = a \in \mathbb{R}
\]

Then let $re^{i\theta} = (x - 1) + iy$, we have

\[
r^ne^{i\theta n}e^{i(\theta n + y)} = a
\]

which implies $n\theta + y = 2\pi k$ for $k \in \mathbb{Z}$. Then I claim the conjugate, $x - iy$ is also a solution, since

\[
(x - 1 - iy)^ne^{-x-iy} = r^ne^xe^{-n\theta-y} = a
\]

Since $-n\theta - y = -2\pi k$. So we know there are two real zeros, since $f(0) = 1$, $f(1) = 0$, and $f(x) > 0$ for large $x$. Thus there are two real zeros.

**Jan 1991 Linear Algebra**

**Problem 1.1**

No, it does hold under scalar multiplication. If $QQ^T = I$, then $(3Q)(3Q)^T = 9I \neq I$.

**Problem 1.2**

YES. $M = UDU^{-1} = U\lambda I U^{-1} = \lambda I$. Hence for $M_1 = \lambda_1 I$ and $M_2 = \lambda_2 I$, we have

\[
M_1 + M_2 = (\lambda_1 + \lambda_2)I \quad \text{and} \quad \alpha M = (\alpha \lambda)I
\]

and also $0 = 0I$. Hence we have the zero matrix. The dimension is 1 since it only depends on $\lambda$. 

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Problem 1.3

NO. Let \( M = U D U^{-1} \), where

\[
D = \begin{pmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{pmatrix}
\]

Then we can’t have the zero matrix, since the zero matrix has \( n \) eigenvalues at 0 counting multiplicity.

Problem 1.4

Let \( v_1 = (1, 0, 0)^T \), \( v_2 = (0, 1, 0)^T \), and \( v_3 = (1, 1, 1)^T \). Since the eigenvectors are linearly independent, \( M \) is diagonalizable. Hence

\[
M = \begin{pmatrix}
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3
\end{pmatrix}^{-1}
\]

where \( \lambda_i \) is arbitrary. Hence we have the zero vector for \( \lambda_i = 0 \forall i \). Scalar multiplication holds since

\[
\alpha M = \begin{pmatrix}
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3
\end{pmatrix}^{-1}
\]

and holds under addition

\[
M_1 + M_2 = \begin{pmatrix}
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3 \\
v_1 & v_2 & v_3
\end{pmatrix}^{-1}
\]

clearly the dimension is 3.

Problem 1.5

YES. If \( M_1 \) and \( M_2 \) hold the following properties, then clearly, \( M_1 + M_2 \) is symmetric and

\[
\text{Tr}(M_1 + M_2) = \text{Tr}(M_1) + \text{Tr}(M_2) = 0
\]

and clearly scalar multiplication holds. Also clearly the zero matrix is included. Now the dimension of the space is \( 5(5 + 1)/2 - 1 = 15 - 1 = 14 \). The minus 1 comes from the fact the the diagonal entries must add up to 0. Let all 14 entries be anything, then the very last diagonal entry must be a certain value.

Problem 1.6

YES. If \( A_1 B = B A_1 \) and \( A_2 B = B A_2 \), then \((A_1 + A_2)B = A_1 B + A_2 B = B A_1 + B A_2 = B(A_1 + A_2)\). Also \( 0 B = B 0 \) and \( \alpha A B = \alpha B A = B(\alpha A) \).
Problem 3
We know that \( \det(B - tI) = t^3 + at + b \). Also notice that \( (B - tI)(B + tI) = B^2 - t^2I \). Hence
\[
\det((B - tI)(B + tI)) = \det(B - tI) \det(B + tI)
\]
\[
= (t^3 + at + b)(-t^3 - at + b) = -t^6 - 2at^4 - a^2t^2 + b = \det(B^2 - t^2I)
\]
Hence \( \det(B^2 - tI) = t^3 - 2at^2 - a^2t + b \)

Problem 5.1
Recall that the trace of \( A^k \) equals the sum of its eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Hence
\[
\det \begin{pmatrix} 4 - \lambda & 3 \\ 2 & 3 - \lambda \end{pmatrix} = (4 - \lambda)(3 - \lambda) - 6 = \lambda^2 - 7\lambda + 6
\]
Hence we have \((\lambda - 6)(\lambda - 1) = 0\), so \( \lambda_1 = 6 \) and \( \lambda_2 = 1 \). Hence
\[
\text{Trace}(I + A + A^2 + \cdots A^{28}) = 2 + (6 + 1) + (6^2 + 1^2) + (6^3 + 1^2) + \cdots (6^{28} + 1) = \sum_{k=0}^{28} (6^k + 1)
\]

Problem 5.2
Notice that if \( A \) is orthogonal, then \( A^H A = D = AA^H \). Hence it is normal, and \( A = U^H DU \) where \( U \) is unitary. So for \( A \) which is 2 \( \times \) 2 and diagonalizable, we have
\[
A = MDM^{-1} = M \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} M^{-1} \Rightarrow e^A = M \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} M^{-1}
\]
Hence it converges for all \( A \).

Sept 1991 Advanced Calculus

Problem 1.a
Since \( \frac{1}{k^{1+\alpha}} \leq \log(1 + 1/k) \leq 1/k \), if \( \alpha \leq 0 \) we have
\[
\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \log(1 + 1/k) \geq \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha}} \to \infty
\]
Now if \( \alpha > 0 \) we have
\[
\sum_{k=1}^{\infty} \frac{1}{k^\alpha} \log(1 + 1/k) \leq \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \frac{1}{k} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha}} < \infty
\]
Hence the summation converges for \( \alpha > 0 \) and diverges for \( \alpha \leq 0 \).
Problem 1.b
Clearly if $\alpha < 0$, the summation diverges. Now recall that for a constant $\alpha$, $(\log k)^\alpha = o(k)$. Thus $\exists N$ such that

$$\sum_{k=N}^{\infty} \frac{1}{k} \leq \sum_{k=N}^{\infty} \frac{1}{(\log k)^\alpha}$$

Hence the summation diverges for all $\alpha \in \mathbb{R}$.

Problem 1.c
For $\alpha = 1$, we have

$$\sum_{k=1}^{\infty} \frac{\alpha^k}{1 + \alpha^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2} \rightarrow \infty$$

Also if $\alpha = -1$, we have

$$\sum_{k=1}^{\infty} \frac{\alpha^k}{1 + \alpha^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2}$$

which simply does not converge. Now if $\alpha < 0$ and $\alpha \neq -1$, we have

$$\sum_{k=1}^{\infty} \frac{\alpha^k}{1 + \alpha^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^k |\alpha|^k}{1 + \alpha^{2k}} < \infty$$

since $\frac{|\alpha|^k}{1 + \alpha^{2k}} \searrow 0$, and thus by the alternating series theorem the summation converges. Now if $0 \leq \alpha < 1$, we have

$$\sum_{k=1}^{\infty} \frac{\alpha^k}{1 + \alpha^{2k}} \leq \sum_{k=1}^{\infty} \alpha^k = \frac{1}{1 - \alpha} - 1 < \infty$$

Finally if $\alpha > 1$, we have

$$\sum_{k=1}^{\infty} \frac{\alpha^k}{1 + \alpha^{2k}} \leq \sum_{k=1}^{\infty} \frac{\alpha^k}{\alpha^{2k}} \leq \sum_{k=1}^{\infty} \frac{1}{\alpha^k} = \frac{1}{1 - 1/\alpha} - 1 < \infty$$

Hence the summation converges for all $\alpha$ except at 1 and $-1$.

Problem 1.d
Clearly if $\alpha = 0$, the summation converges. Now for all $\alpha \neq 0$, $\lim_k |\alpha|^{1/k} = 1$. Hence $|\alpha|^{1/k} \frac{1}{k} \searrow 0$.

Thus by the alternating series theorem, we have

$$\sum_{k=1}^{\infty} (-1)^k |\alpha|^{1/k} \frac{1}{k} < \infty$$

Hence the summation converges for all $\alpha$. 

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Problem 2

Let

\[ F(x, y, z) = f(xy) + g(yz) \quad \text{and} \quad G(x, y, z) = g(xy) + f(yz) \]

and notice

\[
\frac{d(F, G)}{d(y, z)} = \begin{vmatrix}
  f'(xy)x + g'(yz)z & g'(yz)y \\
  g'(xy)x + f'(yz)z & f'(yz)y \\
\end{vmatrix}_{(1,1,1)} = \begin{vmatrix}
  f'(1) + g'(1) & g'(1) \\
  f'(1) + g'(1) & f'(1) \\
\end{vmatrix}
\]

So by the implicit function theorem, there exists continuous functions \( h \) and \( g \) where \( y = h(x) \) and \( z = g(x) \) near \( x = y = z = 1 \) if \( g'(1) \neq f'(1) \) and \( f'(1) + g'(1) \neq 0 \). Now

\[
f'(xy) \left( y + \frac{dy}{dx} x \right) + g'(yz) \left( \frac{dy}{dx} z + \frac{dz}{dx} y \right) = 0
\]

so at \((1,1,1)\) we have

\[
f'(1)(1 + a) + g'(1)(a + b) = 0
\]

likewise

\[
g'(xy) \left( y + \frac{dy}{dx} x \right) + f'(yz) \left( \frac{dy}{dx} z + \frac{dz}{dx} y \right)
\]

which implies

\[
g'(1)(1 + a) + f'(1)(a + b) = 0
\]

combining these two equations we have

\[
(f'(1) + g'(1))(1 + a) + (f'(1) + g'(1))(a + b) = 0
\]

and since \( f'(1) + g'(1) \neq 0 \), we have

\[
1 + 2a + b = 0 \Rightarrow b = -1 - 2a
\]

Thus

\[
a = \frac{-f'(1) + g'(1)}{f'(1) - g'(1)} = -1
\]

and so \( b = 1 \).
Problem 3.a

We have \( f(x, y, z) = \frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} \) and \( g(x, y, z) = xyz = c \). By the Lagrange multiplier theorem, we have \( \nabla f = \lambda \nabla g \):

\[
(x^{p-1}, y^{q-1}, z^{r-1}) = \lambda(yz, xz, xy)
\]

Thus we have

\[
\frac{x^{p-1}}{yz} = \frac{y^{q-1}}{xz} = \frac{z^{r-1}}{xy}
\]

Since \( x, y, z \geq 0 \), we have \( y = x^{p/q} \) and \( z = x^{p/r} \). Thus

\[
x x^{p/q} x^{p/r} = c \Rightarrow x = c^{1 + p/q + p/r}
\]

Thus we have

\[
z = c^{1 + r/p + r/q} \quad \text{and} \quad y = c^{1 + q/p + q/r}
\]

Therefore

\[
\frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} = f(x, y, z) = \frac{c^{1 + p/q + p/r}}{p} + \frac{c^{1 + q/p + q/r}}{q} + \frac{c^{1 + r/p + r/q}}{r}
\]

\[
= c^{\frac{pqr}{q^r + p^r + p^q}} \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) = c^{\frac{pqr}{q^r + p^r + p^q}}
\]

Now notice that

\[
\left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right) = \frac{qr + pr + qp}{pqr} = 1 \Rightarrow \frac{pqr}{qr + pr + qp} = 1
\]

hence

\[
\frac{x^p}{p} + \frac{y^q}{q} + \frac{z^r}{r} = f(x, y, z) \geq c = xyz
\]

Problem 3.b

Now if \( 1/p + 1/q + 1/r = 1 \), we have from above

\[
f(x, y, z) \geq c^{\frac{pqr}{q^r + p^r + p^q}} \left( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \right)
\]
Problem 4

Using Green’s Theorem, we have
\[
\int \int x^2 + y^2 \, dy \, dx = \oint_E -yx^2 \, dx + xy^2 \, dy
\]

Let \( x = a \cos \theta \), \( y = b \sin \theta \), \( dx = -a \sin \theta \, d\theta \), and \( dy = b \cos \theta \). Hence
\[
\oint_E -yx^2 \, dx + xy^2 \, dy = \int_0^{2\pi} (a^3 b + ab^3) \frac{1}{4} \sin^2(2\theta) = \frac{a^3 b + ab^3}{2} \left( \theta - \frac{\sin(4\theta)}{4} \right) \bigg|_0^{2\pi} = \pi(a^3 b + ab^3)
\]

Problem 5

Notice that for constant \( M \), we have
\[
\lim_{n \to \infty} \int_0^{2\pi} M \cos(nx) \, dx = \lim_{n \to \infty} \frac{M \sin(n2\pi)}{n} = 0
\]

So since \( f \) is continuous on \([0, 2\pi]\), \( \exists \) a sequence of step function \( s_k \) such that \( s_k \to f \) on \([0, 2\pi]\).

Now notice by the dominated convergence theorem, we have
\[
\lim_{n \to \infty} \int_0^{2\pi} f(x) \cos(nx) \, dx = \lim_{n \to \infty} \int_0^{2\pi} \lim_{k \to \infty} s_k(x) \cos(nx) \, dx = \lim_{k \to \infty} \lim_{n \to \infty} \int_0^{2\pi} s_k(x) \cos(nx) \, dx
\]

Now let \( s_k(x) = a_j \), a constant, when \( x \in (x_j, x_{j+1}) \). Hence we have
\[
\lim_{n \to \infty} \int_0^{2\pi} f(x) \cos(nx) \, dx = \lim_{n \to \infty} \lim_{k \to \infty} \left( \int_0^{x_1} a_1 \cos(nx) \, dx + \int_{x_1}^{x_2} a_2 \cos(nx) \, dx + \cdots + \int_{x_{m-1}}^{2\pi} a_m \cos(nx) \, dx \right)
\]
\[
= \lim_{k \to \infty} (0 + 0 + \cdots 0) = 0
\]

Sept 1991 Complex Variables

Problem 1

Let \( \gamma \) be the key hold (pacman) contour with outer radius \( R \) and inner radius \( \epsilon \). Then let \( f(z) = \frac{z^{\alpha - 1}}{(z+i)^2} \), and by the Residue Theorem, we have
\[
\int_\gamma f(z) \, dz = 2\pi i (\alpha - 1) e^{\alpha i\pi}
\]

Now notice on the outer radius \( z = Re^{i\theta} \)
\[ \left| \int_{C_R} f(z)dz \right| \leq 2\pi R \frac{R^{\alpha-1}}{R^2} \to 0 \]
as \( R \to \infty \). Also notice on the inner radius
\[ \left| \int_{C_\epsilon} f(z)dz \right| \leq 2\pi \epsilon \epsilon^{\alpha-1} \frac{(\epsilon e^{i\theta} + 1)^2}{2} \to 0 \]
as \( \epsilon \to 0 \). Hence as \( \epsilon \to 0 \) and \( R \to \infty \) we have
\[ (1 - e^{i2\pi(\alpha-1)}) \int_0^\infty f(z)dz = 2\pi i(\alpha - 1)e^{\alpha i\pi} \]
Hence this implies
\[ (e^{-i\pi(\alpha-1)} - e^{i\pi(\alpha-1)}) \int_0^\infty f(z)dz = 2\pi i(\alpha - 1)e^{\alpha i\pi}e^{-i\pi(\alpha-1)} \]
thus
\[ 2i \sin(\pi(\alpha - 1)) \int_0^\infty f(z)dz = 2\pi i(\alpha - 1) \]
therefore
\[ \int_0^\infty f(z)dz = \frac{\pi(\alpha - 1)}{\sin(\pi(\alpha - 1))} \]

**Problem 2**

Notice that \( f(z) = 2z^4 + 3z^2 - 2z + 1 + \frac{9}{z} = (1/z)(2z^5 + 3z^3 - 2z^2 + z + 9) \). Hence the number of zeros inside the annulus \( 1 < |z| < 2 \) is the same for \( f(z) \) and \( 2z^5 + 3z^3 - 2z^2 + z + 9 \). Using Rouche’s Theorem, we have on the circle \( |z| = 2 \)
\[ |2z^5 + 3z^3 - 2z^2 + z + 9 - 2z^5| = |3z^3 - 2z^2 + z + 9| \leq 24 + 8 + 2 + 9 = 43 < 64 = |2z^5| \]
Hence we have 5 zeros inside the the circle of radius 2. Now on the unit circle \( |z| = 1 \), we have
\[ |2z^5 + 3z^3 - 2z^2 + z + 9| = |2z^5 + 3z^3 - 2z^2 + z| \leq 2 + 3 + 2 + 1 = 8 < 9 \]
Hence we have no zeros inside the unit circle. Thus \( f(z) \) has a total of 5 zeros inside the annulus \( 1 < |z| < 2 \).

**Problem 5**

The only singularity in \( |z| < 1 \) is at \( z = 0 \), which is a pole of order 3. Thus by the Residue Theorem, we have
\[ \oint_{|z|=1} \frac{\cos z}{z^2 \sin z/2} dz = 2\pi i \text{Res}(f, 0) \]
Now notice that
\[
\frac{\cos z}{z^2 \sin z/2} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots}{\frac{z}{2} \left( \frac{(z/2)^3}{3!} + \frac{(z/2)^5}{5!} - \cdots \right)} = \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots
\]

Hence
\[
\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right) = \left( \frac{z^3}{2} - \frac{z^5}{23!} + \frac{z^7}{255!} - \cdots \right) \left( \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots \right)
\]
Thus we have \(a_{-3} = 2, a_{-2} = 0, a_{-1} = -22/24 = -11/12\). Hence \(\text{Res}(f, 0) = -11/12\). Thus
\[
\oint_{|z|=1} \frac{\cos z}{z^2 \sin z/2} dz = -2\pi i \left( \frac{11}{12} \right) = -\frac{11}{6}\pi i
\]

**Sept 1991 Linear Algebra**

**Problem 1.1**

By gaussian elimination, we have
\[
\begin{array}{ccc|c}
1 & 2 & 0 & 3 \\
0 & -3 & 0 & b - 6 \\
0 & -5 & a & c - 9 \\
\hline
1 & 2 & 0 & 3 \\
0 & -3 & 0 & b - 6 \\
0 & 0 & a & (-5/3)b + c + 1 \\
\end{array}
\]
Hence we have a solution when if \(a \neq 0, b, c \in \mathbb{R}\) or if \(a = 0, c = (5/3)b - 1\).

**Problem 1.2**

We have a unique solution when \(a \neq 0, b, c \in \mathbb{R}\).

**Problem 1.3**

In order for a solution set to be a vector space, \(v = (0, 0, 0)\) must be a solution. However this is impossible for all \(a, b, c\) since the first equation \(x + 2y = 3\) must be satisfied.

**Problem 2.1**

We need to have \(2(c + di) - i(a + bi) \neq 0\), which implies \(2c + b \neq 0\) and at the same time \(2d - a \neq 0\).

**Problem 2.2**

In order to have rank equal 1, we need \(2(c + di) - i(a + bi) = 0\), which implies \(ac + b = 0\) and at the same time \(ad - 2 = 0\).
Problem 2.3
For $A$ to be self adjoint, we must have $A^H = A$. Hence we must have
\[
\begin{pmatrix}
2 & a + bi \\
0 & c + di
\end{pmatrix} = \begin{pmatrix}
2 & -i \\
a - bi & c - di
\end{pmatrix}
\]
Hence we must have $a = 0, b = -1, d = 0, c \in \mathbb{R}$.

Problem 2.4
Here we must have $A$ to be normal, i.e. $A^H A = AA^H$. Hence
\[
\begin{pmatrix}
2 & a + bi \\
0 & c + di
\end{pmatrix} \begin{pmatrix}
2 & -i \\
a - bi & c - di
\end{pmatrix} = \begin{pmatrix}
4 + a^2 + b^2 & -2i + ac + bci - adi + bd \\
2i + ac + (ad - bc)i + bd & 1 + c^2 + d^2
\end{pmatrix}
\]
\[
= \begin{pmatrix}
2 & -i \\
a - bi & c - di
\end{pmatrix} \begin{pmatrix}
2 & a + bi \\
i & c + di
\end{pmatrix} = \begin{pmatrix}
5 & 2a + 2bi - ic + d \\
2a - 2bi + ci + d & a^2 + b^2 + c^2 + d^2
\end{pmatrix}
\]
Hence we must have $a^2 + b^2 = 1$ and $ac + bd = 2a + d$.

Problem 3.1
We have $P = A(A^T A)^{-1} A^T$ where
\[
A = \begin{pmatrix}
1 & 1 \\
-1 & -1 \\
1 & 1
\end{pmatrix}
\]
therefore
\[
P = \begin{pmatrix}
1/2 & -1/2 & 0 \\
-1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Problem 3.2
Notice for
\[
A = \begin{pmatrix}
3/2 & -3/2 & 0 \\
-3/2 & 3/2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]
satisfies the condition.

Problem 4
Notice
\[
\det(AB - \lambda I) = \det(A^{-1} \det(AB - \lambda I) \det(A) = \det(A^{-1}(AB - \lambda I)A) = \det(BA - \lambda I)
\]
Problem 5.1

Notice

\[
M \left( I + \begin{pmatrix} 0 & 2 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -2 \\ \frac{2}{2!} & \frac{2}{3!} & \cdots \end{pmatrix} M^{-1} \right)
\]

\[
= M \begin{pmatrix} 1 & e^2 \\ e^{-2} & \end{pmatrix} M^{-1} = e^A
\]

Now notice

\[-t(A + 2I) = M \begin{pmatrix} -2t & -4t \\ \end{pmatrix} M^{-1}\]

So

\[
\lim_{t \to \infty} M \begin{pmatrix} 0 & \\ 0 & \end{pmatrix} M^{-1} = 0
\]

Problem 5.2

If \( A \) is diagonalizable, then yes by similar argument as above. Now if we have it in Jordan Form

\[A = M \begin{pmatrix} 0 & -2 & 1 \\ -2 & \end{pmatrix} M^{-1}\]

Then we have

\[-t(A + 2I) = M \begin{pmatrix} -2t & 0 & 0 \\ 0 & 0 & -t \\ 0 & 0 & 0 \end{pmatrix} M^{-1}\]

So

\[e^{-t(A+2I)} = MM \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & 0 & -t \\ 0 & 0 & 0 \end{pmatrix} M^{-1}\]

So

\[
\lim_{t \to \infty} e^{-t(A+2I)} \to \text{Undefined}
\]
Jan 1992 Advanced Calculus

Problem 1.a
We have

\[ a = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} = \sum_{j=1}^{n} (-1)^{j+1} a_j \]

Notice \( a_j \searrow 0 \), hence by the alternating series theorem, the series converges. Hence \( a \) exists.

Problem 1.b
Notice we have

\[ \frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots \]

Thus integrating on both sides we have

\[ \int \frac{1}{1 + x} = \log(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \ldots \]

Hence \( \log(2) = a \).

Problem 2.a
Notice for \( x \neq 0 \) we have

\[ (f'(x))^2 = \frac{1 - e^{-x} - x f''(x)}{3x} \]

at a local extreme we have

\[ 1 - e^{-x} - x f''(x) = 0 \Rightarrow f''(x) = \frac{1 - e^{-x}}{x} \]

So \( f''(x) > 0 \) implies that \( f \) is concave up for all \( x > 0 \). So we have a local minimum.

Problem 2.b
We are given that \( f(0) = f'(0) = 0 \). By Taylor’s theorem, we have

\[ f(x) = f(0) + f'(0) + E(x) \]

where

\[ E(x) = \frac{f''(c)x^2}{2!} \]

Where \( c \in (0, x) \). So if we can bound \( f''(x) \) by above, we are done. Notice

\[ f''(x) = \frac{1 - e^{-x} - 3x(f'(x))^2}{x} \]
is bounded above. So the max occurs for $x$ such that

$$xe^{-x} - 6x^2 f'(x)f''(x) - 1 + e^{-x} = 0$$

implies that $x = 0$ Hence $f''(0)$ achieves maximum. So by L’Hopital’s rule we have

$$\lim_{x \to 0} f''(x) = \lim_{x \to 0} \frac{e^{-x} - 3(f'(x))^2 - 6xf'(x)f''(x)}{1} = \lim_{x \to 0} \frac{e^{-x} - 3(f'(x))^2}{1 + 6xf'(x)} = 1$$

Hence $f''(x) \leq 1$ for all $x \geq 0$. So $f(x) \leq (1/2)x^2$ for all $x$.

**Problem 3**

Assume that $e$ is rational, and $\exists$ integers $p$ and $q$ such that $e = q/p$. Now I claim that for all positive integer $p$, we have

$$0 < p! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} \right) \right] < 1$$

**BASE CASE:** $p = 1$. Then notice we have

$$0 < 1!(e - (1 + 1)) \approx 0.71828 < 1$$

**INDUCTIVE STEP:** We are given that

$$0 < p! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} \right) \right] < 1$$

We need to show that

$$0 < (p + 1)! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} + \frac{1}{(p + 1)!} \right) \right] < 1$$

Clearly we have

$$0 < (p + 1)! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} + \frac{1}{(p + 1)!} \right) \right] = (p + 1)! \sum_{k=p+2}^{\infty} \frac{1}{k!}$$

Now notice that

$$(p + 1)! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} + \frac{1}{(p + 1)!} \right) \right] \leq p! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} \right) \right] < 1$$

Indeed since the first inequality would imply

$$p! \sum_{k=p+1}^{\infty} \frac{1}{k!} \leq (p + 1)! \sum_{k=p+2}^{\infty} \frac{1}{k!} \Rightarrow$$

$$\frac{1}{p + 1} + \frac{1}{(p + 3)(p + 2)} + \frac{1}{(p + 4)(p + 3)(p + 2)} + \cdots \leq \frac{1}{p + 1} + \frac{1}{(p + 2)(p + 1)} + \frac{1}{(p + 3)(p + 2)(p + 1)} + \cdots$$

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The inequality is true since each term in the summation we have
\[ \prod_{k=1}^{p} \frac{1}{p + 1 + k} \leq \prod_{k=1}^{p} \frac{1}{p + k} \]
Hence the inequality is true. Thus notice that
\[ p! \left[ e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{p!} \right) \right] \]
is an integer. However this integer is less than 1 but greater than 0, which is a contradiction. Hence \( e \) is irrational.

**Problem 4.a**

We know that the tangent plane of a surface at \((x_0, y_0, z_0)\) is the equation
\[ \nabla f \cdot (x - x_0, y - y_0, z - z_0) = 0 \]
Hence we have the surface \( f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \), which implies
\[ \left( \frac{2x}{a^2} + \frac{2y}{b^2} + \frac{2z}{c^2} \right) (x - x_0, y - y_0, z - z_0) = 0 \]
which implies
\[ \frac{2x}{a^2} (x - x_0) + \frac{2y}{b^2} (y - y_0) + \frac{2z}{c^2} (z - z_0) = 0 \Rightarrow \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1 \]
since \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).

**Problem 4.c**

The solid generated is a tetrahedron. Hence recall that the volume of a tetrahedron is
\[ V = \frac{1}{6} |v_1 \cdot (v_2 \times v_3)| \]
where \( v_1 = (a^2/x_0, 0, 0) \), \( v_2 = (0, b^2/y_0, 0) \), and \( v_3 = (0, 0, c^2/z_0) \). Thus
\[ V = \frac{1}{6} \left| \left( \frac{a^2}{x_0}, 0, 0 \right) \cdot \left( \frac{b^2}{y_0 z_0}, 0, 0 \right) \right| = \frac{1}{6} \frac{a^2 b^2 c^2}{x_0 y_0 z_0} \]
So the volume of our tetrahedron as a function is \( V(x, y, z) = \frac{1}{6} \frac{a^2 b^2 c^2}{x y z} \), constrained to
\[ f(x, y, z) = \frac{x^2}{z^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]
Now applying the Lagrange multiplier theorem, we have \( \nabla V = \lambda \nabla f \). Thus
\[ \left( -\frac{a^2 b^2 c^2}{6x^2 y z}, -\frac{a^2 b^2 c^2}{6x y^2 z}, -\frac{a^2 b^2 c^2}{6x y z^2} \right) = \lambda \left( \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right) \]
Thus this simplifies to

\[ \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2} \]

Hence we have \( x = a/\sqrt{3}, \ y = b/\sqrt{3}, \) and \( z = c/\sqrt{3} \). Therefore our minimized volume is

\[ V = \frac{abc\sqrt{3}}{2} \]

**Problem 5**

Since \( \nabla \times F = 0 \) \( \exists \phi \) such that \( \nabla \phi = F \). Since \( \nabla \cdot F = 0 \) implies that \( \phi \) is harmonic. So notice that

\[ \nabla^2(\phi^2) = \nabla \cdot (2\phi \nabla(\phi)) = 2(\nabla \phi \cdot \nabla \phi + \phi \cdot \nabla^2 \phi) = 2\nabla \nabla = 2||F|| \]

So by the divergence theorem, we have

\[ \int \int \int \nabla \cdot \nabla(\phi^2)dV = \int \int \nabla(\phi^2) \cdot ndS = \int \int 2\phi \nabla \phi \cdot ndS = 0 \]

since \( F \cdot n = 0 \) on the boundary. Therefore

\[ \int \int \int 2||F||dV = 0 \]

which implies that \( F = 0 \).

**Jan 1992 Complex Variables**

**Problem 1**

Notice we have

\[ \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \]

Hence the radius of converges of the power series is \( 1/e \).

**Problem 2.a**

Note for the integral does not exist for \( Im(\xi) \neq 0 \). So for CASE 1, \( \xi = 0 \), we have

\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx = \arctan(x)|_{-\infty}^{\infty} = \pi \]

**CASE 2, \( \xi > 0 \), then let \( f(z) = \frac{e^{-iz\xi}}{1+z^2} \) and \( \gamma_1 \) be the semi-circle of radius \( R \) in the lower half plane. Then by the Residue Theorem, we have

\[ \int_{\gamma_1} f(z) dz = 2\pi i \frac{e^{-i(-1)\xi}}{-2i} = -\pi e^{-\xi} \]
Then
\[ \int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = -\pi e^{-\xi} \]

now notice
\[ \left| \int_{C_R} f(z)\,dz \right| \leq \int_{0}^{\pi} \left| \frac{e^{-i(-R\cos \theta - iR\sin \theta)\xi Rie^{i\theta}}}{R^2 - 1} \right| \,d\theta \]
\[ \leq \int_{0}^{\pi} \frac{R}{R^2 - 1} e^{-R\xi \sin \theta} \,d\theta \leq \frac{R}{R^2 - 1} \int_{0}^{\pi} e^{-\xi R^2/\pi} \,d\theta \leq \frac{R}{R^2 - 1} \frac{\pi}{2} (e^{-\xi R^2} - 1) \to 0 \]
as \( R \to \infty \). Hence for \( \xi > 0 \), we have as \( R \to \infty \)
\[ \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1 + x^2} \,dx = \pi e^{-\xi} \]

**CASE 3**, for \( \xi < 0 \) then let \( \gamma_2 \) be the semi-circle of radius \( R \) in the upper-half plane. Then
\[ \int_{\gamma_2} f(z)\,dz = 2\pi i \frac{e^{-i(1)\xi}}{2i} = \pi e^{\xi} \]

Hence
\[ \int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = \pi e^{\xi} \]

and notice
\[ \left| \int_{C_R} f(z)\,dz \right| \leq \int_{C_R} \left| \frac{e^{-iR(\cos \theta + i\sin \theta)\xi Ri e^{i\theta}}}{R^2 - 1} \right| \,d\theta \]
\[ \leq \frac{R}{R^2 - 1} \int_{C_R} e^{\xi R \sin \theta} \,d\theta \leq \frac{R}{R^2 - 1} \int_{C_R} e^{\xi R^2/\pi} \,d\theta = \frac{\pi}{(R^2 - 1)\xi^2} (e^{\xi R^2} - 1) \to 0 \]
as \( R \to \infty \) since \( \xi < 0 \). Hence as \( R \to \infty \)
\[ \int_{-\infty}^{\infty} f(z)\,dz = \pi e^{\xi} \]

So for all \( \xi \in \mathbb{R} \),
\[ \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{x^2 + 1} \,dx = \frac{\pi}{e^{\xi|\xi|}} \]
Problem 5.a
We are given that
\[
\lim_{r \to \infty} \frac{\log M(r)}{\sqrt{r}} = 1
\]
This implies
\[
\lim_{r \to \infty} \frac{\log \log M(r)}{\frac{1}{2} \log r} = 1
\]
Hence
\[
\lim_{r \to \infty} \frac{\log \log M(r)}{\log r} = 2
\]
which is the order of \( f \). Then the order \( \lambda = 2 \) By Hadamard's theorem, we have
\[
h \leq \lambda \leq h + 1
\]
where \( h \) is the genus. Hence let \( h = 1 \). Since \( f(z) \) is entire, it has the form
\[
f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} = e^{g(z)} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}
\]
where \( g(z) \) is a polynomial of degree 1. Then we have
\[
f(z) = e^{1+z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}
\]
Problem 5.b
Yes, otherwise it would have genus 0 if the number of 0 if the number of zeros is finite. This contradicts \( h \leq 2 \leq h + 1 \).

Jan 1992 Linear Algebra
Problem 1
We have
\[
p(\lambda) = \det \begin{pmatrix} -\lambda & 1 & a \\ 1 & -\lambda & 0 \\ 0 & 1 & a - \lambda \end{pmatrix} = -\lambda(\lambda - a) - (a - \lambda) + a = -\lambda^3 + a\lambda^2 + \lambda
\]
Hence \( a = -2 \). Notice that \( p(\lambda) = -\lambda(\lambda - (-1 + \sqrt{2}))(\lambda - (-1 - \sqrt{2})) \). For \( \lambda = 0 \), we have
\[
\begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0
\]
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Hence the solution to the system above is

\[ s_1 = t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \]

Thus \( s_1 \) is an eigenvector for \( \lambda_1 = 0 \). Now for \( \lambda_2 = -1 + \sqrt{2} \), we have

\[
\begin{pmatrix}
1 - \sqrt{2} & 1 & -2 \\
1 & 1 - \sqrt{2} & 0 \\
0 & 1 & -1 - \sqrt{2}
\end{pmatrix}
\]

Hence the solution to the system above is

\[ s_2 = t \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 1 \end{pmatrix} \]

Thus \( s_2 \) is an eigenvector for \( \lambda_2 = -1 + \sqrt{2} \). Now for \( \lambda_3 = -1 - \sqrt{2} \) we have

\[
\begin{pmatrix}
1 + \sqrt{2} & 1 & -2 \\
1 & 1 + \sqrt{2} & 0 \\
0 & 1 & -1 + \sqrt{2}
\end{pmatrix}
\]

Hence the solution to the system above is

\[ s_3 = t \begin{pmatrix} 1 \\ 1 - \sqrt{2} \\ 1 \end{pmatrix} \]

Thus \( s_3 \) is an eigenvector for \( \lambda_3 = -1 - \sqrt{2} \). Thus let

\[ P = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 - \sqrt{2} & 1 + \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \]

Then notice that \( P \) is invertible since it has rank 3:

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & -1 - \sqrt{2} & -1 + \sqrt{2}
\end{pmatrix}
\]
Hence it suffice to show that \( AP = PD \), where \( D \) is a diagonal matrix. Notice
\[
\begin{pmatrix}
0 & 1 & -2 \\
1 & 0 & 0 \\
0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 \\
2 & 1 - \sqrt{2} & 1 + \sqrt{2} \\
1 & 1 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 1 \\
2 & 1 - \sqrt{2} & 1 + \sqrt{2} \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & -1 - \sqrt{2} & 0 \\
0 & 0 & -1 + \sqrt{2}
\end{pmatrix}
\]
Note that there are no real values of \( a \) where \( A \) cannot be diagonalized. Indeed since by the quadratic formula, we have for \( \lambda^2 + a\lambda - 1 \)
\[
-\frac{a \pm \sqrt{a^2 + 4}}{2}
\]
Thus we must have \( a = 2i \), for there to be a double eigenvalue. Hence we will always have three distinct eigenvalues which implies that we have three distinct eigenvectors which implies that \( A \) can be diagonalized.

**Problem 4.a**

Rank of the matrix \( B \) is the dimension size of the column (or row) space of \( B \). The nullity of \( B \) is the dimension \( N(B) \) where
\[
N(B) = \{ x \in \mathbb{R}^n : Bx = 0 \}
\]

**Problem 4.b**

We are given that \( A \) is invertible. Then we have \( N(B) = N(AB) \). Indeed since if \( v \in N(B) \), then \( ABv = A0 = 0 \). Hence \( N(B) \subset N(AB) \). Now if \( v \in N(AB) \), Then \( ABv = 0 \). Thus \( Bv = A^{-1}0 = 0 \). Hence \( N(AB) \subset N(B) \Rightarrow N(AB) = N(B) \). We also know that
\[
\dim(N(B)) + \text{Rank}(B) = n
\]
and
\[
\dim(N(AB)) + \text{Rank}(AB) = n
\]
Hence the \( \text{Rank}(AB) = \text{Rank}(B) \).

**Problem 4.c**

We can’t really conclude anything about the eigenvalues of \( AB \) and \( BA \). If \( AB = BA \), then they share the same eigenvalues and vectors (trivial). **Need more!**

**Sept 1992 Advanced Calculus**

**Problem 1.a**

Greens Theorem is as follows: Let \( \Omega \) be a Jordan region with a piecewise-smooth boundary \( C \). If \( p \) and \( Q \) are scalar fields continuously differentiable on an open set that contains \( \Omega \), then
\[
\int \int_{\Omega} \left[ \frac{dQ}{dx}(x, y) - \frac{dP}{dy}(x, y) \right] \, dxdy = \oint_{C} P(x, y) \, dx + Q(x, y) \, dy
\]

Now we define the rectangular region \( R \) consisting of \( C_1 : (x, c) \) for \( x : a \to b \), \( C_2 : (b, y) \) for \( y : c \to d \), \( C_3 : (x, d) \) for \( x : b \to a \) and finally \( C_4 : (a, y) \) for \( y : d \to c \). Notice

\[
\int \int_{\Omega} \left[ \frac{dQ}{dx}(x, y) - \frac{dP}{dy}(x, y) \right] \, dxdy = \int_{a}^{b} P(x, c) \, dx - \int_{b}^{c} P(x, d) \, dx + \int_{b}^{a} P(x, d) \, dx = \oint_{C} P(x, y) \, dy
\]

Likewise notice

\[
\int \int_{\Omega} dQ \, dx - \int_{a}^{b} \frac{dP}{dy}(x, y) \, dy \, dx = \int_{c}^{d} Q(b, y) \, dy - \int_{c}^{d} Q(a, y) \, dy + \int_{a}^{b} P(x, c) \, dx = \oint_{C} P(x, y) \, dy
\]

Hence

\[
\int \int_{\Omega} \left[ \frac{dQ}{dx}(x, y) - \frac{dP}{dy}(x, y) \right] \, dxdy = \oint_{C} P(x, y) \, dx + Q(x, y) \, dy
\]

**Problem 1.b**

Let \( \epsilon > 0 \). Then we have

\[
\int \int_{\Omega} \frac{dQ}{dx} - \frac{dP}{dy} \, dxdy = \int_{C_{1}} \left( \frac{x}{x^2 + y^2} + x \right) \, dy - \int_{C_{2}} \frac{y}{x^2 + y^2} \, dx - \int_{C_{3}} \left( \frac{x}{x^2 + y^2} + x \right) \, dy - \int_{C_{4}} \frac{y}{x^2 + y^2} \, dx
\]

This implies

\[
\int_{C_{1}} \left( \frac{x}{x^2 + y^2} + x \right) \, dy - \int_{C_{2}} \frac{y}{x^2 + y^2} \, dx = \int \int_{\Omega} \frac{dQ}{dx} - \frac{dP}{dy} \, dxdy + \int_{C_{3}} \left( \frac{x}{x^2 + y^2} + x \right) \, dy - \int_{C_{4}} \frac{y}{x^2 + y^2} \, dx
\]

So notice

\[
\frac{dQ}{dx} - \frac{dP}{dy} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + 1 - \left( \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right) = 1
\]

So we define

\[
V = \int \int_{\Omega} dxdy
\]
Then on the small circle, we let $x = \epsilon \cos \theta$ and $y = \epsilon \sin \theta$. Then we have

$$\int_{C_\epsilon} \left( \frac{x}{x^2 + y^2} + x \right) dy - \frac{y}{x^2 + y^2} dx = \int_0^{2\pi} \left( \epsilon \cos \theta + \epsilon \cos \theta \right) \epsilon \cos \theta + \epsilon \sin \theta \epsilon \sin \theta d\theta$$

$$= \int_0^{2\pi} 1 + \epsilon^2 \cos^2 \theta = \theta + \epsilon^2 \left[ \frac{\theta}{2} + \frac{1}{4} \sin(2\theta) \right]_0^{2\pi} = 2\pi + \epsilon^2 \pi \to 2\pi$$
as $\epsilon \to 0$. So

$$\int_{C} \left( \frac{x}{x^2 + y^2} + x \right) dy - \frac{y}{x^2 + y^2} dx = V + 2\pi$$

**Problem 2.a**

Now notice that

$$\lim_{n} \sum_{k=1}^{n} a_k b_k = \lim_{n} a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n = \lim_{n} A_1 b_1 + (A_2 - A_1) b_2 + (A_3 - A_2) b_3 + \cdots + (A_n - A_{n-1}) b_n$$

$$= \lim_{n} A_1(b_1 - b_2) + A_2(b_2 - b_3) + \cdots + A_{n-1}(b_{n-2} - b_{n-1}) + b_n A_n$$

$$= \lim_{n} A_n b_n + \sum_{k=1}^{n-1} A_k(b_k - b_{k+1}) \leq \lim_{n} M \sum_{k=1}^{n-1}(b_k - b_{k+1})$$

Now notice that

$$\sum_{k=1}^{n-1}(b_k - b_{k+1}) = b_1 - b_2 + b_2 - b_3 + b_3 - b_4 + \cdots + b_{n-2} - b_{n-1} + b_{n-1} - b_n = b_1 - b_n$$

Hence notice that $\lim_{n} \sum_{k=1}^{n-1}(b_{k+1} - b_k) = \lim_{n} b_1 - b_n = b_1$. Thus

$$\lim_{n} \sum_{k=1}^{n} a_k b_k \leq b_1 M$$

Therefore

$$\lim_{n} \sum_{k=1}^{n} a_k b_k < \infty$$

and the summation converges.
Problem 2.b

First let’s recall a lemma from buck:

\[ \sum_{k=1}^{n} \sin(kx) = \frac{\cos(x/2) - \cos((n + 1/2)x)}{2\sin(x/2)} \]

for all \( x \) with \( \sin(x/2) \neq 0 \). Proof: We have

\[ \sin(x/2) \sum_{k=1}^{n} \sin(kx) = \sin(x/2) \sin x + \sin(x/2) \sin 2x + \cdots + \sin(x/2) \sin nx \]

using the identity \( 2 \sin A \sin B = \cos(B - A) - \cos(B + A) \). Hence we have

\[ 2 \sin(x/2) \sum_{k=1}^{n} \sin(kx) = (\cos(x/2) - \cos(3x/2)) + (\cos(3x/2) - \cos(5x/2)) + \cdots \]

\[ + (\cos((n - 1/2)x) - \cos((n + 1/2)x)) = \cos(x/2) - \cos(n + 1/2)x \]

Thus

\[ \sum_{k=1}^{n} \sin(kx) = \frac{\cos(x/2) - \cos((n + 1/2)x)}{2\sin(x/2)} \]

So for \( k = 1 \), we have

\[ \sum_{k=1}^{n} \sin k \leq \frac{1}{\sin(1/2)} \]

for all \( n \). Thus by applying the previous result, the summation

\[ \sum_{k=1}^{\infty} \sin \ln n \]

converges.

Problem 4

Let’s assume that \( a_0 + a_1x + a_2y + a_3z = 0 \) and \( b_0 + b_1x + b_2y + b_3z = 0 \) do indeed intersect. Now let \( P_0 \) be any point on the intersection where \( P_0 = (P_x, P_y, P_z) \). Then the intersection is the line \( l \)

\[ l = P_0 + t(v_1, v_2, v_3) \]

where

\[ (v_1, v_2, v_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_3 - a_3b_2) \]

since the cross product of the normal vectors of the planes. Hence we want to minimize the function

\[ f(t) = (P_x + tv_1)^2 + (P_y + tv_2)^2 + (P_z + tv_3)^2 \]
so
\[ \frac{df}{dt} + 2(P_x + tv_1)v_1 + 2(P_y + tv_2)v_2 + 2(P_z + tv_3)v_3 = 0 \]
Hence
\[ (P_x + tv_1)v_1 + (P_y + tv_2)v_2 + (P_z + tv_3)v_3 = 0 \Rightarrow t_0 = \frac{-P_1v_1 - P_2v_2 - P_3v_3}{v_1^2 + v_2^2 + v_3^2} \]
Thus the minimum distance is
\[ D = \sqrt{(P_x + t_0v_1)^2 + (P_y + t_0v_2)^2 + (P_z + t_0v_3)^2} \]

**Problem 5**
Assume for all intervals \([a, b]\) \(\{f_n(x)\}\) is not uniformly bounded. So let \([a, b]\) be any interval. Since \(\{f_n(x)\}\) is not uniformly bounded and for all \(x_0 \in [a, b]\), \(\{f_n(x_0)\} \leq M\) implies that \(\exists\) limit point \(x_1\) such that for \(x_n \in [a, b]\) such that \(x_n \to x_1\)
\[ \lim_{n \to \infty} f(x_n) = \infty \]
But since \([a, b]\) is compact, it contains all of its limit points. Hence this implies that \(x_1 \in [a, b]\) and \(f(x_1)\) is not continuous. Thus we have a contradiction.

**Sept 1992 Complex Variables**

**Problem 1**
Let \(C\) be the semicircle in the upper-half plane with radius \(R\). Let \(f(z) = \frac{1}{(1+z)^2}\). By the Residue Theorem we have
\[ \int_C f(z)dz = 2\pi i \left( \text{Res}(f, e^{i\pi/4}) + \text{Res}(f, -e^{-i\pi/4}) \right) \]
Notice
\[ f(z) = \frac{1}{(z + e^{i\pi/4})^2(z - e^{i\pi/4})^2(z - e^{-i\pi/4})^2(z + e^{-i\pi/4})^2} \]
Hence
\[ \text{Res}(f, e^{i\pi/4}) = \lim_{z \to e^{i\pi/4}} \left( \frac{1}{(z^2 + i)^2(z + e^{i\pi/4})^2} \right)' \]
\[ \lim_{z \to e^{i\pi/4}} \frac{-2(z^2 + i)(z + e^{i\pi/4})(2z(z + e^{i\pi/4}) + (z^2 + i))}{(z^2 + i)^4(z + e^{i\pi/4})^4} = \frac{-3e^{i\pi/4}}{16} \]
Likewise
\[ \text{Res}(f, -e^{-i\pi/4}) = \lim_{z \to -e^{-i\pi/4}} \frac{1}{(z^2 - i)^2(z - e^{-i\pi/4})^2} \]

\[ \lim_{z \to -e^{-i\pi/4}} \frac{-2(z^2 - i)(z - e^{-i\pi/4})(2z(z - e^{-i\pi/4}) + (z^2 + i))}{(z^2 - i)^4(z - e^{-i\pi/4})^4} = \frac{3e^{-i\pi/4}}{16} \]

Thus
\[
\int_C f(z) \, dz = 2\pi i \left( \frac{3}{16} \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) - \frac{3}{16} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right) = \frac{3\sqrt{2}\pi}{8}
\]

now the integral along the arc approaches zero since
\[
\left| \int_{C_R} f(z) \, dz \right| \leq \pi R \frac{1}{(R^4 - 1)^2} \to 0
\]
as \( R \to \infty \). Hence
\[
\int_{-\infty}^{\infty} \frac{dx}{(1 + x^4)^2} = \frac{3\sqrt{2}\pi}{8}
\]

Finally since \( f(z) \) is an even function, we have
\[
\int_{0}^{\infty} \frac{dx}{(1 + x^4)^2} = \frac{3\pi}{8\sqrt{2}}
\]

**Problem 2.a**

Using Rouches’ Theorem, on the circle \(|z| = 1/2\), we have
\[
|z^7 + 2z^3 + 1 - 1| = |z^7 + 2z^3| \leq |z^7| + |2z^3| = \left( \frac{1}{2} \right)^7 + 2 \left( \frac{1}{2} \right)^3 < 1 = |1|
\]

Hence there are no zeros inside the circle \(|z| = 1/2\).

**Problem 2.b**

Let \( \epsilon > 0 \) be really small. Notice on the circle \(|z| = 1 - \epsilon\), we have
\[
|z^7 + 2z^3 + 1 - 2z^3| = |z^7 + 1| \leq |z^7| + 1 = (1 - \epsilon)^7 + 1 < 2(1 - \epsilon)^3 = |2z^3|
\]

Since \( \epsilon \) is arbitrary, by above there are 3 zeros inside the annulus \( 1/2 < |z| < 1 \).

**Problem 2.c**

For \(|z| = 2\), we have
\[
|z^7 + 2z^3 + 1 - z^7| = |2z^3 + 1| \leq |2z^3| + 1 = 17 < 128 = |z^7|
\]

Thus by above there are 4 zeros inside the annulus \( 1 \leq |z| < 2 \).
Problem 3

We first map \( D_1 = \{x + iy : -1 < x < 1\} \) to \( D_2 = \{x + iy : -\pi/2 < x < \pi/2\} \) by \( w = (\pi/2)z \).

Then we map \( D_2 \) to \( D_3 = \{x + iy : -\pi/2 < y < \pi/2\} \) by \( w' = i(\pi/2)z \).

Then we map \( D_3 \) to \( D_4 = \{x + iy : 0 < y < \pi\} \) by \( w'' = i(\pi/2)z + i\pi/2 \).

Then we map \( D_4 \) to the upper-half plane \( U \) by

\[ w''' = e^{i(\pi/2)z + i\pi/2} \]

Finally we map the upper half plane \( U \) to the unit circle by

\[ w'''' = e^{i(\pi/2)z + i\pi/2} - i \over e^{i(\pi/2)z + i\pi/2} + i \]

Sept 1992 Linear Algebra

Problem 2.a

Let \( \lim_{k \to \infty} B_k = B = e^A \). This implies that \( b_{k,i,j} \to b_{i,j} \) \( \forall i, j \). Thus we need to check if \( b_{i,j} < \infty \).

Let \( || \cdot || \) be the usual Euclidean norm for vector and for \( A \in \mathbb{F}^{n \times n} \)

\[ ||A|| \triangleq \max_{x \neq 0} \frac{||Ax||}{||x||} \]

Recall that \( ||Ax|| \leq ||A|| \cdot ||x|| \) and that

\[ ||A||^2 \triangleq \max_{x \neq 0} \frac{||Ax||^2}{||x||^2} = \max_{x \neq 0} \frac{x^T A^T A x}{x^T x} = \lambda_{\text{max}}(A^T A) \]

Then let \( c = \sqrt{\lambda_{\text{max}}(A^T A)} = ||A||. \) Thus for any \( x \neq 0 \), we have

\[ \lim_{k \to \infty} B_k = \lim_{k \to \infty} \left| \sum_{j=0}^{k} \frac{A^j x}{j!} \right| \leq \lim_{k \to \infty} B_k = \lim_{k \to \infty} \sum_{j=0}^{k} \frac{||A^j x||}{j!} = \lim_{k \to \infty} \sum_{j=0}^{k} \frac{c^j ||x||}{j!} = ||x|| e^c \]

Hence the series does indeed converge.

Problem 2.b

Clearly the matrices \( A \) and \( (-A) \) commute, \( A(-A) = (-A)A \). Now I claim that for any two square matrices \( A \) and \( B \) that commute, we have \( e^{A+B} = e^A e^B \). Indeed since

\[ e^{A+B} = I + (A + B) + \frac{(A + B)^2}{2!} + \frac{(A + B)^3}{3!} + \cdots \]

\[ = \left( I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \right) \left( I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots \right) = e^A e^B \]

Hence

\[ \det(e^0) = \det\left(e^{A+(-A)}\right) = \det(e^A e^{-A}) = \det(e^A) \det(e^{-A}) = 1 \]

Thus \( \det(e^A) \neq 0 \) which implies that it is nonsingular.
Problem 3.a

Using Gaussian elimination we have

\[
\begin{array}{cccc|c}
1 & 3 & 2 & 1 & a \\
2 & 5 & 3 & 0 & a \\
1 & 0 & -1 & 0 & -b \\
0 & 1 & 1 & 0 & b \\
\end{array}
\]

\[
\begin{array}{cccc|c}
1 & 3 & 2 & 1 & a \\
0 & -1 & -1 & -2 & -a \\
0 & -3 & -3 & -1 & -b - a \\
0 & 1 & 1 & 0 & b \\
\end{array}
\]

\[
\begin{array}{cccc|c}
1 & 3 & 2 & 1 & a \\
0 & 1 & 1 & 2 & a \\
0 & 0 & 0 & 5 & -b + 2a \\
0 & 0 & 0 & -2 & b - a \\
\end{array}
\]

Hence we must have \(3b = a\) for a solution to exist.

Jan 1993 Advanced Calculus

Problem 1

Recall the theorem, for \(f \in C^2[1, \infty)\) such that \(f \geq 0\), \(f' \geq 0\), and \(f'' \leq 0\), we have

\[
\left| \sum_{k=1}^{n} f(k) - \int_{1}^{n} f(x)dx - \frac{f(n)}{2} \right| \leq C
\]

Hence let \(f(x) = \log x\), we have

\[
\sum_{k=1}^{n} \log k = \int_{1}^{n} \log xdx - \frac{\log n}{2} + C
\]

Now since \(\int_{1}^{n} \log xdx = n \log n - n + 1\), we have

\[
\sum_{k=1}^{n} \log k = n \log n - n + 1 - \frac{\log n}{2} + C = n \log n - n + O(\log n)
\]

Jan 1993 Complex Analysis

Problem 1.a

Let \(f(z) = \frac{e^{ikz}}{1+z^2}\), and let \(\gamma\) be the upper semi-circle of radius \(R\). Then by the residue theorem, we have

\[
\int_{\gamma} f(z)dz = 2\pi i \frac{e^{-k}}{2i} = \pi e^{-k}
\]
Now notice on the arc $z = Re^{i\theta}$ for $\theta: 0 \to \pi$, we have

$$\left| \int_{C_R} \frac{e^{ik(R\cos \theta + iR\sin \theta)}}{1 + R^2 e^{2i\theta}} R e^{i\theta} d\theta \right| \leq \frac{Re^{-Rk\sin \theta}}{R^2 - 1} \pi \to 0$$

as $R \to \infty$. Hence

$$\int_{-R}^{R} \frac{e^{ikz}}{1 + z^2} dz = \int_{-R}^{R} \frac{\cos(kz)}{1 + z^2} dz + i \int_{-R}^{R} \frac{\sin(kz)}{1 + z^2} dz = \pi e^{-k}$$

Thus since $\cos(kz)/(1 + z^2)$ is an even function, we have as $R \to \infty$

$$\int_{0}^{\infty} \frac{\cos(kz)}{1 + z^2} dz = \frac{\pi e^{-k}}{2}$$

**Problem 1.b**

Let $\log z$ be the principle branch. Then notice if $F(z) = z \log z - z$, $F'(z) = \log z$. Hence $F(z)$ is the antiderivative of $f(z)$. Thus

$$\int_{C} \log zdz = \int_{1}^{i} \log zdz = z \log z - z |_{1}^{i} = i (\log 1 + i\pi/2) - i + 1 = \left(-\frac{\pi}{2} + 1\right) - i$$

**Jan 1993 Linear Algebra**

**Problem 1.a**

Notice that $(1, 1, 1, 1)^T$ is in the null space of $A^T$. Hence the Rank of $A$ must be less than 4. Notice by gaussian elimination

\[
\begin{array}{cccc}
-5 & 3 & 1 & 6 \\
1 & -4 & 1 & 4 \\
2 & 1 & -3 & 0 \\
2 & 0 & 1 & -1 \\
\end{array}
\]

\[
\begin{array}{cccc}
-5 & 3 & 1 & 6 \\
0 & -17 & 6 & 20 \\
0 & 1 & -13 & 12 \\
0 & 6 & 7 & -38 \\
\end{array}
\]

Hence if we continued, we can see that it’s not going to be all zeros below the second line. Hence Rank of $A$ is 3.

**Problem 1.b**

$(\Rightarrow)$ Notice $Ax = b$ and $x_1 + x_2 + x_3 + x_4 = 0$. Then notice that

$$(1, 1, 1, 1)Ax = (1, 1, 1, 1)b = 0$$
Hence \( b_1 + b_2 + b_3 + b_4 = 0 \). (\( \Leftarrow \)). Now suppose that \( b_1 + b_2 + b_3 + b_4 = 0 \). Then for \( V = \{ x \in \mathbb{R}^4 : v_1 + v_2 + v_3 + v_4 = 0 \} \), \( V \) is a subspace since \( x_1, x_2 \in V \) implies that \( x_1 + x_2 \in V \), and \( \alpha v_1 \in V \), and \( 0 \in V \). Also notice that \( V \) has dimension 3. So if we define the linear transformation \( T : V \mapsto V \), then

\[
\text{Im}(T) + \text{Ker}(T) = 3
\]

So if \( \text{Ker}(T) = 0 \), implies that \( T \) is a one to one and onto mapping of \( V \). Notice if \( Tx = 0 \), implies that \( x = 0 \). Hence \( T \) is one-to-one and onto.

**Problem 2.a**

\[
\det\begin{pmatrix}
7 - x & -9 \\
11 & -13 - x
\end{pmatrix} = (7 - x)(-13 - x) + 99 = (x + 4)(x + 2) = 0
\]

Hence the eigenvalues are \( \lambda_1 = -2 \) and \( \lambda_2 = -4 \).

**Problem 2.b**

We have \( M(k) = \begin{pmatrix} 7 - k^2 & -9 \\ 11 & -13 - k^2 \end{pmatrix} \). Thus

\[
\det\begin{pmatrix}
7 - k^2 - x & -9 \\
11 & -13 - k^2 - x
\end{pmatrix} = (7-k^2-x)(-13-k^2-x) + 99 = x^2 + (2k^2+26)x + k^4 + 6k^2 + 8 = 0
\]

using the quadratic formula we have

\[
x = \frac{-(2k^2 + 26) \pm \sqrt{(2k^2 + 26)^2 - 4(k^2 + 4)(k^2 + 2)}}{2}
\]

Notice that \( 4(k^2 + 4)(k^2 + 2) > 0 \ \forall k \in \mathbb{R} \). Thus if \( 4(k^2 + 4)(k^2 + 2) \geq (2k^2 + 26)^2 \), then \( \text{Re}(x) = -(2k^2 + 26) < 0 \). If \( 4(k^2 + 4)(k^2 + 2) < (2k^2 + 26)^2 \), then \( \text{Re}(x) < -(2k^2 + 26) + 2k^2 + 26 = 0 \). Hence the eigenvalues have a positive real part for all real \( k \).

**Problem 3.a**

We are given that \( AB = BA \). Then notice

\[
A^2B = AAB = ABA = BAA = BA^2
\]

Hence \( A^2 \) commutes with \( B \).

**Problem 3.b**

Let \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Then \( A^2 = 0 \). Hence regardless what \( B \) is, we have \( A^2B = BA^2 \). However notice if \( B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), then we have

\[
BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = AB
\]

Hence the converse is not true.
Problem 4.a

Using Gaussian Elimination, we have

\[
A = \begin{pmatrix}
1 & 2 & 3 & -2 \\
2 & 5 & 4 & -5 \\
3 & 4 & 14 & -2 \\
-2 & -5 & -2 & 10
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & -2 \\
0 & 1 & -2 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 5
\end{pmatrix}
\]

\[
\rightarrow \begin{pmatrix}
1 & 2 & 3 & -2 \\
0 & 1 & -2 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Now let’s just check to see if \( A = L^T L \):

\[
L^T L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & -2 & 1 & 0 \\
-2 & -1 & 2 & 1
\end{pmatrix} \begin{pmatrix}
1 & 2 & 3 & -2 \\
0 & 1 & -2 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 & -2 \\
2 & 5 & 4 & -5 \\
3 & 4 & 14 & -2 \\
-2 & -5 & -2 & 10
\end{pmatrix}
\]

Hence we found \( L \) by luck.

Problem 4.b

\( A \) is indeed positive semi-definite since for any \( x \in \mathbb{R}^4 \), except \( x = 0 \), we have

\[
x^T A x = x^T L^T L x = (Lx)^T L x = \sum_{i=1}^{n} a_i^2 > 0
\]

where \( Lx = (a_1, a_2, a_3, a_4) \).

Problem 4.c

Since \( A = L^T L = L^T IL \), the eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1 \). Thus

\[
\det(A) = \prod_{i=1}^{4} \lambda_i = 1
\]

Problem 5.a

Geometrically we can see that the vertices of the pentagon at

\[
(0, 0, 0), (0, 1, 2/3), (1, 0, 2/3), (1, 1/2, 1), (1/2, 1, 1)
\]

.
Problem 5.b
Let \( d(x, y, z) = c \) for some constant \( c \). Then this is a plane in \( \mathbb{R}^3 \). So as \( c \) sweeps the real line, \( d(x, y, z) = c \) sweeps \( \mathbb{R}^3 \) which implies that max of \( d(x, y, z) \) must occur at a vertex of the pentagon since it is convex. Thus we can see that the max is

\[
d(1/2, 1, 1) = 1 + 2 + 3 = 5
\]

Sept 1993 Advanced Calculus

Problem 1.a
TRUE. Assume that the sequence of integers \( s_n \) converges to a non-integer \( k \). Then for all \( \epsilon > 0 \), \( \exists N \) such that \( \forall n > N \), we have

\[
|s_n - k| < \epsilon
\]

Let \( \epsilon = \frac{[|k|] - k}{2} \). Then we have

\[
|s_n - k| < \frac{|[k] - k|}{2}
\]

which is not possible and hence we have a contradiction. Thus every Cauchy sequence of integers has an integer limit.

Problem 1.b
FALSE. Let our sequence of rational numbers \( s_n \) be defined as

\[
s_n = \sum_{k=1}^{n} \frac{1}{k!}
\]

However notice that \( \lim s_n = e \) which is irrational.

Problem 1.c
FALSE. Notice that the sequence \( s_n = e^{-n} \) is irrational. However \( \lim s_n = 0 \) which is rational.

Problem 1.d
FALSE. Notice that the sequence \( s_n = 1/n \) is strictly positive. However \( \lim s_n = 0 \), which is not.

Problem 1.e
TRUE. Assume that \( s_n \) is a sequence of non-negative numbers that converges to \( k < 0 \). Then \( \forall \epsilon > 0, \exists N \) such that \( \forall n > N \), we have

\[
|s_n - k| < \epsilon
\]

Thus let \( \epsilon = |k|/2 \). Then we have
\[ |s_n - k| < |k|/2 \]

which is not possible, and hence we have a contradiction.

**Problem 2**

No, not true. Let \( f_n(x) = x^n \) on \([0, 1]\). Clearly for all \( n \), \( f_n \) is continuous. However

\[
\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 
0 & x \in [0, 1) \\
1 & x = 1
\end{cases}
\]

Thus \( f(x) \) is not continuous.

**Problem 3.a**

On the circle of radius \( R \), we have \( x = R \cos \theta \) and \( y = R \sin \theta \). Hence

\[
C(R) = \int_{x^2 + y^2 = R^2} (u \, dx + v \, dy) = \int_0^{2\pi} \frac{KR^2 \sin^2 \theta}{2\pi(R^2 + r_0^2)} + \frac{KR^2 \cos^2 \theta}{2\pi(R^2 + r_0^2)} \, d\theta
\]

\[
= \int_0^{2\pi} \frac{KR^2}{2\pi(R^2 + r_0^2)} \, d\theta = \frac{KR^2}{R^2 + r_0^2}
\]

**Problem 3.b**

Notice we have

\[
\frac{du}{dy} = -\frac{K2\pi(x^2 + y^2 + r_0^2) + 4\pi y^2 K}{(2\pi(x^2 + y^2 + r_0^2))^2}
\]

and

\[
\frac{dv}{dx} = \frac{K2\pi(x^2 + y^2 + r_0^2) - 4\pi x^2 K}{(2\pi(x^2 + y^2 + r_0^2))^2}
\]

Hence

\[
w(x, y) = -\frac{du}{dy} + \frac{dv}{dx} = \frac{4K\pi(x^2 + y^2 + r_0^2) - 4\pi K(x^2 + y^2)}{4\pi(2\pi(x^2 + y^2 + r_0^2))^2} = \frac{Kr_0^2}{\pi(x^2 + y^2 + r_0^2)^2}
\]

Now using Stoke’s Theorem and polar coordinates, we have

\[
C(R) = \int_S \left( -\frac{du}{dy} + \frac{dv}{dx} \right) \, dS = \int_S \frac{Kr_0^2}{\pi(x^2 + y^2 + r_0^2)^2} \, dx \, dy
\]

\[
= \int_{r=0}^{R} \int_{\theta=0}^{2\pi} \frac{Kr_0^2}{\pi(r^2 + r_0^2)^2} \, r \, d\theta \, dr = \int_{r=0}^{R} \frac{2Kr_0^2}{(r^2 + r_0^2)^2} \, r \, dr
\]

Now applying \( u \) substitution we have \( u = r^2 + r_0^2 \) and \( du = 2r \)
\[
\begin{align*}
\int \frac{Kr_0^2}{u^2} du &= \left. \frac{-Kr_0^2}{u} \right|_{r = 0}^{R} = \frac{KR^2}{R^2 + r_0^2}
\end{align*}
\]

**Problem 3.d**

\[
w_0(x, y) = \lim_{r_0 \to 0} w(x, y) = \lim_{r_0 \to 0} \frac{Kr_0^2}{\pi (x^2 + y^2 + r_0^2)^2} = 0
\]

and

\[
\lim_{r_0 \to 0} C(R) = \lim_{r_0 \to 0} \frac{KR^2}{R^2 + r_0^2} = K
\]

**Problem 5.a**

We want to show that

\[
1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1 + x} < 1 + \frac{1}{2}x
\]

for all \( x > 0 \). Notice that the last two terms are positive. Hence

\[
\sqrt{1 + x} < 1 + \frac{1}{2}x \Rightarrow 1 + x < a + x + \frac{1}{4}x^2 \Rightarrow 0 < \frac{1}{4}x^2
\]

Hence the inequality is true. Now notice that \( 1 + \frac{1}{2}x - \frac{1}{8}x^2 \) is a parabola with a local max and it approaches \(-\infty\) as \( x \to \pm\infty \). Also notice that for

\[
x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{2}} = 2 \pm 2\sqrt{3}
\]

Thus for \( f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \) and \( x \in (0, 2 + 2\sqrt{3}) \), \( f(x) \geq 0 \). Hence

\[
\left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)^2 < 1 + x \Rightarrow \frac{x^4}{64} < \frac{x^3}{8}
\]

which is true for \( x \in (0, 2 + 2\sqrt{3}) \). Also for \( x > 2 + 2\sqrt{3} \), we have

\[
1 + \frac{1}{2}x - \frac{1}{8}x^2 < 0 < \sqrt{1 + x}
\]

Thus for \( x > 0 \) we have

\[
1 + \frac{1}{2}x - \frac{1}{8}x^2 < \sqrt{1 + x} < 1 + \frac{1}{2}x
\]

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Problem 5.b
By using the Ratio Test, we have
\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{\frac{1}{2} \left[(1 + \frac{1}{2}x_n) - \sqrt{1 + x_n}\right]}{x_n} = \lim_{n \to \infty} \frac{2}{x_n^2} \left[(1 + \frac{1}{2}x_n) - \sqrt{1 + x_n}\right] \]
By applying part (a), we have
\[ < \lim_{n \to \infty} \frac{2}{x_n^2} \left[(1 + \frac{1}{2}x_n - 1 - \frac{1}{2}x_n + 1)\right] = \lim_{n \to \infty} \frac{2}{x_n^2} \frac{x_n^2}{8} = \frac{1}{4} < 1 \]
Hence \( x_n \) converges. Again using the Ratio Test we have
\[ \lim_{n \to \infty} \frac{y_{n+1}}{y_n} = \frac{2^{n+1} \sqrt{x_{n+1}}}{2^n \sqrt{x_n}} = 2 \frac{2}{x_n} \left[(1 + \frac{1}{2}x_n) - \sqrt{1 + x_n}\right] \]
\[ < 2 \frac{2}{x_n} \left[(1 + \frac{1}{2}x_n - 1 - \frac{1}{2}x_n + 1)\right] = 2 \frac{2}{x_n} \frac{1}{8} \frac{x_n^2}{8} = 2 \frac{\sqrt{1}}{4} = 1 \]
Thus \( y_n \) converges.

Problem 5.c
By part (a) and (b), we have \( x_{n+1} \leq \frac{2}{x_n} \frac{x_n^2}{8} = \frac{1}{4} x_n \). Since \( x_2 = 1 \), we have
\[ x_{n+1} \leq \left(\frac{1}{4}\right)^{n-1} \]
Hence \( x_n \to 0 \).

Sept 1993 Linear Algebra

Problem 1.a
Using Gaussian Elimination, we have
\[
\begin{array}{ccc|c}
 a & 1 & 1 & b \\
 1 & a & 1 & c \\
 1 & 1 & 1 & 2 \\
\hline
 a & 1 & 1 & b \\
 0 & 1 - a^2 & 1 - a & b - ac \\
 0 & 1 - a & 1 - a & b - 2a \\
\hline
 a & 1 & 1 & b \\
 0 & 1 - a^2 & 1 - a & b - ac \\
 0 & 0 & a^2 - a & 2a^2 + 2a - ac - ab \\
\end{array}
\]
Thus the system of equations have a unique solutions when \( a \neq 0, a \neq 1, \) and \( b, c \in \mathbb{R} \).
Problem 1.b
We have a non-unique solutions when \( a = 0, \) and \( b, c \in \mathbb{R}. \) Or when \( a = 1, \) and \( b + c = 4. \)

Problem 2
Notice that \( A \) is a symmetric real invertible matrix. Thus it is diagonalizable. Thus

\[
\det \begin{pmatrix}
\frac{7\pi}{4} - \lambda & -3 \\
-3 & \frac{7\pi}{4} - \lambda
\end{pmatrix} = (\frac{7\pi}{4} - \lambda)^2 - (3\pi/4)^2 = \lambda^2 - \frac{7\pi^2}{2} + \frac{10\pi^2}{4} = (\lambda - 5\pi/2)(\lambda - \pi)
\]

Hence our eigenvalues are \( 5\pi/2 \) and \( \pi. \) Thus

\[
D = \begin{pmatrix}
\pi & 0 \\
0 & 5\pi/2
\end{pmatrix}
\]

Thus solving for the eigenvectors, we have for \( \lambda = \pi \)

\[
\frac{\pi}{4} \begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \pi \begin{pmatrix} x \\ y \end{pmatrix}
\]

Which implies that \( 7x - 3y = 4x, \) and \( x = y. \) Hence our first eigenvector is \( u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \) And since our eigenvectors must be orthogonal, we have \( u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \) Thus \( A = UDU^{-1} \) where

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

Hence

\[
\sin A = U \sin DU^{-1} = U \begin{pmatrix} \sin \pi & 0 \\ 0 & \sin 5\pi/2 \end{pmatrix} U^{-1}
\]

Problem 3
No such matrix exists since

\[
Tr(AB - BA) = Tr(AB) - Tr(BA) = Tr(AB) - Tr(AB) = 0 \neq Tr(I) = n
\]

Problem 5
We know that \( \det A \) is a polynomial of \( \lambda_i \) of degree \( \leq n(n - 1)/2. \) Also notice that if \( a_i = a_j \) for \( i \neq j, \) then clearly we have \( \det A = 0. \) Thus by the fundamental theorem of algebra (Factor Theorem), we know that

\[
\prod_{n \geq i > j \geq 1} (\lambda_i - \lambda_j) \quad \det A
\]

Since we know it’s a polynomial of degree \( \leq n(n - 1)/2, \) we have
\[ \det A = c_n \prod_{n \geq i > j \geq 1} (\lambda_i - \lambda_j) \]

for some constant \( c_n \). We will know show that \( c_n = 1 \) for all \( n \) by induction. **BASE CASE:** for \( n = 2 \), we have

\[ \det A = \lambda_2 - \lambda_1 \]

and so \( c_2 = 1 \). **INDUCTIVE STEP:** Now we know that

\[ \det A = \sum_{k=1}^{n} (-1)^{k+1} \lambda_n^{n-k} \det A_{n+1-k,n} = c_n \prod_{n \geq i > j \geq 1} (\lambda_i - \lambda_j) \]

which implies that \( c_{n-1} = c_n \). Thus this completes the proof.

**Jan 1994 Advanced Calculus**

**Problem 1.a**

By using integration by parts, we set

\[ u = x^{-1/2}, \quad v = (1/i)e^{ix} \]
\[ du = (-1/2)x^{-3/2}dx, \quad dv = e^{ix}dx \]

and we have

\[ \int_{1}^{\infty} \frac{e^{ix}}{\sqrt{x}} dx = \left[ \frac{e^{ix}}{i\sqrt{x}} \right]_{1}^{\infty} + \frac{1}{2} \int_{1}^{\infty} \frac{e^{ix}}{x^{3/2}} dx \]

and clearly \( \frac{1}{2} \int_{1}^{\infty} \frac{e^{ix}}{x^{3/2}} dx \) converges. Hence the integral exists.

**Problem 1.b**

By applying \( u \) substitution, we set \( u = \ln x, \ du = (1/x)dx, \) and \( x = e^u \). Then we have

\[ \int_{2}^{\infty} \left( 1 - \frac{2}{x} \right)^{x \ln x} dx = \int_{\ln 2}^{\infty} (e^u - 2)e^u du \]

which clearly diverges. Hence the integral does not exist.

**Problem 2.a**

We want to show that

\[ f(x) = \sum_{n=1}^{\infty} \frac{x}{(1+n^2)x^{n^2}} \]

is continuous for \( x > 0 \). If \( \alpha > 1/2 \), then notice for each fixed \( n \),
is continuous. Now

$$\sum_{n=1}^{\infty} \frac{x}{(1 + nx^2)n^\alpha}$$

is continuous if it converges uniformly. We will prove this by the Weierstrass $M$-test. Recall that if $|f_n(x)| \leq a_n$ and $\sum |a_n| < \infty$, then $\sum f_n(x)$ converges uniformly. So let

$$f_n(x) = \frac{x}{(1 + nx^2)n^\alpha}$$

then by taking the derivative and setting it to zero we have

$$f'_n(x) = \frac{(1 + nx^2)n^\alpha - 2n^\alpha nx^2}{(1 + nx^2)^2n^{2\alpha}} = 0$$

which implies

$$1 - nx^2 = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Thus for all $n$,

$$|f_n(x)| \leq \frac{1}{\sqrt{n}} = \frac{1}{2n^{\alpha+1/2}}$$

Now we just apply the Weierstrass $M$-test, which implies that $f$ converges uniformly for $x \geq 0$.

**Problem 2.b**

By above, since $f$ is continuous, $f(0) = 0$ implies that $\lim_{x \to 0} f(x) = 0$.

**Problem 2.c**

Notice we have

$$\sum_{n=1}^{\infty} \frac{x}{(1 + nx^2)\sqrt{n}} \geq \int_{1}^{\infty} \frac{x}{(1 + nx^2)\sqrt{n}} dn$$

Then by applying $u$-substitution, we have $u = x\sqrt{n}$ and $du = (1/2)xu^{-1/2}$. Hence

$$\geq \int_{x}^{\infty} \frac{2}{1+u^2} du = 2\arctan u|_{0}^{\infty} = \pi > 0$$
Problem 4
Let $x = r \cos \theta$ and $y = (r/2) \sin \theta$. By Taylor’s theorem, we have

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!} (x^2 f_{xx}(0, 0) + xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0))$$

$$+ \frac{1}{3!} (x^3 f_{xxx}(0, 0) + x^2 y f_{xxy}(0, 0) + x y^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)) + \cdots$$

So

$$\int \int_{x^2 + 4y^2 \leq r^2} f(x, y) \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^r f(0, 0) \rho + \rho^2 \cos \theta f_x(0, 0) + \rho^2 \sin \theta f_y(0, 0)$$

$$+ \frac{1}{2!} (\rho^3 \cos^2 \theta f_{xx}(0, 0) + \rho^3 \cos \theta \sin \theta f_{xy}(0, 0) + \rho^3 \sin^2 \theta f_{yy}(0, 0)) + E(\rho, \theta) \, d\rho \, d\theta$$

Notice that

$$\int_0^{2\pi} \cos \theta \, d\theta = \int_0^{2\pi} \sin \theta \, d\theta = 0$$

so

$$= \int_0^{2\pi} \int_0^r \rho f(0, 0) + \frac{1}{2!} (\rho^3 \cos^2 \theta f_{xx}(0, 0) + \rho^3 \cos \theta \sin \theta f_{xy}(0, 0) + \rho^3 \sin^2 \theta f_{yy}(0, 0)) + E(\rho, \theta) \, d\rho \, d\theta$$

$$= \int_0^{2\pi} \int_0^r \frac{\rho^2}{2} f(0, 0) + \frac{\rho^4}{8} \cos^2 \theta f_{xx}(0, 0) + \frac{\rho^4}{8} \cos \theta \sin \theta f_{xy}(0, 0) + \frac{\rho^4}{8} \sin^2 \theta f_{yy}(0, 0) + F(r, \theta) \, d\rho \, d\theta$$

$$= r^2 f(0, 0) \pi + \frac{\pi r^4}{8} f_{xx}(0, 0) + \frac{\pi r^4}{8} f_{yy}(0, 0) + O(r^5)$$

So

$$a = f(0, 0) \pi \quad b = \frac{\pi f_{xx}(0, 0)}{8} + \frac{\pi f_{yy}(0, 0)}{8}$$

Problem 5
Using the Lagrange multiplier theorem, we have $\nabla f = \lambda \nabla g$, thus

$$(4x_1^3, 4x_2^3, 4x_3^3, 4x_4^3) = \lambda (1, 1, 1, 1)$$

which implies $x_1 = x_2 = x_3 = x_4 = 1/4$. Therefore the minimum $f = 1/4^4$. 

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Jan 1994 Complex Variables

Problem 1

The domain between the two circles \((x - 1)^2 + y^2 = 1\) and \((x - 2)^2 + y^2 = 4\) will be mapped to the infinite strip \(D_2 = \{x + iy : 1/4 < x < 1/8\}\) by \(w = 1/z\). Indeed since \(w : 0 \to \infty\), \(w : 2 \to 1/2\), and \(w : 1 + i \to 1/2 - i/2\). Hence the inner circle maps to the line \(x = 1/2\). Likewise \(w : 4 \to 1/4\) and \(w : 2 + 2i \to 1/4 - i/4\). Hence the outer circle maps to the line \(x = 1/4\). Hence we will map \(D_2\) to \(D_3 = \{x + iy : -1/8 < x < 1/8\}\) by \(w = 4\pi z\). The rotate and shift the region up by \(w = iz + i\pi/2\). Finally we will map \(D_5 = \{x + iy : 0 < y < \pi\}\) to the upper-half plane by \(w = e^z\). Thus our mapping is

\[ w = e^{4\pi i(1/z - 3/8)} + i\pi/2 \]

Problem 2.a

Let \(f(z) = \frac{1}{(z^2 - 4)\log(z + 2)}\) where \(\log\) is taken on the principle branch, and let \(\gamma\) be the right-half semi-circle with radius \(R\). Then by the Residue Theorem, we have

\[ \int_{\gamma} f(z)dz = 2\pi i \frac{1}{4\log 4} \]

Then on the arc where \(z = Re^{i\theta}\), we have

\[ \left| \int_{C_R} f(z)dz \right| \leq R\pi \frac{1}{R^2 \log R} \to 0 \]

as \(R \to \infty\). Hence

\[ \int_{-i\infty}^{i\infty} \frac{1}{(z^2 - 4)\log(z + 2)}dz = 2\pi i \frac{1}{4\log 4} \]

Problem 2.b

Let \(\gamma\) be the key hole contour, and let \(f(z) = \frac{z^\alpha}{z^{1+i}}\). Here we define \(z^\alpha = e^{\alpha \log z}\) where \(\log z = \ln |z| + i \arg z\) and \(0 < \arg z < 2\pi\). So we are cutting out the nonnegative real axis. Then by the residue theorem, we have

\[ \int_{\gamma} f(z)dz = 2\pi i \text{Res}(f, e^{i\pi/3}) + 2\pi i \text{Res}(f, e^{i\pi}) + 2\pi i \text{Res}(f, e^{i5\pi/3}) = \frac{2\pi ie^{i\pi i/3}}{3e^{2i\pi/3}} + \frac{2\pi ie^{i\pi}}{3e^{2i\pi}} + \frac{2\pi ie^{i5\pi/3}}{3e^{2i5\pi/3}} \]

Notice that on \(C_R\), we have

\[ \left| \int_{C_R} f(z)dz \right| \leq 2\pi R \frac{R^\alpha}{R^3 - 1} \to 0 \]

as \(R \to \infty\) and on \(C_\epsilon\), we have
\[ \left| \int_{C_\epsilon} f(z) \, dz \right| \leq 2\pi \epsilon \frac{\epsilon^\alpha}{e^\epsilon - 1} \to 0 \]
as \(\epsilon \to 0\). Thus on the rest of the contour we have
\[ \int_0^\infty \frac{r^\alpha e^{\alpha i \delta}}{r^3 e^{3i \delta} + 1} e^{i \delta} \, dr + \int_0^\infty \frac{r^\alpha e^{\alpha i 2\pi - \alpha i \delta}}{r^3 e^{3i 2\pi - 3i \delta} + 1} e^{i 2\pi - \delta} \, dr \]
as \(\delta \to 0\), we have
\[ (1 - e^{2\pi i \alpha}) \int_0^\infty \frac{r^\alpha}{r^3 + 1} \, dr = \frac{2\pi i e^{\alpha i \pi/3}}{3e^{2\pi i/3}} + \frac{2\pi i e^{\alpha i 5\pi/3}}{3e^{2i 5\pi/3}} \]
and so
\[ \int_0^\infty \frac{r^\alpha}{r^3 + 1} \, dr = \frac{2\pi i e^{\alpha i \pi/3}}{3e^{2\pi i/3}} + \frac{2\pi i e^{\alpha i 5\pi/3}}{3e^{2i 5\pi/3}} \]

**Problem 3.a**

Let \(z = x + iy\) and \(\zeta = a + bi\). Then by comparing
\[ |z - \zeta| \leq |1 - \overline{\zeta} z| \]
it works out.

**Problem 3.b**

By part (a), it’s obvious.

**Problem 3.c**

Let \(g(z) = \frac{z - \zeta}{1 - \overline{\zeta} z} f(z)\). Then \(g\) is analytic inside the unit disk. Also \(g(z) \leq 1\) on \(|z| = 1\). We are also given the \(\exists z_0\) such that \(g(z_0) = 1\). By the Maximum Modulus principle, the maximum of \(|g(z)|\) occurs at the boundary \(|z| = 1\). This implies that \(g\) is a constant with radius 1. Since \(g(1)\) is real, this implies \(g = 1\).

**Problem 4**

Using Rouche on \(|z| = 1\), we have
\[ |15z + 10z^{100} - z^{101} + 1 - 15z| = |10z^{100} - z^{101} + 1| \leq 10 + 1 + 1 = 12 < 15 = |15z| \]
So we have exactly 1 root inside \(|z| < 1\). Now on \(|z| = 2\), we have
\[ |15z + 10z^{100} - z^{101} + 1 - 10z^{100}| = |15z - z^{101} + 1| \leq 30 + 2^{101} + 1 < 10 \cdot 2^{101} = |10z^{100}| \]
Hence we have 99 zeros inside \(1 \leq |z| < 2\). Since we have a polynomial of degree 101, by the Fundamental Theorem of Algebra, we know that we have 101 roots C.M. total. So there is one root outside \(|z| \geq 2\).
Jan 1994 Linear Algebra

Problem 1.a
Matrices $A$ and $B$ are similar if $\exists$ an invertible matrix $M$ such that $A = M^{-1}BM$. Thus if $\lambda_i$ is an eigenvalue of $A$ with $x_i$ as its eigenvector, then

$$Ax_i = M^{-1}BMx_i = \lambda_i x_i \Rightarrow B(Mx_i) = \lambda Mx_i$$

Hence they share the same eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$. Thus they have the same characteristic equation

$$P_A(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) = P_B(x)$$

Problem 1.b

$(\Rightarrow)$ Suppose $A$ has $n$ linearly independent eigenvectors $x_1, ..., x_n$ with eigenvalues $\lambda_1, ..., \lambda_n$. Then we define the $n \times n$ matrix $S$ as

$$S = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

Then $S$ is invertible, and notice

$$AS = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} = SD$$

Hence since $S$ is invertible, we have $A = SDS^{-1}$. Hence $A$ is similar to a diagonal matrix.

$(\Leftarrow)$ Now suppose that $A$ is similar to a diagonal matrix $D$. Then $\exists$ an invertible matrix $S$ such that

$$S = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

where $x_i$ are all linearly independent. Then notice

$$AS = SD = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{pmatrix}$$

Hence $x_i$ for $i \leq 1 \leq n$ are all eigenvectors of $A$ with $\lambda_i$ as its eigenvalue. Hence $A$ has $n$ linearly independent eigenvectors.
Problem 2.a

By applying Gaussian Elimination, we have

\[
\begin{array}{cccc|c}
2 & 2 & 4 & 1 & b_1 \\
-3 & -3 & -6 & 2 & b_2 \\
-6 & -6 & -12 & 3 & b_3 \\
1 & 1 & 2 & 1 & b_4 \\
\end{array}
\]

\[
\begin{array}{cccc|c}
2 & 2 & 4 & 1 & b_1 \\
0 & 0 & 0 & 7/2 & (3/2)b_1 + b_2 \\
0 & 0 & 0 & 6 & 3b_1 + b_3 \\
0 & 0 & 0 & 1/2 & b_4 - (1/2)b_1 \\
\end{array}
\]

\[
\begin{array}{cccc|c}
2 & 2 & 4 & 1 & b_1 \\
0 & 0 & 0 & 7 & 3b_1 + 2b_2 \\
0 & 0 & 0 & 6 & 3b_1 + b_3 \\
0 & 0 & 0 & 1 & 2b_4 - b_1 \\
\end{array}
\]

Thus for there to exist solutions, we must have

\[3b_1 - 12b_2 + 7b_3 = 0 \quad \text{and} \quad -10b_1 - 2b_2 + 14b_4 = 0\]

Problem 3.b

By above we can see that the basis for \(V\) is \(\{(1, -1/3, -1), (1, 2, 1)\}\), which is linearly independent.

Problem 3.a

Notice

\[
\left\| \sum_{k=0}^{\infty} P^k \right\| \leq \sum_{k=0}^{\infty} \| P^k \| < \infty
\]

which implies

\[
\sum_{k=0}^{\infty} P^k
\]

exists. Now notice

\[I = (I - P)(I + P + P^2 + \cdots) \Rightarrow \det(I) = 1 = \det(I - P) \det(I + P + P^2 + \cdots)\]

which implies \(\det(I - P) \neq 0\). Hence \((I - P)\) is invertible, and

\[(I - P)^{-1} = I + P + P^2 + P^3 + \cdots\]
Problem 3.b

Notice by (3a) we have

\[ ||(I - P)^{-1}|| = \left| \sum_{k=0}^{\infty} P^k \right| \leq \sum_{k=0}^{\infty} ||P^k|| \leq \sum_{k=0}^{\infty} ||P||^k = \frac{1}{1 - ||P||^2} \]

Problem 3.c

Notice

\[ A = I - \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

So the inverse by above is

\[ A^{-1} = I + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 0 = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Problem 4.a

By letting \( x = (x_1, x_2, ..., x_9) \), we have

\[ x^T Ax = (-2x_1 + x_2, x_1 - 2x_2 + x_3, x_2 - 2x_3 + x_4, ..., x_7 - 2x_8 + x_9, x_8 - 2x_9) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_9 \end{pmatrix} \]

\[ = (-2x_1^2 + x_1x_2) + (x_1x_2 - 2x_2^2 + x_2x_3) + \cdots + (x_7x_8 - 2x_8^2 + x_8x_9) + (x_8x_9 - 2x_9^2) \]

\[ = -(x_1^2 + (x_2 - x_1)^2 + \cdots + (x_9 - x_8)^2 + x_9^2) \]

Problem 4.b

By above we can see that \(-A\) is positive definite. Hence since \(-A\) is symmetric, it has all positive eigenvalues. Hence \(A\) must have all negative eigenvalues.

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Problem 4.c
We have
\[
B = \begin{pmatrix}
1 & -3 & 0 & \cdots \\
-3 & 0 & 0 & \\
0 & 0 & 0 & \ddots \\
\vdots & & & & \\
0 & & & & A
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \\
\vdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\]
And by the Min/Max Theorem, we have
\[
\lambda_{\text{max}}(B) = \max_{x \in \mathbb{R}^{10}} \frac{x^T B x}{x^T x} = \frac{1}{x^T x} x^T \begin{pmatrix}
1 & -3 & 0 & \cdots \\
-3 & 0 & 0 & \\
0 & 0 & 0 & \ddots \\
\vdots & & & & \\
0 & & & & A
\end{pmatrix} x + x^T \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \\
\vdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} x
\]
So for \( x = (x_1, x_2, \ldots, x_{10}) \) we have
\[
x^T B x = x_1^2 - 3x_1x_2 - 3x_1x_2 - (x_2^2 + (x_3 - x_2)^2 + \cdots + (x_{10} - x_9)^2 + x_{10}^2)
\]
So let \( x = (1, -1, 0, \ldots) \). Then
\[
\frac{x^T B x}{x^T x} = \frac{5}{2} \leq \max_{x \in D} \frac{x^T B x}{x^T x}
\]
which implies \( \lambda_{\text{max}} > 1 \).

Problem 4.d
So if we have the eigenvalues of \( B \), \( \lambda_n, \lambda_{n-1}, \ldots, \lambda_1 \) such that \( \lambda_{i+1} \geq \lambda_i \) for all \( i \). Then
\[
\lambda_{n-1} = \min_{\dim(S) = 9} \max_{x \in S} \frac{x^T B x}{x^T x}
\]
notice that \( V = \{x \in \mathbb{R}^{10} : x_1 = 0\} \) is a vector space of dimension 9. Hence
\[
\lambda_{n-1} = \min_{\dim(S) = 9} \max_{x \in S} \frac{x^T B x}{x^T x} \leq \max_{x \in V} \frac{x^T B x}{x^T x} = -\frac{(x_2^2 + (x_3 - x_2)^2 + \cdots + x_{10}^2)}{|x|^2} < 0
\]
which implies that \( B \) has 9 negative eigenvalues.

Sept 1994 Advanced Calculus

Problem 1.a
Using L’Hospital’s rule we have
\[
\lim_{x \to \infty} x \sqrt{x^2 + 1 - x^2} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2} - 1}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{1}{2}\left(1 + \frac{1}{x^2}\right)^{-1/2}(-2x^{-3})}{-2x^{-3}} = \lim_{x \to \infty} \frac{1}{2} \left(1 + \frac{1}{x^2}\right)^{-1/2} = \frac{1}{2}
\]
Problem 1.b

Using L’Hospital’s rule twice we have

\[
\lim_{x \to \pi^-} \frac{\ln(\pi - x)}{\tan(x/2)} = \lim_{x \to \pi^-} \frac{-1}{\frac{x - \pi}{2} \sec^2(x/2)} = \lim_{x \to \pi^-} -\frac{2}{\pi - x} \cos^2(x/2) = \lim_{x \to \pi^-} \frac{-2 \cos(x/2) \sin(x/2)}{-1} = 0
\]

Problem 1.c

Notice we have

\[
\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \sum_{i=0}^{\lfloor \log_2 n \rfloor - 1} \sum_{j=0}^{2^i - 1} \frac{1}{\sqrt{2^i + j}} \leq \sum_{i=0}^{\lfloor \log_2 n \rfloor - 1} 2^i \frac{1}{\sqrt{2^i}} \leq \log n 2^{\frac{\log n}{2}} = \sqrt{n} \log n
\]

Hence

\[
\lim_{n \to \infty} n^{-3/2} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq \lim_{n \to \infty} n^{-3/2} \sqrt{n} \log n \to 0
\]

Problem 2.a

Let \( f_k(x) = \prod_{n=1}^{k} g(x/2^n) \). Then

\[
\log(f_k(x)) = \log \left( \prod_{n=1}^{k} g(x/2^n) \right)
\]

since the product if finite, we know its differentiable, and so

\[
\frac{1}{f_k(x)} f'_k(x) = \sum_{n=1}^{k} \frac{g'(x/2^n)}{g(x/2^n)} \frac{1}{2^n}
\]

which implies

\[
f'_k(x) = f_k(x) \sum_{n=1}^{k} \frac{g'(x/2^n)}{g(x/2^n)} \frac{1}{2^n}
\]

Thus

\[
\lim_{k \to \infty} f'_k(0) = f'(0) = \lim_{k \to \infty} f_k(0) \sum_{n=1}^{k} \frac{g'(0)}{g(0)} \frac{1}{2^n} = \sum_{n=1}^{\infty} 3 \frac{1}{2^n} = 3
\]
Problem 3.a
notice for $\theta < \pi/2$, we have $\sin \theta \leq \theta$. Hence
\[
\sum_{n=1}^{\infty} \sin(1/n^2) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
\]
Hence it converges.

Problem 3.b
Notice
\[
\lim_{n \to \infty} \frac{(n+1)^n}{(n+1)!} \frac{n!}{n^n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e > 1
\]
Hence the summation diverges.

Problem 3.c
Recall the Dirichlet’s Test: Given that $|\sum_{n=1}^{p} a_n| < K$ where $K$ and $p$ are independent, and if $b_n \geq b_{n+1} > 0$ for all $n$ and $\lim_n b_n = 0$, then we have
\[
\sum_{n=1}^{\infty} a_n f_n < \infty
\]
So let $a_n = \cos(\log n)$. Then notice
\[
\sum_{n=2}^{k} = \Re \sum_{n=2}^{k} e^{i \log n} = \sum_{n=2}^{k} \left(e^{\log n}\right)^i = \sum_{n=2}^{k} n^i = \sum_{n=2}^{k} \frac{1}{n^{-i}} \leq \zeta(-i)
\]
Thus it is bounded. and clearly we have $b_n = \frac{1}{n \log n}$ \(\searrow 0\) and so by the Dirichlet Test,
\[
\sum_{n=2}^{\infty} \frac{\cos(\log n)}{n \log n}
\]
This completes the proof. Also recall for future reference Abel’s Test: Given that $\sum a_n$ converges and $b_n(x) \searrow$ something and if $b_n(x) \leq M$ for $x \in A$, then
\[
\sum a_n b_n(x)
\]
converges uniformly for $x \in A$. 

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Problem 4

Recall that **Dominated Convergence Theorem**: Given the $f_n$ is a sequence of measurable functions and $f_n \to f$ pointwise and $\exists g$ such that $|f_n| < g$ where $\int g < \infty$, then

$$\lim_{n \to \infty} \int f_n = \int f$$

Let's first start by change of variables: Let $u = x/\sqrt{\epsilon}$ and $v = y/\sqrt{\epsilon}$. Then taking the Jacobian we have

$$dxdy = \begin{vmatrix} \sqrt{\epsilon} & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} \end{vmatrix} dudv = \epsilon dudv$$

Hence

$$\lim_{\epsilon \to 0} \frac{1}{2\pi\epsilon} \int \int f(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \frac{\sqrt{\epsilon}v}{\epsilon} e^{-(u^2+v^2)/2} \epsilon dudv = \frac{1}{2\pi} \int \int f(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \frac{v}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2} dvdu$$

Now we integrate by parts over $v$.

$$f(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \quad -\frac{1}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2}$$

$$\sqrt{\epsilon} f_v(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \quad \frac{v}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2}$$

Thus we have

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int \left[ \frac{-1}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2} f(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \right]^{\infty}_{-\infty} - \int f_g(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \frac{-1}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2} dvdu$$

notice that $\left[ \frac{-1}{\sqrt{\epsilon}} e^{-(u^2+v^2)/2} f(\sqrt{\epsilon}u, \sqrt{\epsilon}v) \right]^{\infty}_{-\infty} \to 0$. Hence

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int \int f_g(\sqrt{\epsilon}u, \sqrt{\epsilon}v) e^{-(u^2+v^2)/2} dvdu$$

now by applying the dominated convergence theorem, we have

$$= \frac{1}{2\pi} \int \int f_g(0,0) e^{-(u^2+v^2)/2} dvdu = f_g(0,0)$$

**Sept 1994 Complex Variables**

**Problem 1.a**

Notice

$$1 - z - z^2 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{5}}{2}$$

Hence our radius of convergence is $R = \frac{-1 + \sqrt{5}}{2}$. 

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Problem 1.b

Thus we have

\[ f(z) = \frac{1}{1 - z - z^2} = 1 + c_1 z + c_2 z^2 + \cdots \]

which implies

\[ (1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots)(1 - z - z^2) = 1 \]

Notice we have \( c_0 = 1 \). By the equation above we have \( c_1 z - z = 0 \Rightarrow c_1 = 1 \) and \( c_2 z^2 - z^2 - z^2 = 0 \Rightarrow c_2 = 2 \). Hence \( c_2 = c_1 + c_0 \). In general one can see that by multiplication

\[ c_{n+2} z^{n+2} - c_{n+1} z^{n+2} - c_n z^{n+2} = 0 \Rightarrow c_{n+2} = c_{n+1} + c_n \]

Problem 2.a

It’s the number of zeros of \( f \) minus the number of poles of \( f \) inside \( |z| < r \).

Problem 2.b

indent for large enough \( n \) it’s zero. Let \( g(z) = f(1/z) = 1 + z + \frac{z^2}{1!} + \cdots + \frac{z^n}{n!} \). Now clearly \( z_0 = re^{i\theta} \) is a zero of \( f \) if and only if \( (1/r)e^{-i\theta} \) is a zero of \( g \). So we will show that all the zeros of \( g \) lie outside the circle with radius \( \sqrt{n} \) for all \( n \). Notice by Rouche we have

\[ |g(z) - e^z| = \left| \frac{z^{n+1}}{(n+1)!} + \frac{z^{n+2}}{(n+2)!} + \cdots \right| \leq \frac{(\sqrt{n})^{n+1}}{(n+1)!} + \frac{(\sqrt{n})^{n+2}}{(n+2)!} + \cdots < \frac{(\sqrt{n})^n}{n!} \]

We can see the last inequality since

\[ \frac{n!}{(\sqrt{n})^n} \left( \frac{(\sqrt{n})^{n+1}}{(n+1)!} + \frac{(\sqrt{n})^{n+2}}{(n+2)!} + \cdots \right) = \frac{n}{n + 1} + \frac{(\sqrt{n})^2}{(n + 1)(n + 2)} + \cdots < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 \]

Now since

\[ \lim_{n \to \infty} \frac{\sqrt{n}}{(n!)^{1/n}} \to 0 \]

We have

\[ \frac{\sqrt{n}}{(n!)^{1/n}} + \frac{1}{\sqrt{n}} \leq 1 \]

for large enough \( n \). Hence

\[ \frac{\sqrt{n}}{(n!)^{1/n}} \leq 1 - \frac{1}{\sqrt{n}} = 1 - \frac{\sqrt{n}}{n} \]

which implies
\[
\left(\frac{\sqrt{n}}{n!}\right)^n \leq \left(1 - \frac{\sqrt{n}}{n}\right)^n \leq e^{-\sqrt{n}}
\]

Thus

\[|g(z) - e^z| < |e^{-\sqrt{n}}| \leq |e^z|
\]

for \(|z| = \sqrt{n}\). Hence there are no zeros inside the circle \(|z| = \sqrt{n}\) which implies all zeros of \(f\) are inside the circle \(|z| = 1/\sqrt{n}\). SO for any \(r > 0\), \(\exists N\) such that for \(n > N\) \(f\) has all zeros inside \(|z| = r\).

**Problem 2.b again**

Here is the Workshop and the better method. Now let

\[f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}
\]

Now I claim that \(f_n \to e^{1/z}\) uniformly on \(D = \{z : |z| \geq r/2\}\). This is clear since \(f_n\) is bounded for all \(n\) on \(D\). So this implies that \(f_n' \to (e^{1/z})'\). Therefore

\[\frac{f_n'(z)}{f_n(z)} \to -\frac{1}{z^2}
\]

and

\[
\lim_{n \to \infty} \int_{|z|=r} \frac{f_n'(z)}{f_n(z)} \, dz = \int_{|z|=r} \frac{-1}{z^2} \, dz = 0
\]

This implies that there must be \(n\) zeros inside \(|z| = r\) for any \(r\). Check out Ahlfors p.177.

**Problem 2.c**

All the zero’s are approaching the origin.

**Problem 3.a**

We have

\[p(z) = a(z - z_1)(z - z_2)\cdots(z - z_d)
\]

Then

\[\frac{1}{p(z)} = \frac{c_1}{z - z_1} + \frac{c_2}{z - z_2} + \cdots + \frac{c_d}{z - z_d}
\]

Then

\[c_n = \lim_{z \to z_n} \frac{z - z_n}{p(z)} = \frac{1}{p'(z_n)}
\]
Problem 3.b

We have
\[
\frac{1}{p(z)} = \frac{1}{a(z - z_1)(z - z_2) \cdots (z - z_n)}
\]
where \( a \in \mathbb{C} \). Then
\[
\frac{1}{p(z)} = \left( \frac{1}{z - z_1} \right) \left( \frac{1}{a(z - z_2) \cdots (z - z_n)} \right)
\]
now
\[
\left( \frac{1}{a(z - z_2) \cdots (z - z_n)} \right)
\]
is analytic inside \( B(z_1, \epsilon) \) for some \( \epsilon > 0 \). Hence it has a Taylor expansion
\[
\left( \frac{1}{a(z - z_1)(z - z_2) \cdots (z - z_n)} \right) = \left( \frac{1}{z - z_1} \right) (a_{1,0} + a_{1,1}(z - z_1) + a_{1,2}(z - z_2) + \cdots)
\]
therefore
\[
\frac{1}{p(z)} = \frac{a_{1,0}}{z - z_1} + a_{1,1} + a_{1,2}(z - z_1) + \cdots
\]

So let \( H_1(z) = a_{1,1} + a_{1,2}(z - z_1) + \cdots \), which is valid only when \( z \) in \( B(z_1, \epsilon) \). Notice \( H(z_1) = a_{1,1} < \infty \). So lets look at
\[
\frac{1}{p(z)} - \sum_{j=1}^d \frac{a_{j,0}}{z - z_j}
\]
Now the only poles in the equation above is a rational polynomial, and can only be \( z_1, \ldots, z_d, \infty \). Notice however at \( z = \infty \) we have
\[
\frac{1}{p(\infty)} - \sum_{j=1}^d \frac{a_{j,0}}{\infty - z_j} = 0 < \infty
\]
and
\[
\frac{1}{p(z_k)} - \sum_{j=1}^d \frac{a_{j,0}}{z_k - z_j} = - \sum_{j=1, j \neq k}^d \frac{a_{j,0}}{z - z_j} + H_k(z_k) < \infty
\]
Hence we have a rational polynomial with no poles, and hence it must be a constant. Thus
\[
\frac{1}{p(z)} = c + \sum_{j=1}^d \frac{a_{j,0}}{z - z_j}
\]
Problem 3.c

\[ \frac{1}{p(z)} = \sum_{n=3}^{d} \frac{c_n}{z-z_n} + \frac{c_1}{z-z_1} + \frac{c_2}{(z-z_1)^2} \]

Problem 5.c

By performing \( u \) substitution, we can see that

\[ \int_{0}^{\infty} \frac{\sin x}{\sqrt{x}} \, dx = 2 \int_{0}^{\infty} \sin x \, dx \]

Thus let \( f(z) = e^{iz^2} \), and let \( \gamma \) be the curve along \([0,R], Re^{it}\) for \( 0 \leq t \leq \pi/4 \), and \( te^{i\pi/4} \) for \( 0 \leq t \leq R \). Thus by the residue theorem we have

\[ \int_{\gamma} f(z) \, dz = \int_{0}^{R} e^{iz^2} \, dz + \int_{0}^{\pi/4} iRe^{i\theta} e^{i(RE^{i\theta})^2} \, d\theta + \int_{R}^{0} e^{i(Re^{i\pi/4})^2} e^{i\pi/4} \, dt \]

Now notice

\[ \left| \int_{0}^{\pi/4} iRe^{i\theta} e^{i(RE^{i\theta})^2} \, d\theta \right| \leq \int_{0}^{\pi/4} \left| Re^{iR^2e^{2\theta}} \right| \, d\theta = \int_{0}^{\pi/4} Re^{-R^2\sin(2\theta)} \, d\theta \]

Now recall that for \( 0 < \theta < \pi/2 \) we have \( 2\theta/\pi \leq \sin \theta \). Thus our situation we have \( 0 < 2\theta < \pi/2 \) \( \Rightarrow 4\theta/\pi \leq \sin 2\theta \). Hence

\[ \int_{0}^{\pi/4} Re^{-R^24\theta/\pi} \, d\theta = \frac{\pi}{4R} \left( e^{-R^2} - 1 \right) \to 0 \]

as \( R \to \infty \). Hence we have

\[ \int_{0}^{R} e^{iz^2} \, dz + \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \int_{R}^{0} e^{-z^2} \, dz = 0 \]

now recall that \( \int_{0}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}/2 \). Thus as \( R \to \infty \) we have

\[ \int_{0}^{\infty} \cos z^2 + i \sin z^2 \, dz = \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \frac{\sqrt{\pi}}{2} \]

Which implies that

\[ \int_{0}^{\infty} \sin z^2 \, dz = \frac{\sqrt{2\pi}}{4} \]

Hence

\[ \int_{0}^{\infty} \frac{\sin x}{\sqrt{x}} \, dx = \frac{\sqrt{2\pi}}{2} \]
Sept 1994 Linear Algebra

Problem 1.a

We have $x = r \cos \theta_0$ and $y = r \sin \theta_0$. We want

$$T_\theta \begin{pmatrix} r \cos \theta_0 \\ r \sin \theta_0 \end{pmatrix} = \begin{pmatrix} r \cos(\theta + \theta_0) \\ r \sin(\theta + \theta_0) \end{pmatrix}$$

Hence

$$T_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and clearly

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 1.b

Just by matrix multiplication, we have

$$VT_\theta U = T_\eta = \begin{pmatrix} -\sin \theta & \cos \theta \\ -\cos \theta & -\sin \theta \end{pmatrix}$$

since

$$T_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

we have $\eta = \theta - \pi/2$

Problem 2

$A$ is $n \times n$ real matrix. Since $A$ has $n$ distinct eigenvalues, $A$ is diagonalizable.

$$A = M D M^{-1}$$

where

$$M = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & & \\ & \lambda_2 & \cdots \\ & & \lambda_n \end{pmatrix}$$

Hence

$$A^k = M \begin{pmatrix} 1 & & \\ & \lambda_2^k & \cdots \\ & & \lambda_n^k \end{pmatrix}$$
note that bad notation in the exam. Thus
\[ \lim_{k \to \infty} A_k = M \begin{pmatrix} 1 & 0 & \cdots & 0 \\ & & & \\ & & & \\ 0 & & & \end{pmatrix} \]
\[ M^{-1} = \begin{pmatrix} \alpha_1 v_1 & \alpha_2 v_1 & \cdots & \alpha_n v_1 \end{pmatrix} \]
Thus dim ker $T = n - 1$ and dim(Im($T$)) = 1. The basis of the Image is \{v_1\} and the bases of the kernel is
\[ \{(-\alpha_2, \alpha_1, 0, \ldots, 0), (0, -\alpha_3, \alpha_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, -\alpha_n, \alpha_{n-1})\} \]
where $\alpha_i$ are the elements in the first row of $M^{-1}$.

**Problem 3.a**

Let $v_1 = 1$ and $v_2 = \sqrt{3}x$. Then notice
\[ \frac{1}{2} \int_{-1}^{1} \sqrt{3}xdx = 0 \]
\[ \frac{1}{2} \int_{-1}^{1} dx = 1 \]
\[ \frac{1}{2} \int_{-1}^{1} 3x^2 dx = 1 \]
Hence so far we have an orthonormal basis. So to figure out the last polynomial of degree 2, we have
\[ a + bx + cx^2 \]
we want
\[ \frac{1}{2} \int_{-1}^{1} a + bx + cx^2 dx = 0 \Rightarrow a + c/3 = 0 \]
We also want
\[ \frac{1}{2} \int_{-1}^{1} \sqrt{3}x(a + bx + cx^2) dx = 0 \Rightarrow b = 0 \]
Finally we want
\[ \frac{1}{2} \int_{-1}^{1} (a + cx^2)^2 dx = 1 \Rightarrow a^2 + 2ac/3 + c^2/5 = 1 \]
Thus we have $c = 3\sqrt{5}/2$ and $a = -\sqrt{5}/2$. Therefore
\[ v_3 = -\frac{\sqrt{5}}{2} + \frac{3\sqrt{5}}{2} x^2 \]
and $v_1, v_2, v_3$ is our orthonormal basis $P_2$. 

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Problem 3.b

In order to minimize
\[ \int_{-1}^{1} (p(x) - x^3)^2 \, dx \]
we need to find the projection vector (polynomial) \( p(x) \in P_2 \) of \( x^3 \). Notice we do have a basis for \( P_3, v_1, v_2, v_3, x^3 \). Now by applying Gram-Schmidt, we have the orthogonal basis \( u_1, u_2, u_3, u_4 \) where
\[ u_1 = v_1, u_2 = v_2, u_3 = v_3 \]
and
\[ u_4 = x^3 - (v_1, x^3)v_1 - (v_2, x^3)v_2 - (v_3, x^3)v_3 \]
Notice that \( (v_1, x^3) = (v_3, x^3) = 0 \). Hence
\[ (v_2, x^3) = \frac{\sqrt{3}}{2} \int_{-1}^{1} x^4 \, dx = \frac{\sqrt{3}}{5} \]
Therefore \( u_4 = x^3 - (3/5)x \). Hence \( p(x) = (3/5)x \).

Problem 4.a

Notice that functions in \( V_{n,r} \) are essentially functions of the form \( p(x)e^{rx} \), where \( p(x) \) is a polynomials of degree at most \( n \). Hence
\[ Tf = p(x)e^{rx} + 2(p'(x)e^{rx} + rp(x)e^{rx}) = e^{rx}(p(x) + 2p'(x) + 2tp(x)) = e^{rx}q(x) \]
where \( q(x) \) is a polynomial of degree at most \( n \). Hence \( T(f) \in V_{n,r} \).

Problem 4.b

For \( n = 4 \), we have the basis \( \{e^{rx}, xe^{rx}, x^2e^{rx}, x^3e^{rx}, x^4e^{rx}\} \). So for \( f = p(x)e^{rx} \), we have
\[ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \]
which implies
\[ T(f) = (1 + 2r) [a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4] + 2(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3) \]
Hence our transformation matrix \( T \) is
\[
T = \begin{pmatrix}
(1 + 2r) & 2 & 0 & 0 & 0 \\
0 & (1 + 2r) & 4 & 0 & 0 \\
0 & 0 & (1 + 2r) & 6 & 0 \\
0 & 0 & 0 & (1 + 2r) & 8 \\
0 & 0 & 0 & 0 & (1 + 2r)
\end{pmatrix}
\]
Problem 4.c

(⇒) We are given the $T : V_{n,r} \mapsto V_{n,r}$ onto. Assume that $r = 1/2$. Then for $x^n e^{-x/2}$, there should exist a polynomial $p$ of degree at most $n$ such that

$$2p'(x) = x^n$$

However this is impossible, and hence we have a contradiction.

(⇒) We are given that $r = -1/2$. Then notice

$$T(f) = e^{-x/2}(2p'(x))$$

where $p$ is a polynomial of degree at most $n$. Now I claim that for any $q(x)e^{-x/2} \in V_{n,-1/2}$, $\exists$ a polynomial $p$ such that

$$T(p(x)e^{-x/2}) = q(x)e^{-x/2}$$

To see this, let

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

and

$$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$$

Thus by letting $a_n = b_n/(1 + 2r)$, and solving

$$b_{n-1} = (1 + 2r)a_{n-1}2na_n$$

$$
\vdots
$$

$$b_1 = (1 + 2r)a_1 + 4a_2$$

$$b_0 = (1 + 2r)a_0 + 2a_1$$

Then we have our polynomial.

Problem 4.d

For $r = -1/2$, we know that

$$T(p(x)e^{-x/2}) = e^{-x/2}(2p'(x))$$

Clearly $T : V_{n+1,r} \mapsto V_{n,r}$. To see that this is onto, let $q(x)$ be some polynomial of degree $n$, and $p(x)$ be a polynomial of degree of at most $n + 1$. Then let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n+1} x^{n+1}$$
and

\[ q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \]

So for \( q(x)e^{-x/2} \in V_{n,r} \), we just let

\[ a_0 = a_0, \quad a_1 = b_0/2, \quad a_2 = b_1/4, \ldots, a_{n+1} b_n/(n+1) \]

### Problem 5

We will first show that

\[ \max_{V, \dim(V) = 2} \min_{x \in V - \{0\}} \frac{T x}{|x|} \leq \lambda_2 \]

where \( V \) is 2 dimensional. Notice for \( V = S(v_1, v_2) \) \( \exists w \in V \) such that \( w \perp x_1 \) where \( x_1, \ldots, x_n \) are the eigenvectors of \( T \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Indeed since \( v_1 \perp x_1 \), then let \( w = v_1 \). Else let

\[ w = v_1 - \frac{v_1 \cdot x_1}{v_2 \cdot x_1} v_2 \in V \]

Either way we have \( w \perp x_1 \). Now since \( T \) is positive semi-definite symmetric matrix, \( w = c_2 x_2 + \cdots + c_n x_n \). Hence

\[ \frac{|T w|}{|x|} = \sqrt{\frac{c_2^2 \lambda_2^2 + \cdots + c_n^2 \lambda_n^2}{c_2^2 + \cdots + c_n^2}} \leq \lambda_2 \]

Hence all \( V \) that is two dimensional has such a vector implies

\[ \max_{V} \min_{x \in V - \{0\}} \frac{|T x|}{|x|} \leq \lambda_2 \]

Now we want to show that

\[ \max_{V} \min_{x \in V - \{0\}} \frac{|T x|}{|x|} \geq \lambda_2 \]

Just let \( V = \{x_1, x_2\} \). Then

\[ \min_{x \in V} \frac{|T x|}{|x|} = \lambda_2 \]

Thus

\[ \max_{x \in V - \{0\}} \frac{|T x|}{|x|} \geq \lambda_2 \]

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Jan 1995 Advanced Calculus

Problem 1

Here we have $-1 \leq z \leq 1$, $\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$, and $\sqrt{1-z^2} \leq z \leq \sqrt{1-z^2}$. Thus our volume $V$ of the intersection of the two cylinders is

$$V = \int_{z=-1}^{1} \int_{y=-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{x=-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dx dy dz = \frac{16}{3}$$

Problem 2.a

Notice

$$\lim_{x \to 0} \frac{x^x}{x} = \lim_{x \to 0} x^{x-1}$$

So notice that

$$\lim_{x \to 0} \log (x^{x-1}) = \lim_{x \to 0} (x^x - 1) \log x = \lim_{x \to 0} \frac{\log x}{(x^x - 1)^{-1}}$$

Since $\lim_{x \to 0} x^x = 1$, we can apply L’Hôpital’s rule, and have

$$= \lim_{x \to 0} \frac{1}{x} \frac{-(x^x - 1)^2}{x^x(x \log x + x)} = \lim_{x \to 0} -\frac{(x^x - 1)^2}{x^x(x \log x + x)}$$

Since $\lim_{x \to 0} x \log x = 0$, we can apply L’Hôpital’s rule again and have

$$= \lim_{x \to 0} \frac{-2(x^x - 1)x^x(\log x + 1)}{x^x(\log x + 1) + (x^x(\log x + 1)^2 + x^x - 1)x} = \lim_{x \to 0} \frac{-2(x^x - 1)}{1 + x(\log x + 1) + \frac{\log x}{\log x}} = 0/1 = 0$$

Thus

$$\lim_{x \to 0} \frac{x^x}{x} = 1$$

Problem 2.b

$$\lim_{n \to \infty} \frac{1 + \cos(x/n) + \cos(2x/n) + \cdots + \cos((n-1)x/n)}{n} = \text{Re} \left( \lim_{n \to \infty} \frac{1 + e^{ix/n} + e^{i2x/n} + \cdots + e^{i(n-1)x/n}}{n} \right)$$

$$= \text{Re} \left( \lim_{n \to \infty} \frac{1 - e^{ix}/n}{n} \right) = \text{Re} \left( \lim_{n \to \infty} \frac{n^2(1 - e^{ix})}{e^{i2x/n}(-i\pi n^2)} \right) = \text{Re} \left( \frac{-1 - e^{ix}}{ix} \right) = \frac{\sin x}{x}$$

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Problem 3.a

Notice that

\[
\frac{d(F, G)}{d(x, y)} = \begin{vmatrix} 1 & 0 \\ 0 & a \end{vmatrix} = a \neq 0
\]

So by the implicit function theorem, there exists continuous functions \( p \) and \( q \) such that \( x = p(z) \) and \( y = q(z) \) for \( z \in B(0, \epsilon) \). Hence \( z = s \in B(0, \epsilon) \) and \( x = x(s), y = y(s), z = s, \) and \( dz/ds = 1 \).

Problem 3.b

\( f(x(s), y(s), s) = 0 \) for all \( s \in B(0, \epsilon) \). So for a small neighborhood in 0, we have

\[
x + y^2 + z^3 = 0 \quad \text{and} \quad ay + x^3 + z^2 = 0
\]

Then the first tangent plane is \( x = 0 \) and the second is \( ay = 0 \). Thus the tangent vector is

\[
\begin{vmatrix} 1 & 0 & 0 \\ 0 & a & 0 \end{vmatrix} = (0, 0, a)
\]

the so the unit tangent vector is \( (0, 0, 1) \).

Problem 3.c

Now for \( a = 0 \), we have

\[
F(x, y, z) = x + y^2 + z^3 \quad \text{and} \quad G(x, y, z) = z^2 + x^3
\]

which is 0 at \( x = y = z = 0 \). Now notice by algebra \( x = -z^{2/3} \) and \( y = \pm \sqrt[2]{z^{2/3} - z^3} \), which is a curve but not unique since \( y = \pm \sqrt[2]{z^{2/3} - z^3} \). Also notice that \( dx/dz \) is not defined at \( z = 0 \).

Hence there is no tangent vector at \( z = 0 \).

Problem 4

Here we let \( f(x, y, z) = 8xyz \), and \( g(x, y, z) = (x/a)^2 + (y/b)^2 + (z/c)^2 \). Using the lagrange multiplier theorem, we have \( \nabla f = \lambda \nabla g \):

\[
(8yz, 8xz, 8xy) = \lambda \left( \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)
\]

which implies

\[
\frac{yz a^2}{x} = \frac{x z b^2}{y} = \frac{y c^2}{z}
\]

which implies \( x = \frac{a}{\sqrt[3]{3}}, y = \frac{b}{\sqrt[3]{3}}, \) and \( z = \frac{c}{\sqrt[3]{3}} \). Thus the maximum volume inside the ellipsoid is

\[
f(x, y, z) = \frac{8abc}{3\sqrt[3]{3}}
\]

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Problem 5
Recall that
\[ \Delta(\phi) = \nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2} \]
Thus
\[ \Delta(\phi^2) = \nabla \cdot \nabla \phi^2 = \nabla \cdot \left( 2\phi \frac{d\phi}{dx}, 2\phi \frac{d\phi}{dy}, 2\phi \frac{d\phi}{dz} \right) \]
\[ = \left( 2 \left( \frac{d\phi}{dx} \right)^2 + 2\phi \frac{d^2 \phi}{dx^2}, 2 \left( \frac{d\phi}{dy} \right)^2 + 2\phi \frac{d^2 \phi}{dy^2}, 2 \left( \frac{d\phi}{dz} \right)^2 + 2\phi \frac{d^2 \phi}{dz^2} \right) = \phi^2 \Delta \phi + 2 |\nabla \phi|^2 \]
Hence
\[ \Delta(\phi^2) = \phi^2 \Delta \phi + 2 |\nabla \phi|^2 = 2\phi \Delta \phi + 2(4\phi) = 20\phi \]
Hence \( \phi \Delta \phi = 6\phi \). Since \( \phi \) is a strictly positive number, we have
\[ \Delta \phi = \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2} = 6 \]
Now notice by the Divergence theorem,
\[ \int_S \frac{d\phi}{dn} dS = \int_S \nabla \phi \cdot ndS = \int \int \int_V \nabla \cdot \nabla \phi dxdydz \]
\[ = \int \int \int_V \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} + \frac{d^2 \phi}{dz^2} dxdydz = \int \int \int_V 6dxdydz = \frac{4}{3} \pi = 8\pi \]
Jan 1995 Complex Variables
Problem 1
Notice
\[ \sum_{n=0}^{\infty} \cos(\alpha \sqrt{1+n^2})z^n = \frac{1}{2} \sum_{n=0}^{\infty} e^{i\alpha \sqrt{1+n^2}}z^n + \frac{1}{2} \sum_{n=0}^{\infty} e^{-i\alpha \sqrt{1+n^2}}z^n \]
Now if \( \alpha \) is real, then clearly the radius of converges of each is 1 and so the radius of convergence is 1. Now if \( \alpha = a + bi \), then
\[ \frac{1}{R_1} = \lim_{n \to \infty} \left(e^{-b\sqrt{1+n^2}}\right)^{1/n} \to e^{-b} \]
Hence the radius of convergence is \( \min(e^b, e^{-b}) \).
Problem 2

Assume $\xi \geq 0$. Then let

$$f(z) = \frac{e^{i\xi z}}{z^2 - 2z + 2} = \frac{e^{i\xi z}}{(z - (1 + i))(z - (1 - i))}$$

Let $\gamma_1$ be the contour of a semi-circle with radius $R$ in the LHP. Then by the residue theorem we have

$$\int_{\gamma_1} f(z)\,dz = 2\pi i \frac{e^{-i\xi(1-i)}}{-2i} = -\pi e^{-i\xi-\xi}$$

So

$$\int_{\gamma_1} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2 - 2x + 2} \,dx + \int_{C_R} f(z)\,dz = -\pi e^{-i\xi-\xi}$$

And notice

$$\left| \int_{C_R} f(z)\,dz \right| \leq \pi R \frac{e^{-i\xi(R \cos \theta + iR \sin \theta)}}{R^2 - 2R - 2} = \pi Re^{\xi R \sin \theta} \frac{R}{R^2 - 2R - 2} \to 0$$

as $R \to \infty$. Since $\pi \leq \theta \leq 2\pi$, we have $\sin \theta \leq 0$. So

Hence

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2 - 2x + 2} \,dx = \pi e^{-i\xi-\xi}$$

Now if $\xi < 0$ then let $\gamma_2$ be the semicircle of radius $R$ in the UHP. Then by the residue theorem, we have

$$\int_{\gamma_2} f(z)\,dz = 2\pi i \frac{e^{-i\xi(1+i)}}{2i} = \pi e^{i\xi+\xi}$$

and

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2 - 2x + 2} \,dx + \int_{C_R} f(z)\,dz$$

But notice

$$\left| \int_{C_R} \frac{e^{-i\xi z}}{z^2 - 2z + 2} \,dz \right| \leq \frac{\pi Re^{\xi R \sin \theta}}{R^2 - 2R - 2} \to 0$$

Since $0 \leq \theta \leq \pi$ since $\sin \theta \geq 0$. Hence for this case we have

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2 - 2x + 2} \,dx = \pi e^{-i\xi+\xi}$$

Problem 3.a

$$e^z = -2 \Rightarrow z = \log 2 + i(\pi + 2\pi k)$$

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Problem 3.b

Let $C_R = \{z : |z| = R\}$. Then let $\gamma$ be the square contour inside $C_R$. Then we partition $f(\gamma)$ by $\gamma_1, \ldots, \gamma_{R/2\pi}$, which are rectangular contours as described in the hint. Then as we traverse $\gamma_n$ from $-R + 2\pi ni$ to $R + 2\pi ni$, notice

$$f(z) = e^z + z = e^{x+2\pi ni} + x + 2\pi ni = e^x + x + i2\pi n$$

as $R \to \infty$, the argument changes about $-\pi$. Now as we traverse $R + 2\pi ni$ to $R + 2\pi(n + 1)i$ we have

$$f(z) = e^{R+2\pi(n+y)i} + R + 2\pi(n + y)i$$

for $y : 0 \to 1$. Then

$$= e^{Re^{2\pi ni + 2\pi yi}} \left( 1 + \frac{R + \pi(n + y)i}{e^{Re^{2\pi ni + 2\pi yi}}} \right)$$

so as $R \to \infty$, $\arg(f) \sim e^{2\pi ni + 2\pi yi}$. So the argument changes about $2\pi$. Now when we traverse from $R + 2\pi(n+1)i$ to $-R + 2\pi(n + 1)i$ we have

$$f = e^{x+2\pi(n+1)i} - x + 2\pi(n + 1)i = e^x + x + i(2\pi(n + 1))$$

Then the argument changes about $\pi$. Then from $-R + 2\pi(n + 1)i$ to $-R + 2\pi ni$, we have

$$f = e^{-R+2\pi(n+y)i} + -R + 2\pi(n + y)i$$

as $R \to \infty$, the argument does not change. Hence the total change in the argument is $\pi - \pi + 2\pi = 2\pi$. By the argument theorem, the number of zeros inside $\gamma_n$ is 1. So there are about $R/\pi$ zeros inside the circle $C_R$ for very large $R$. This also show that there are an infinite number of zeros’ in $\mathbb{C}$.

Problem 4.a

Since $f$ is an entire function, then $h(z) = e^{i f(z)}$ is an entire function as well. Notice for $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ we have

$$|h(z)| = |e^{i(u(x,y)+iv(x,y))}| = |e^{-v(x,y)}| < 1$$

Hence by Liouville’s Theorem, $h$ is constant. Thus $e^{i f(z)} = c$, which implies

$$i f(z) = \log c + i(\arg(c) + 2\pi n)$$

hence since $f$ is continuous

$$f(z) = \arg c + 2\pi n_0 - i \log c$$

for some constant $n_0$. Therefore $f(z)$ is indeed constant.
Problem 4.b

Since $u(x, y)$ is harmonic, $\exists$ an analytic function $g$ such that $\text{Re}(g) = u$. Then we know that the linear fractional transformation $\phi(z) = \frac{x - 1}{x + 1}$ maps the right-half plane to the unit circle (one to one and analytic). Thus $h = \phi \circ g$ is an entire function such that $|h(z)| \leq 1$. By Liouville’s Theorem, $h = \phi \circ g = c$ for some constant $c$. Thus $g = \phi^{-1}(c)$ which is constant. Clearly $\text{Re}(g) = u$ is constant.

Problem 5.a

Notice for $z \in \Delta(0, 1)$, we have

$$w = x + iy + \frac{1}{x + iy} = x + iy + \frac{x - iy}{x^2 + y^2} = \frac{x(x^2 + 1)}{r^2} + i\frac{y(y^2 - 1)}{r^2}$$

So we can see that $w$ is not on the interval $[-2, 2]$ since that would imply either $r^2 = x^2 + y^2 = 1$ (which is not possible), or $y = 0 \Rightarrow x + 1/x \in [-2, 2]$. But that’s not possible since

$$|x + 1/x| \geq 2$$

Now to show that is onto, let $w \in \mathbb{C} \setminus [-2, 2]$. Then notice the pre-image

$$w = z + 1/z \Rightarrow z = \frac{w \pm \sqrt{w^2 - 4}}{2}$$

so if

$$\frac{w + \sqrt{w^2 - 4}}{2} = re^{i\theta}$$

Then

$$\frac{w - \sqrt{w^2 - 4}}{2} = \frac{1}{r}e^{-i\theta}$$

and $r \neq 1$ since that would imply that $w = \pm 2$. Thus one of these is inside the unit circle.

Problem 5.b

Notice

$$w = re^{i\theta} + \frac{1}{r}e^{-i\theta} = r \cos \theta + ir \sin \theta + \frac{1}{r} \cos \theta - \frac{1}{r}i \sin \theta$$

which implies

$$w = \left( r + \frac{1}{r} \right) \cos \theta + i \left( r - \frac{1}{r} \right) \sin \theta = u + iv$$

so in the $u, v$ plane, we have an ellipse with the major axis on the real axis of length $2(r + 1/r)$ and minor axis on the imaginary axis of length $2(r - 1/r)$ centered at 0. And the foci are at

$$\pm \sqrt{(r + 1/r)^2 - (r - 1/r)^2} = \pm \sqrt{4} = \pm 2$$

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Jan 1995 Linear Algebra

Problem 1.a
Using Gaussian elimination, we have

\[
\begin{array}{ccc|c}
1 & 2 & 3 & 1 \\
4 & 5 & 6 & 5 \\
7 & 8 & 9 & 11 \\
\hline
1 & 2 & 3 & 1 \\
0 & -3 & -6 & 1 \\
0 & -6 & -12 & 4 \\
\hline
1 & 2 & 3 & 1 \\
0 & -3 & -6 & 1 \\
0 & 0 & 0 & 2 \\
\end{array}
\]

which implies that 0 = 2. Hence there are no solutions.

Problem 1.b
For \( Ax = Bx \) implies \((A - B)x = 0\). Thus using Gaussian elimination again, we have

\[
\begin{array}{ccc|c}
0 & -2 & -4 & 0 \\
2 & 0 & -2 & 0 \\
4 & 2 & 0 & 0 \\
\hline
2 & 0 & -2 & 0 \\
0 & -2 & -4 & 0 \\
0 & 2 & 4 & 0 \\
\end{array}
\]

Thus we have \( x = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \), for \( t \in \mathbb{R} \).

Problem 2.a
Let

\[
A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}
\]

Then we have

\[
A^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}
\]

Thus \( A^2 \neq 0, B^2 \neq 0, \) and \( C^2 \neq 0, \) but \( AB = BC = CA = 0. \)
Problem 2.b

\[ A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \]

Problem 3

Notice

\[ RH = \begin{pmatrix} -i & 2i & 0 \\ 0 & 0 & -1 + 2i \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} i & 0 & 2 \\ 2i & 0 & -1 \\ 0 & -1 - 2i & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5I \]

Thus

\[ R = i\sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2i}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{i-2}{\sqrt{5}} & 0 \end{pmatrix} \]

Now I claim

\[ U = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2i}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{i-2}{\sqrt{5}} & 0 \end{pmatrix} \]

is unitary. Indeed since

\[ U^H U = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{2-i}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2i}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & \frac{i-2}{\sqrt{5}} \end{pmatrix} = I \]

Also notice \( \lambda = -\sqrt{5}i = -\lambda \).

Problem 4.a

Notice that \( A - B \) is symmetric. Now for eigenvalues \( \lambda_1, ..., \lambda_n \) of \( A - B \). Then \((A - B)^2\) is symmetric with eigenvalues \( \lambda_1^2, \lambda_2^2, ..., \lambda_n^2 \). Hence all eigenvalues are non-negative real, and so \((A - B)^2\) is a positive symmetric semi-definite matrix. So

\[ ((A - B)^2 x, x) \geq 0 \Rightarrow ((A^2 + B^2) x, x) \geq ((AB + BA) x, x) \]

Problem 4.b

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

Then \( A^2 = B^2 = 0, \) But \( AB + BA = I \)
Sept 1995 Advanced Calculus

Problem 1

Notice that $S_1$ converges for any $\alpha > 0$ by the alternating series theorem. Now notice

$$S_1 = \frac{1}{3^\alpha + 1} - \frac{1}{4^\alpha + 1} + \frac{1}{5^\alpha + 1} \pm \cdots$$

and

$$S_2 = \frac{1}{3^\alpha - 1} - \frac{1}{4^\alpha + 1} + \frac{1}{5^\alpha + 1} \pm \cdots$$

Hence

$$S_1 - S_2 = \sum_{n=1}^{\infty} \left( \frac{1}{(2n+1)^\alpha + 1} - \frac{1}{(2n+1)^\alpha - 1} \right)$$

$$= \sum_{n=1}^{\infty} \frac{(2n+1)^\alpha - 1 - (2n+1)^\alpha - 1}{(2n+1)^{2\alpha} - 1} = \sum_{n=1}^{\infty} \frac{-2}{(2n+1)^{2\alpha} - 1} \to \infty$$

for $\alpha \leq 1/2$. Thus $S_2$ diverges for $\alpha = 1/2$ and $S_2$ converges for $\alpha > 1/2$.

Problem 2.a

Let $F_k(x) = \sum_{n=1}^{k} a_n(x)$ and let $F(x) = \lim_{k \to \infty} F_k(x)$. Then $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly on $\mathbb{R}$ if $\forall \epsilon > 0, \exists N$ such that $\forall k > N$ we have

$$|F_k(x) - F(x)| < \epsilon$$

for all $x \in \mathbb{R}$.

Problem 2.b

Assume that the sum does converge uniformly on $\mathbb{R}$. Then $\forall \epsilon > 0 \exists N$ such that

$$\sum_{n=N}^{\infty} \frac{x^2}{n^2 + x^2} < \epsilon$$

for all $x$. So fix such $\epsilon$ and $N$. Then

$$x^2 \sum_{n=N}^{\infty} \frac{1}{n^2 + x^2} < \epsilon$$

Notice

$$\sum_{n=N}^{\infty} \frac{x^2}{n^2 + x^2} = \sum_{n=N}^{\infty} \frac{1}{1 + \left(\frac{n}{x}\right)^2} \geq \int_{N}^{\infty} \frac{1}{1 + \left(\frac{n}{x}\right)^2} dn = x \cdot \arctan(n/x)|_{N}^{\infty} = \pi \cdot x/2 - x \cdot \arctan(N/x)$$

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Hence as we let $x \to \infty$, $\frac{\pi}{2}x - x \cdot \text{arctan}(N/x) \to \infty \not< \epsilon$. Hence we have a contradiction and so the summation does not converge uniformly.

Problem 4

By geometry the area of a pyramid is $(1/3)Abh$. Since the base is an equilateral triangle, we have $A_b = (\sqrt{3}/4)l^2$. Now in order to compute the height, we know that the distance from the centroid of the base triangle to any corner is $l/\sqrt{3}$. Hence by Pythagorean theorem, we have

$$h = \sqrt{l^2 - \frac{l^2}{3}} = \sqrt{\frac{2}{3}}l$$

Hence

$$V = \frac{1}{3} \frac{l^2 \sqrt{3}}{4} \sqrt{\frac{2}{3}}l = \frac{\sqrt{2}}{12}l^3$$

Sept 1995 Complex Variables

Problem 1

First notice that by integration by parts we have

$$\lim_{m \to \infty} \int_0^m \frac{\cos(\alpha x)}{x + \beta} dx = \lim_{m \to \infty} \frac{\sin(\alpha x)}{\alpha(x + \beta)} \bigg|_0^\infty + \int_0^\infty \frac{\sin(\alpha x)}{\alpha(x + \beta)^2} dx < \infty$$

since the integral clearly converges absolutely. Then let $x = \beta t$, and $dx = \beta dt$ so

$$\int_0^\infty \frac{\cos(\alpha x)}{x + \beta} dx = \int_0^\infty \frac{\cos(\alpha \beta t)}{t + 1} dt$$

Now let $\gamma$ be the contour quarter circle in the first quadrant of radius $R$ and $f(z) = e^{i\alpha \beta z}/(z + 1)$. Then by the residue theorem we have

$$\int_\gamma f(z)dz = 0 = \int_0^\infty \frac{e^{i\alpha \beta z}}{z + 1} dz + \int_0^\infty \frac{e^{-\alpha \beta z}}{iz + 1} idz$$

Hence

$$\int_0^\infty \frac{e^{i\alpha \beta z}}{z + 1} dz = \int_0^\infty \frac{e^{-\alpha \beta z}}{iz + 1} idz = \int_0^\infty \frac{e^{-\alpha \beta z}}{iz + 1} \frac{1 - iz}{1 - iz} idz$$

$$= \int_0^\infty \frac{(1 - iz)e^{-\alpha \beta z}}{z^2 + 1} idz = \int_0^\infty \frac{ze^{-\alpha \beta z}}{z^2 + 1} dz + i \int_0^\infty \frac{e^{-\alpha \beta z}}{z^2 + 1} dz$$

Hence

$$\int_0^\infty \frac{\cos(\alpha x)}{x + \beta} dx = \int_0^\infty \frac{te^{-\alpha \beta t}}{t^2 + 1} dt$$
Problem 2

Let $C$ be the contour of two semicircles with radius $R$ and $\epsilon$ and $f(z) = \frac{(\log z)^2}{z^2 + 1}$. By the Residue Theorem, we have

$$\int_C f(z)\,dz = 2\pi i \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}$$

Now notice on $C_R$ we have $z = Re^{i\theta}$. Hence

$$\left|\int_C f(z)\,dz\right| \leq R\pi \frac{(\log R)^2}{R^2 - 1} \to 0$$
as $R \to \infty$. Also notice on $C_\epsilon$, we have

$$\left|\int_C f(z)\,dz\right| \leq \epsilon\pi \frac{(\log(\epsilon))^2}{\epsilon^2 - 1} \to 0$$
as $\epsilon \to 0$. Thus we have

$$\int_{\epsilon}^R \frac{(\log z)^2}{z^2 + 1}\,dz + \int_{-R}^{\epsilon} \frac{(\log z)^2}{z^2 + 1}\,dz = \int_{\epsilon}^R \frac{(\log z)^2}{z^2 + 1}\,dz + \int_{\epsilon}^R \frac{(\log z + i\pi)^2}{z^2 + 1}\,dz$$

$$= \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}\,dz + \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1}\,dz + 2\pi i \int_{\epsilon}^{R} \frac{\log z}{z^2 + 1}\,dz + \int_{\epsilon}^{R} \frac{-\pi^2}{z^2 + 1}\,dz$$

Thus as $R \to \infty$ and $\epsilon \to 0$, we have

$$\int_{0}^{\infty} \frac{\log z}{z^2 + 1}\,dz = 0$$

Now Recall that

$$\int_{0}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$$

Indeed since we can take the integral around the semicircle of radius $R$ and apply the Residue Theorem. Thus by above

$$\int_{0}^{\infty} \frac{(\log z)^2}{z^2 + 1}\,dz = \frac{-\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{8}$$

Problem 3

So we map $D$ to the upper-half unit disc by $z^2$ and then map that to the upper half plane by

$$w = \left(\frac{1 + z^2}{1 - z^2}\right)^2$$

Now notice that $|z| = 1$ on $D$ maps to the positive real axis and the rest of the boundary of $D$ maps to the negative real axis. So we want to find a harmonic function $\phi$ such that for $x \in R$, we have $\phi(x, 0) = 1$ for $x > 0$ and $\phi(x, 0) = 0$ for $x \leq 0$. By Poisson’s formula we have
\[ \phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \phi(\zeta, 0)}{y^2 + (x - \zeta)^2} d\zeta = \frac{1}{\pi} \int_{0}^{\infty} \frac{y}{y^2 + (x - \zeta)^2} d\zeta \]

Then applying \( u \)-substitution we have \( u = (x - \zeta)/y, \) \( du = -d\zeta/y, \) and have

\[ \frac{1}{\pi} \int_{1}^{\infty} -\frac{1}{1+u^2} du = \frac{1}{\pi} \arctan \left( \frac{x - \zeta}{y} \right) \Bigg|_{0}^{\infty} = \frac{1}{2} - \arctan(y/x) \]

So our \( \phi \) function is

\[ \phi(u, v) = \frac{1}{2} - \arctan(v/u) \]

Then for

\[ w = \left( \frac{1 + z^2}{1 - z^2} \right)^2 = \left( \frac{1 + (x + iy)^2}{1 - (x + iy)^2} \right)^2 = \left( \frac{1 - (x + y)^2}{(1 - x^2 + y^2)^2 + 4x^2y^2} + i \frac{4xy}{(1 - x^2 + y^2)^2 + 4x^2y^2} \right)^2 \]

Hence

\[ u = \left( \frac{1 - (x + y)^2}{(1 - x^2 + y^2)^2 + 4x^2y^2} \right)^2 - \frac{16x^2y^2}{(1 - x^2 + y^2)^2 + 4x^2y^2} \]

and

\[ v = \frac{8xy - 8xy(x + y)^2}{((1 - x^2 + y^2)^2 + 4x^2y^2)^2} \]

Hence

\[ \phi(x, y) = \frac{1}{2} - \arctan \left( \frac{8xy - 8xy(x + y)^2}{(1 - (x + y)^2)^2 - 16x^2y^2} \right) \]

**Problem 4.a**

By the Weierstrass Theorem for infinite products, we have

\[ f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{e^{\frac{z}{n}} + \frac{1}{2} (\frac{z}{n}) + \cdots + \frac{1}{n} (\frac{z}{n})^n} \]

for \( n \in \mathbb{N} \cup \{0\}. \) Now for our situation, we have zeros at the negative integers. So the question is when does

\[ \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) \]

converge. By we know that it does converge if and only if

\[ \sum_{n=1}^{\infty} \log(1 + z/n) \]

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converges. Fix \( z \), for \( n > N, |n| > |z| \) and \( |z/n| < 1 \). Now recall the Taylor Expansion of \( \log(1 - h) \), which is
\[
\log(1 - h) = -(h + \frac{h^2}{2} + \frac{h^3}{3} + \cdots) = -h + E(h)
\]
since in our case, \( \log(1 + z/n) \), will not converge because the first term is essentially the harmonic series. Thus we need to take
\[
\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}
\]
will converge since it will kill off the first term. To see take the log. Thus this implies that our function \( f \) has a genus of 1. So our answer is
\[
f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}
\]

**Problem 4.b**

Since any entire function has a product representation by the Weierstrass Theorem, we want to know if \( \exists f \) such that
\[
|f(z)| \leq C_1 e^{C_2|z|^p}
\]
Notice that would imply that
\[
\log(|f(z)|) \leq \log C_1 + c_2|z|^p
\]
So the the canonical representation must have genus 0. Hence
\[
f(z) = C_1 \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)
\]
but
\[
\left| \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \right| = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)
\]
converges if and only if \( \sum(1/n) \) converges, which it doesn’t. Hence no such function exists.

**Problem 5**

Let \( f \) be analytic in the upper half plane. By the reflection principle, \( \exists \) an analytic function \( \overline{f(z)} \) in the lower half plane. Then notice since \( f \) is real on the real axis except at \( z = 0 \), this implies that \( \overline{f(z)} \) is also real on the real axis and
\[
\overline{f(z)} = f(z)
\]
for \( z \in \mathbb{R} \setminus \{0\} \). Then by Morera’s Theorem, we can show that this \( f \) extended is analytic on \( \mathbb{C} \setminus \{0\} \), and is unique. Thus \( f \) has a Laurent series about \( z = 0 \)

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n z^n
\]

Now since \( |z^3 f(z)| \leq C \), this implies that \( f(z) = b/z^3 \), since as \( a \to 0 \), implies we can’t have \( z^i \) for \( i < -3 \) and as \( z \to \infty \), implies we can’t have \( z^i \) for \( i > -3 \). Hence \( f(z) = b/z^3 \) for \( |b| \leq c \). Finally since \( f(i) = 4i \Rightarrow \)

\[
f(z) = \frac{4}{z^3}
\]

and is unique.

**Sept 1995 Linear Algebra**

**Problem 1.a**

Using Gaussian Elimination, we have

\[
\begin{array}{cccc|c}
1 & 2 & 0 & 3 & 0 \\
0 & -3 & 0 & 1 - 6 & b \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 3 & 0 \\
\hline
1 & 2 & 0 & 3 & 0 \\
0 & -3 & 0 & a - 6 & b \\
0 & 0 & 3 & a - 6 & 3 + b \\
0 & 0 & 1 & 3 & 0 \\
\hline
1 & 2 & 0 & 3 & 0 \\
0 & -3 & 0 & a - 6 & b \\
0 & 0 & 3 & a - 6 & 3 + b \\
0 & 0 & 0 & a - 15 & 3 + b \\
\end{array}
\]

For the system to have a solution, then if \( a = 15 \), we have \( b = -3 \), or if \( a \neq 15 \) we have \( b \in \mathbb{R} \).

**Problem 1.b**

To have a unique solution, we have \( a \neq 15 \) and \( b \in \mathbb{R} \).

**Problem 1.c**

For there to be infinite number of solutions, we must have \( a = 15 \) and \( b = -3 \). Then using Gaussian Elimination on the original system, we have

\[
\begin{array}{cccc|c}
1 & 2 & 0 & 3 & 0 \\
0 & -3 & 0 & 9 & -3 \\
0 & 0 & 3 & 9 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Hence have \( z = t, \ y = -3t, \ x = 3t + 1, \) and \( w = -2 - 9t. \) Thus all solutions are in the form of

\[
s = t \begin{pmatrix} -9 \\ 3 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\]

**Problem 2**

Notice

\[
(-2 - \lambda)(1 - \lambda) - 4 = (\lambda + 3)(\lambda - 2) = 0
\]

so we have \( \lambda_1 = -3 \) and \( \lambda_2 = 2. \) So

\[
A = M \begin{pmatrix} -3/2 & 0 \\ 0 & 1 \end{pmatrix} M^{-1}
\]

where \( v_1 \) and \( v_2 \) are the eigenvectors that are linearly independent. So for \( x = a_1 v_1 + a_2 v_2, \) we have

\[
\lim_{N \to \infty} ||A^N x|| = \lim_{N \to \infty} ||a_1 \lambda_1^N v_1 + a_2 \lambda_2^N v_2|| < \infty
\]

if and only if \( a_1 = 0. \) So the only vector that satisfies that is \( S((1, 1)^T). \) Hence \( y = S((-1, 1)^T). \)

**Problem 3.a**

Since \( T \) is a linear transformation, we have

\[
T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = T \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = T \begin{pmatrix} 1 \\ 1 \end{pmatrix} - T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}
\]

**Problem 3.b**

By above we see that \( Te_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}. \) Now notice

\[
T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T \left( -\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}
\]

and notice

\[
T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = T \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]
Thus the matrix $T$ is

$$T = \begin{pmatrix} -2 & 3 & -1 \\ 2 & 5 & -4 \\ 1 & 0 & 0 \end{pmatrix}$$

**Problem 4**

By the min/max theorem, we have $\lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1$ and

$$\lambda_i = \max_{y \in \mathbb{R}^i} \frac{y^T Ay}{y^T y} \geq \frac{x^T Ax}{x^T x} = \frac{180}{30} = 6$$

and notice that $x$ is not an eigenvector since

$$Ax = \begin{pmatrix} 5 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

so we have strict inequality.

**Problem 5.a**

Let $v$ be a vector in $\mathbb{F}^n$ for some field $\mathbb{F}$. Then let $\{v, u_2, ..., u_n\}$ be the orthonormal basis of $\mathbb{F}^n$. Then let

$$A = \begin{pmatrix} \mid & \mid & \mid & \mid \\ u_2 & u_3 & \cdots & u_n \end{pmatrix}$$

The $P = A(A^T A)^{-1}A^T$. But $A^T A = I$, so $P = AA^T$. $Px$ is the point in the hyperplane closest to $x$.

**Problem 5.b**

Let $R$ denote the matrix that is the reflection of the unit vector $u$ in the opposite direction. Let $w$ be the vector perpendicular to $u$ through the origin. Then $R = I - 2uu^T$ is the reflection across the $w$ vector in the hyperplane. Notice $uu^T$ is symmetric, which implies $R$ is symmetric. Hence

$$RR^T = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T - 2uu^T 2uu^T = I = R^T R$$

Hence $R$ is orthogonal and $R = R^T = R^{-1}$. Clearly the determinant of $R$ is -1. Notice

$$\det(I) = \det(RR^T) = \det(R) \det(R^T) = \det(R)^2 = 1$$

which implies $\det(R) = \pm 1$. But the only eigenvectors of $R$ are the vectors perpendicular to $u$, i.e. $\alpha w$. Hence

$$(I - 2uu^T)w = w - 2uu^T w = w$$
Hence all the eigenvalue are 1, and the determinant is 1.

**Problem 5.c**

Clearly $P^2 = P$ since we are projecting a vector twice on the plane. Algebraically, we have

$$P^2 = (AA^T)(AA^T) = AA^T$$

and $R^2 = I$ by part (b), and $PR = RP$ since

$$PR = AA^T(I - 2uu^T) = AA^T - 2AA^T uu^T = RP$$

Since $P$ and $R$ are symmetric, they commute.

**Problem 5.d**

The eigenvalues of $R$ are all 1 by part (b). Now the eigenvectors of $P$ are the vectors in the hyperplane with eigenvalues 1. Hence all the eigenvalues of $P$ are 1.
Jan 1996 Advanced Calculus

Problem 1

Now we know that

\[ \int \int_S f(ax + by + cz) dS = \text{Area}(S) \cdot \text{(average of } f(ax + by + cz) \text{ on } (x, y, z) \in S) \]

\[ = 4\pi \text{(average of } f(ax + by + cz) \text{ on } (x, y, z) \in S) \]

Now lets see what the domain is for \( f(ax + by + cz) \), where \( x^2 + y^2 + z^2 = 1 \). Let \( h(x, y, z) = ax + by + cz \) and \( g(x, y, z) = x^2 + y^2 + z^2 = 1 \). Then by the Lagrange Multiplier theorem, we have \( \nabla h = \lambda \nabla g \)

\[ (a, b, c) = \lambda (2x, 2y, 2z) \Rightarrow \frac{a}{2x} = \frac{b}{2y} = \frac{c}{2z} \]

So let \( x = t \), and we have \( y = (b/a)t \) and \( z = (c/a)t \). Hence

\[ t^2 + \frac{b^2}{a^2} t^2 + \frac{c^2}{a^2} t^2 = 1 \Rightarrow t = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}} \]

Therefore

\[ x = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \]
\[ y = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} \]
\[ z = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}} \]

Hence the minimum is

\[ h_{\text{min}} = -\frac{a^2}{\sqrt{a^2 + b^2 + c^2}} + -\frac{b^2}{\sqrt{a^2 + b^2 + c^2}} + -\frac{c^2}{\sqrt{a^2 + b^2 + c^2}} = -\sqrt{a^2 + b^2 + c^2} \]

and the maximum is

\[ h_{\text{max}} = \frac{a^2}{\sqrt{a^2 + b^2 + c^2}} + \frac{b^2}{\sqrt{a^2 + b^2 + c^2}} + \frac{c^2}{\sqrt{a^2 + b^2 + c^2}} = \sqrt{a^2 + b^2 + c^2} \]

Therefore the average of \( f(ax + by + cz) \) on \( (x, y, z) \in S \) is

\[-1\]
\[
\frac{\int_{-\sqrt{a^2+b^2+c^2}}^{\sqrt{a^2+b^2+c^2}} f(t)\,dt}{2\sqrt{a^2+b^2+c^2}} = \frac{\int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})\,dt}{2}
\]

Hence

\[
\int \int_{S} f(ax + by + cz)\,dS = 4\pi \frac{\int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})\,dt}{2} = 2\pi \int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})\,dt
\]

**Workshop method:** We know that \( v = \frac{1}{r} = \frac{1}{|x-x'|} \) is harmonic, i.e. \( \Delta v = 0 \). Now let \( B_\epsilon \) be the punctured sphere at \( x \) with radius \( \epsilon \). Then

\[
d(\Omega) = S + \lim_{\epsilon \to 0} d(B_\epsilon)
\]

Then we have

\[
\int_{\Omega} (u\Delta v - v\Delta u)\,dV = \int_{S} \left( \frac{du}{dn} - v \frac{du}{dn} \right)\,dS + \lim_{\epsilon \to 0} \int_{d(B_\epsilon)} \left( \frac{dv}{dn} - v \frac{du}{dn} \right)\,dS
\]

Since \( u \) and \( v \) are harmonic, we know that it equals zero. Hence

\[
\int_{S} \left( \frac{du}{dn} - v \frac{du}{dn} \right)\,dS = -\lim_{\epsilon \to 0} \int_{d(B_\epsilon)} \left( \frac{dv}{dn} - v \frac{du}{dn} \right)\,dS
\]

Now

\[
\nabla v = -\frac{1}{|x-x'|} \cdot \hat{r} \quad \text{where} \quad \hat{r} = \frac{x-x'}{|x-x'|}
\]

Then \( \frac{dv}{dn} = \frac{-1}{|x-x'|^2} \cos \beta \). So on \( d(B_\epsilon) \), \( \hat{n} = -\hat{r} \)

\[
\int_{d(B_\epsilon)} u(x') \frac{-1}{|x-x'|^2} \hat{r} \cdot d\hat{n} - \frac{1}{|x-x'|} \nabla v \cdot d\hat{n} = \int_{d(B_\epsilon)} u(x') \frac{1}{\epsilon^2} \,dS - \int_{d(B_\epsilon)} \frac{1}{\epsilon} \nabla v dS
\]

Since \( u \) is harmonic, \( |\nabla u| < m \). Hence

\[
\left| \int_{d(B_\epsilon)} \frac{1}{\epsilon} \nabla u d\hat{n} \right| \leq 4\pi \epsilon^2 \frac{m}{\epsilon} \to 0
\]

as \( \epsilon \to 0 \), and

\[
\int_{d(B_\epsilon)} u(x') \frac{1}{\epsilon^2} d\hat{n} \to 4\pi u(x)
\]

Since \( u \) is continuous.
Problem 3

Notice

\[
\int_0^1 x^{-x} \, dx = \int_0^1 e^{-x \log x} \, dx = \int_0^1 \sum_{n=0}^\infty \frac{(-x \log x)^n}{n!} \, dx = \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 (x \log x)^n \, dx \\
= \sum_{n=0}^\infty \frac{1}{n!} (-1)^n \int_0^1 x^n (\log x)^n \, dx
\]

Now notice that by performing \(u\)-substitution, we have \(u = -(n+1) \log x\), \(du = -(n+1)/x \, dx\)

\[
\int_0^1 x^n (\log x)^n \, dx = \int_0^\infty \frac{x^{n+1}}{-(n+1)} \left( \frac{u}{-(n+1)} \right)^n \, du = \frac{(-1)^n}{(n+1)^{n+1}} \int_0^\infty e^{-u} u^n \, du
\]

and recall the gamma function

\[
\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} \, du
\]

so

\[
\Gamma(n+1) = \int_0^\infty u^n e^{-u} \, du = n!
\]

recall the other properties of the gamma function

\[
\Gamma(x+1) = x\Gamma(x)
\]

\[
\Gamma(1/2) = \sqrt{\pi}
\]

\[
\Gamma(n+1/2) = \frac{(2n)!}{4^n n!} \sqrt{\pi}
\]

So

\[
\int_0^1 x^{-x} \, dx = \sum_{n=0}^\infty \frac{1}{n!} (-1)^n \frac{1}{(n+1)^{n+1}} \sum_{n=0}^\infty \frac{1}{(n+1)^{n+1}} = \sum_{n=0}^\infty \frac{1}{n^n}
\]

Problem 4.a

Now we know that

\[
\frac{dv}{dn} = \nabla v \cdot n \quad \text{and} \quad \frac{du}{dn} = \nabla u \cdot n
\]

Thus we have

\[
\int_S \frac{dv}{dn} - v \frac{du}{dn} \, dS = \int_S u \nabla v \cdot n - v \nabla \cdot ndS = \int_S (u \nabla v - v \nabla u) \cdot ndS
\]

and by the Divergence theorem, we have

-3
\[
\begin{align*}
= \int_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) dV = \int_{\Omega} u \Delta v - v \Delta u dV
\end{align*}
\]

**Problem 4.b**

Now let \( v = 1/|x - x'| = 1/r \). Then

\[
\frac{dv}{dn} = \nabla v \cdot n = ||\nabla v|| \cdot ||n|| \cos \beta
\]

and since \( ||n|| = 1 \), we have

\[
\frac{dv}{dn} = ||\nabla v|| \cos \beta = \left| \frac{d}{dr} \frac{r}{r_u} \right| = \frac{1}{r^2}
\]

Hence

\[
\frac{1}{4\pi} \int_{x' \in S} u(x') \frac{\cos \beta}{|x - x'|^2} + \frac{1}{|x - x'|} \frac{du}{dn}(x')dS(x') = \frac{1}{4\pi} \int_{x' \in S} u(x') \frac{dv}{dn} + v \frac{du}{dn}(x')dS(x')
\]

then by part (a) we have

\[
\frac{1}{4\pi} \int_{\Omega} u \Delta v - v \Delta u dV
\]

Since \( \Delta u = 0 \), we have

\[
\frac{1}{4\pi} \int_{\Omega} u \nabla v dV
\]

Now notice that in any volume

\[
\int_{V} \Delta \frac{1}{r} dV = \int_{V} \nabla \cdot \nabla \frac{1}{r} dV = \int_{S} \nabla \frac{1}{r} \cdot ndS
\]

\[
= \int_{S} \frac{d}{dr} \frac{1}{r} \cdot ndS = \int_{S} \frac{-1}{r} ndS = 4\pi \frac{R^2}{r^2}
\]

where \( R \) is the radius of the small sphere we are integrating around. So as \( r > 0 \) and \( R \to 0 \), the integral goes to zero. And if \( r = R \) and \( R \to 0 \), the integral goes to \(-4\pi\). Hence

\[
\Delta \frac{1}{|x - x'|} = -4\pi \delta(x - x')
\]

Hence

\[
\frac{1}{4\pi} \int_{x' \in \Omega} u(x') \nabla \frac{1}{|x - x'|} dV(x') = \frac{1}{4\pi} \int_{x' \in \Omega} u(x') (-4\pi \delta(x - x')) dV(x') = u(x)
\]
Problem 5.a
For \( x \in [0,1] \), we have
\[
f(x) = \sum_{n=0}^{\lfloor \log(1/x) \rfloor} 2^n x + \sum_{n=\lfloor \log(1/x) \rfloor +1}^{\log(2/x)} 2 - 2^n x + 0 < \infty
\]
Thus it converges pointwise.

Problem 5.b
No. Let \( \epsilon = 2^{-k} \) for \( k \in \mathbb{Z} \). Then notice \( f(0) = \sum h(0) = 0 \). But
\[
\left| f(2^{-k}) \right| = \left| \sum_{n=0}^{\infty} h(2^n 2^{-k}) \right| \geq h(1) = 1
\]
Thus it is not continuous at \( x = 0 \).

Problem 5.c
No. Assume that \( f \) did converge uniformly. Since \( f_k(x) = \sum_{n=0}^{k} h(2^n x) \) is continuous for all \( k \), that would imply that \( f \) is continuous. However as we have seen in part (b), \( f \) is not continuous, and thus we have a contradiction.

Problem 5.d
Notice that
\[
h(x) = \begin{cases} 
0 & x = 0 \\
2 - x & x \in (0, 2]
\end{cases}
\]
Hence for all \( \epsilon > 0 \) \( \exists N \) such that
\[
\left| \sum_{n=N}^{\infty} \frac{1}{n} h(2^n x) \right| \leq \frac{1}{N} \sum_{n=N}^{\infty} \frac{1}{n} h(2^n x) \leq \frac{2}{N} < \epsilon
\]

Jan 1996 Complex Variables

Problem 1.a
CASE 1: if \( p \) is in region \( I \), then by the residue theorem, we have
\[
\oint_{C} \frac{dz}{z-p} = 2\pi i
\]
CASE 2: if \( p \) is in region II, then by the residue theorem, we
\[
\oint_{C} \frac{dz}{z - p} = \oint_{C_1} \frac{dz}{z - p} + \oint_{C_2} \frac{dz}{z - p} = 4\pi i
\]
where \( C_1 \) is the outer circle and \( C_2 \) is in the inner circle.

CASE 3: if \( p \) is in region III, then by the residue theorem, we have
\[
\oint_{C} \frac{dz}{z - p} = 2\pi i
\]

CASE 4: if \( p \) is in region IV we have
\[
\oint_{C} \frac{dz}{z - p} = 0
\]

Problem 1.b
Notice for \( y_0 \in \mathbb{R} \), by the residue theorem, we have
\[
\int_{\gamma} f(z) dz = 0
\]
So
\[
\int_{-R}^{R} f(z) dz + \int_{0}^{y_0} e^{i(R+iy) - (R+iy)^2} idy + \int_{R}^{-R} e^{i(R+iy) - (R+iy)^2} idy + \int_{y_0}^{0} e^{i(-R+iy) - (-R+iy)^2} dy
\]
and notice
\[
\left| \int_{0}^{y_0} e^{i(R+iy) - (R+iy)^2} idy \right| \leq y_0 e^{-R^2} M \to 0
\]
likewise
\[
\left| \int_{y_0}^{0} e^{i(-R+iy) - (-R+iy)^2} dy \right| \to 0
\]
as \( R \to \infty \). Therefore
\[
\int_{-\infty}^{\infty} e^{iz} e^{-z^2} dz = \int_{-\infty}^{\infty} e^{i(x+iy_0) - (x+iy_0)^2} dx
\]

Problem 3.a
Notice
\[
\lim_{z \to 0} \cot z \sim \frac{1}{z} = \lim_{z \to 0} \frac{z \cos z - \sin z}{z \sin z} = \left( -\frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots \right) - \left( -\frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots \right) \to 0
\]
Problem 3.b
So by solving
\[
\left(-\frac{z^3}{2!} + \frac{z^5}{4!} \pm \cdots\right) - \left(\frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots\right) = (a_0 + a_1 z + a_2 z^2 + \cdots) \left(\frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)
\]
Then we get \(a_0 = 0, a_1 = -1/3,\) and \(a_2 = (1/4! - 1/5!)/(1 + 1/18).\)

Problem 3.c
It converges for all \(z,\) since the function is analytic. Hence the radius of convergence is \(\infty.\)

Problem 3.d
Recall the infinite product representation of \(\sin z\) is
\[
\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 k^2}\right) = z \prod_{k=-\infty, k\neq 0}^{\infty} \left(1 - \frac{z}{\pi k}\right)
\]
Then
\[
\log(\sin z) = \log z + \sum_{k=-\infty, k\neq 0}^{\infty} \log \left(1 - \frac{z}{\pi k}\right)
\]
using the principle branch cut. Thus taking the derivatives, we have
\[
\frac{1}{\sin z} \cos z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{1 - \frac{z}{\pi k}} \left(-\frac{1}{\pi k}\right)
\]
Hence
\[
\cot z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{z - \pi k}
\]
so
\[
f(z) = \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{z - \pi k}
\]

Problem 4.a
We want to solve \(1 + z^6 = 0,\) hence \(z = e^{i(\pi + 2\pi k)/6}.\) Thus the 6 poles are \(z_1 = e^{i\pi/6}, z_2 = e^{i\pi/2}, z_3 = e^{i5\pi/6}, z_4 = e^{i7\pi/6}, z_5 = e^{i9\pi/6}, z_6 = e^{i11\pi/6}.\)
Problem 4.b

Thus we have

\[ f(z) = g(z) \cdot h(z) = \frac{1}{(z - e^{i\pi/6})(z - e^{i5\pi/6})(z - e^{i7\pi/6})(z - e^{i9\pi/6})(z - e^{i11\pi/6})} \]

also recall the properties of complex conjugates

**CASE 1:** \( \overline{uv} = (\overline{u})(\overline{v}) \)

**CASE 2:** \( \overline{u + v} = \overline{u} + \overline{v} \)

**CASE 3:** \( (\overline{\pi})^{-1} = \overline{\pi}^{-1} \)

Hence

\[ h(z) = \frac{1}{(z - e^{i\pi/6})(z - e^{i5\pi/6})(z - e^{i7\pi/6})(z - e^{i9\pi/6})(z - e^{i11\pi/6})} = g(z) \]

Problem 5.a

Notice the intersection of \(|z| = 1\) and \(|z - 1| = 1\) is at \(1/2 + i\sqrt{3}/2\) and at \(1/2 - i\sqrt{3}/2\). Then let

\[ T(z) = \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 - i\sqrt{3}/2)} \]

Then we map \(1/2 + i\sqrt{3}/2 \mapsto 0\), \(1/2 - i\sqrt{3}/2 \mapsto \infty\), and \(0 \mapsto 1\). Hence the \(|z - 1| = 1\) circle maps to the real axis. Now after some heavy computation, \(1 \mapsto -1/2 + i\sqrt{3}/2\), which is a line with \(\theta = 2\pi/3\). Hence the mapping

\[ w = \left( \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 - i\sqrt{3}/2)(-1/2 - i\sqrt{3}/2)} \right)^{3/4} \]

maps \(\mathcal{L}\) to the first quadrant. Now the heat equation is

\[ \frac{du}{dt} = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} \]

To find the steady state, we need \(u\) such that

\[ \frac{du}{dt} = 0 = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} \]

Hence we need \(u\) to be harmonic. Since the right boundary of \(L\) maps to the imaginary axis, and the left maps to the real axis, let’s find \(u\) such that it is 20 on the negative real axis, and 0 on the
positive real axis. By Poisson’s Formula in finding harmonic functions in the upper half plane, we have

\[ u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\zeta, 0)y}{y^2 + (x-\zeta)^2} d\zeta = \frac{1}{\pi} \int_{-\infty}^{0} \frac{20y}{y^2 + (x-\zeta)^2} d\zeta = 10 + \frac{20}{\pi} \arctan(y/x) \]

Then for

\[ w_2 = \left( \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 + i\sqrt{3}/2)} \right)^{3/2} \]

We set \( x = \text{Re}(w) \) and \( y = \text{Im}(w) \).

Jan 1996 Linear Algebra

Problem 2.a

SVD of an \( m \times n \) matrix \( A \) is when \( \exists \) orthonormal matrices \( U \), \( V \), and \( \Sigma \) such that

\[ A = U\Sigma V^T \]

where \( U \) is \( m \times m \), \( \Sigma \) is \( m \times n \), and \( V \) is \( n \times n \). If \( \text{Rank}(A) = r \), the the first \( r \) column of \( V \) is the row space, and the last \( n - r \) columns of \( V \) is the null space of \( A \). The first \( r \) columns of \( U \) is the column space of \( A \) and the last \( m - r \) columns is the null space of \( A^T \). And

\[ \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \]

where \( \sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0 \).

Problem 2.b

We have

\[ A = U\Sigma V^T \quad \text{and} \quad A^T = V\Sigma U^T \]

Then

\[ AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T \]

and

\[ A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T \]

since \( U^T = U^{-1} \) and \( V^T = V^{-1} \). Hence

\[ (AA^T)U = U\Sigma^2 \quad \text{and} \quad (A^T A)V = V\Sigma^2 \]

Hence \( \sigma_i^2 \) are the eigenvalues of \( AA^T \) and \( A^T A \) and have the same multiplicity.
Problem 3.a

Recall the minmax theorem

\[ \lambda_i = \min_{S, \dim(S) = i} \max_{x \in S, x \neq 0} \frac{x^T Ax}{x^T x} \]

Let \( S = \{ x \in \mathbb{R}^4 : x_1 + x_2 = 0 \} \). Notice that \( \dim(S) = 3 \). By the minmax theorem, notice

\[ 0 = \lambda_3 = \min_{S, \dim(S) = 3} \max_{x \in S, x \neq 0} \frac{x^T Ax}{x^T x} \leq \max_{x \neq 0} \frac{x^T Ax}{x^T x} = \lambda_4 = 10 \]

Now notice that for \( x = (0, 0, 0, 1), x \in S \). Then

\[ \frac{x^T Ax}{x^T x} = 1 > 0 \]

Hence we have a strict inequality for the lower bound. Now it suffices to show that the eigenvector \( v \) with eigenvalues 10 is not in \( S \). Assume that \( x = (x_1, -x_1, x_3, x_4) \) has an eigenvalue of 10. Then notice \( Ax = 10x \) implies \( x = 0 \). Hence we have a contradiction. Thus we have strict inequality.

Problem 3.b

We have

\[ u = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{array} \right) \quad v_1 = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{array} \right) \quad v_2 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right) \quad v_3 = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right) \]

Problem 3.c

We need to make a change a basis. Let

\[ M = \left( \begin{array}{cccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \]

Then \( M = M^{-1} \), and we will apply the change of basis

\[ MAM^{-1} = \left( \begin{array}{cccc} -1 & -6 & 5/\sqrt{2} & 11/\sqrt{2} \\ -6 & -9 & -11/\sqrt{2} & 5/\sqrt{2} \\ 5/\sqrt{2} & -11/\sqrt{2} & -4 & 4 \\ 11/\sqrt{2} & 5/\sqrt{2} & 4 & 1 \end{array} \right) \]

Thus

\[ \max_{x_1 + x_2 = 0, x \neq 0} \frac{x^T Ax}{x^T x} = \max_{x_1 = 0} \frac{x^T A_{x_1} x}{x^T x} = \max_{x \in \mathbb{R}^3} \frac{x^T \tilde{A} x}{x^T x} \]

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where

\[
A' = \begin{pmatrix}
-9 & -11/\sqrt{2} & 5/\sqrt{2} \\
-11/\sqrt{2} & -4 & 4 \\
5/\sqrt{2} & 4 & 1
\end{pmatrix}
\]

**Problem 4.a**

Notice indeed \(A\) and \(B\) have the same characteristic polynomial

\[
\det(A - \lambda I) = \det \begin{pmatrix}
-\lambda & 1 & 0 \\
0 & -\lambda & 1 \\
-2 & 3 & -\lambda
\end{pmatrix} = -\lambda(\lambda^2 - 3) - 2 = -\lambda^3 + 3\lambda - 2
\]

and

\[
\det(B - \lambda I) = \det \begin{pmatrix}
1 - \lambda & 0 & 0 \\
0 & 1 - \lambda & 0 \\
0 & 0 & -2 - \lambda
\end{pmatrix} = (1 - \lambda)^2(-2 - \lambda) = -\lambda^3 + 3\lambda - 2
\]

However if \(A\) and \(B\) were similar, then they would have the same: determinant, trace, size, nullity, characteristic polynomial, eigenvalues and vectors. So notice

\[
\det A = 0 \neq -2 = \det B
\]

**Problem 4.b**

Give that the characteristic polynomial of \(C\) is \((\lambda^2 - \sqrt{2}\lambda + 1)(\lambda - 1)\), we see that it’s eigenvalues are \(\lambda_1 = 1, \lambda_2 = \sqrt{2}/2 + i\sqrt{2}/2,\) and \(\lambda_3 = \sqrt{2}/2 - i\sqrt{2}/2.\) Let \(Q\) be defined as

\[
Q = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{2}/2 + i\sqrt{2}/2 & 0 \\
0 & 0 & \sqrt{2}/2 - i\sqrt{2}/2
\end{pmatrix}
\]

Then notice \(QQ^T = Q(Q)^T = I = (Q)^TQ = Q^TQ.\) Hence \(Q\) is indeed orthogonal (in the complex way). Also since \(C\) has 3 distinct eigenvalues, it has 3 independent eigenvectors \(x_1, x_2, \) and \(x_3.\) Thus we define

\[
M = \begin{pmatrix}
| & | & | \\
x_1 & x_2 & x_3 \\
| & | & |
\end{pmatrix}
\]

\(M\) is invertible and \(CM = MQ.\) Hence \(C\) and \(Q\) are similar.
Problem 5.a

So we have

\[
\begin{pmatrix}
2 & 1 & 0 & \cdots & 0 \\
1 & 4 & 2 & 0 & \ddots \\
2 & 6 & \ddots & 0 & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
n-1 & n-1 & \cdots & n-1 & 2n
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{pmatrix}
= 0
\]

Now I claim that the matrix (we will call \( A \)) is invertible. Notice that the determinant \( D_n \) is

\[
D_n = 2nD_{n-1} - (n-1)^2D_{n-2} = \prod \lambda_i > 0
\]

for all \( n \) and \( D_1 = 2, D_2 = 7, D_3 = 36 \). Hence \( D_n > 0 \).

Problem 5.b

Recall that for symmetric matrices TFAE: 1) all \( n \) eigenvalues are greater than 0, 2) all upper left determinant is greater than 0, 3) all \( n \) pivots are greater than 0, 4) \( x^T Ax > 0 \) for \( x \neq 0 \). By part (a), 2 is satisfies which implies 1 is satisfied.

Sept 1996 Advanced Calculus

Problem 1.a

We know that the product

\[
\prod_{n=1}^{\infty} (1 + a_n)
\]

converges absolutely if and only if

\[
\sum_{n=1}^{\infty} a_n
\]

converges absolutely. Thus assume that \( \lim_{n} f(p,n) \) converges. Then

\[
\sum_{k=1}^{\infty} \frac{p}{k} < \infty
\]

which is a contradiction. Hence \( f(p,n) \to \infty \) as \( n \to \infty \).
Problem 1.b
Recall that
\[
\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} \, dx = \log(n + 1)
\]
Hence
\[
\lim_{n \to \infty} \mid \log n - \sum_{k=1}^{\infty} \frac{1}{k} \mid = \lim_{n \to \infty} \mid \log(n) - \log(n + 1) \mid = \lim_{n \to \infty} \mid \log(n/(n + 1)) \mid \to 0
\]
Hence it converges and the limit exists.

Problem 1.c
We know that \( \lim_{n \to \infty} \frac{f(p,n)}{n^p} \) exists if and only if
\[
\lim_{n \to \infty} \log \left( \frac{f(p,n)}{n^p} \right)
\]
notice
\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \log(1 + p/k) - \log(n^p) \right) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \frac{(p/k)^s}{s} (-1)^{s+1} - \log n^p
\]
\[
= \left( \sum_{k=1}^{\infty} \frac{p}{k} - \log n^p \right) + \sum_{k=2}^{\infty} \sum_{s=1}^{\infty} \frac{p^s}{sk^s} (-1)^{s+1}
\]
by part (b) it converges.

Problem 2.a
For \( f(x) = xe^x \), we have \( f(0) = 0 \) and \( \lim_{x \to -\infty} f(x) \to \infty \). Also \( f \) is continuous on \( x > 0 \). Therefore for any positive \( t \), \( \exists x_*(t) \) such that \( t = x_*(t)e^{x_*(t)} \).

Problem 2.b
\[
\lim_{t \to -\infty} x_*(t) = \lim_{x \to -\infty} \frac{x}{\log(xe^x)} = \lim_{x \to -\infty} \frac{x}{x + \log x} = 1
\]

Problem 4
Now if \( \nabla \times E = 0 \) implies that \( \exists \) a potential function, that is gradient to it. I.e. \( \exists \phi \) such that
\[
E = \nabla \phi
\]
So
\[
\nabla \cdot (\phi J) = \nabla \phi \cdot J + \phi \nabla \cdot J = \nabla \phi \cdot J = E \cdot J
\]

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since $\nabla \cdot J = 0$. So

$$\int_{\Omega} E \cdot J \, dx = \int_{\Omega} \nabla \cdot (\phi J) \, dx = \int_{\delta(\Omega)} \phi J \cdot dS$$

Since $E = 0$ on the neighborhood of the boundary, then $\phi$ is constant on the boundary. So there is a potential $\psi$ such that $\psi = 0$ on the boundary. Hence

$$\int_{\Omega} E \cdot J \, dx = \int_{\delta(\Omega)} \psi J \, dS = 0$$

Sept 1996 Complex Variables

Problem 1

Here we fall into two cases. **CASE 1**: let $a$ be purely imaginary, $a = iv$ where $v \in \mathbb{R}$. Then notice

$$\int_{-\infty}^{\infty} e^{-t^2/2-v^2} dt = e^{v^2} \int_{-\infty}^{\infty} e^{-(t+v)^2/2} dt = e^{v^2} \sqrt{2\pi}$$

**CASE 2**: Now assume that $a = u + iv$ where $v \neq 0$. Then let $f(z) = e^{-z^2/2}$ and $\gamma$ be the contour from $-R$ to $R$, to $R - ia$, to $-R - ia$, to $-R$. Then by the Residue Theorem, we have

$$\int_{\gamma} f(z) \, dz = \int_{-\infty}^{\infty} e^{-t^2/2} dt + \int_{1}^{0} e^{-(R-iat)^2/2} iadt + \int_{0}^{1} e^{-(R-iat)^2/2} iadt + \int_{-\infty}^{\infty} e^{-(t-ia)^2/2} dt = 0$$

Now notice

$$\left| \int_{0}^{1} e^{-(R-iat)^2/2} iadt \right| \leq \sqrt{u^2 + v^2} e^{-R^2/2 - Rv + u^2/2} \rightarrow 0$$
as $R \rightarrow \infty$. Likewise we have

$$\left| \int_{1}^{0} e^{-(R-iat)^2/2} iadt \right| \rightarrow 0$$
as $R \rightarrow \infty$. Hence as $R \rightarrow \infty$ we have

$$\int_{-\infty}^{\infty} e^{-t^2/2} dt + \int_{-\infty}^{\infty} e^{-(t-ia)^2/2} dt = 0$$

Hence

$$\int_{-\infty}^{\infty} e^{-t^2/2-2\pi itw} dt = \sqrt{2\pi} e^{-2\pi^2 w^2}$$
Problem 2

By the Poisson Integral formula, we have

\[ \phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{-y}{y^2 + (x-w)^2} \, dw + \frac{1}{\pi} \int_{0}^{\infty} \frac{y}{y^2 + (x-w)^2} \, dw \]

By letting \( u = (x-w)/y \) and \( du = -dw/y \), we have

\[ \phi(x, y) = \frac{1}{\pi} \left[ \frac{0}{1+u^2} - \frac{\infty}{1+u^2} \right] = \frac{2}{\pi} \arctan \left( \frac{x}{y} \right) \]

which is bounded. Notice if \( \phi \) did not have to be bounded, we have

\[ \phi(x, y) = \frac{2}{\pi} \arctan \left( \frac{x}{y} \right) + Cy \]

for constant \( C \).

Problem 3

First we will map \( D = \{ z : |z| < 1, 0 < \arg(z) < \pi/2 \} \) to the upper-half unit circle by \( w_1 = z^2 \). Then we will map the upper-half unit circle to the first quadrant by

\[ w_2 = i \left( \frac{1 - z^2}{1 + z^2} \right) \]

Then we will map that to the UHP by

\[ w_3 = - \left( \frac{1 - z^2}{1 + z^2} \right)^2 \]

Then we map the UHP to the interior of the unit circle by

\[ w_f = \frac{- \left( \frac{1-z^2}{1+z^2} \right)^2 - i}{- \left( \frac{1-z^2}{1+z^2} \right)^2 + i} \]

Problem 5

Let \( f \) be an entire function. Then assume \( \exists \) a fixed \( \epsilon > 0 \) and \( a \in \mathbb{C} \) such that for all \( z \in \mathbb{C} \) we have

\[ |f(z) - a| \geq \epsilon \]

Now let \( g(z) = 1/(f(z) - a) \). Then \( g \) is an entire function such that

\[ |g(z)| = \frac{1}{|f(z) - a|} \leq \frac{1}{\epsilon} \]

Thus \( g \) is an entire bounded function. By Liouville’s Theorem, \( g \) must be constant.
\[ g(z) = c = \frac{1}{f(z) - a} \]

which implies \( f \) is constant. But since \( f \) assumes 1 and 0, we have a contradiction.

**Sept 1996 Linear Algebra**

**Problem 1**

If \( v_1 \) and \( v_2 \) are the eigenvectors of \( A \) with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then \( v_1 \) and \( v_2 \) are eigenvectors of \( A^2 \) with eigenvalues \( \lambda_1^2 \) and \( \lambda_2^2 \). So by applying the quadratic equation, we have

\[ t = -1 \pm i\sqrt{2} \]

Thus we have \( \lambda_1 = -1+i\sqrt{2} \) and \( \lambda_2 = -1-i\sqrt{2} \). So this implies \( \lambda_1^2 = -1-2i\sqrt{2} \) and \( \lambda_2^2 = -1+2i\sqrt{2} \).

Therefore the characteristic equation for \( A^2 \) is

\[ (t - (-1 - 2\sqrt{2}i))(t - (-1 + 2\sqrt{2}i)) = t^2 + 2t + 9 \]

**Problem 2.1**

notice

\[
\det \begin{pmatrix}
\alpha + 1 - \lambda & \alpha \\
\alpha - 1 & \alpha - \lambda
\end{pmatrix} = \lambda^2 + (-2\alpha - 1)\lambda + 2\alpha
\]

Thus \( \lambda_1 = 2\alpha \) and \( \lambda_2 = 1 \). Hence

\[ Tr(B) = \sum_{i=0}^{m} (2\alpha)^i + 1 = m + \frac{1 - (2\alpha)^{m+1}}{1 - 2\alpha} \]

**Problem 2.2**

Now if \( \alpha \neq 1/2 \), then we can diagonalize \( A \), and notice

\[ A^i = M \begin{pmatrix} 1 \\ (2\alpha)^i \end{pmatrix} M^{-1} \neq 0 \]

as \( i \to \infty \). Thus the sum does not converge. Now if \( \alpha = 1/2 \), then

\[ A^n = \frac{1}{2} \begin{pmatrix} n + 2 & n \\
-n & 2 - n \end{pmatrix} \neq 0 \]

as \( n \to \infty \). Thus the summation does not converge for any \( \alpha \).
Problem 3.1
Let $x \in Im(A)$. Then $\exists y$ such that $Ay = x$. Thus notice
\[
Ax = A(Ay) = A^2y = 0y = 0
\]
hence $x \in \ker(A) \Rightarrow Im(A) \subset \ker(A)$.

Problem 4
Notice by Gaussian elimination we have
\[
\begin{array}{ccc|c}
2 & s + 1 & 1 & -a \\
s + 4 & 2 & 2 & 3 \\
2s & 2s & s & 4 \\
\hline
2 & s + 1 & 1 & -a \\
0 & -(s + 4)(s + 1)/2 & -(s + 4)/2 + 2 & 3 + a(s + 4)/2 \\
0 & -s^2 + s & 0 & 4 + as \\
\end{array}
\]
Hence in order to have an infinite number of solutions, we must have $s = 1$ and $a = -4$. Hence by back substitution we have
\[
s = (3/5, 7/5, 0) + t(-3/5, 1/10, 1)
\]

Problem 5.1
If the entries on the diagonal of $L$ are positive and real, then all pivots are positive, and thus it is positive definite.

Problem 5.2
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Problem 5.3
We want to show that the Cholesky Decomposition is unique. Notice
\[
A = LL^T = BB^T
\]
Since $L$ and $B$ have all positive entries, it is invertible. Now
\[
L^T(B^T)^{-1} = L^{-1}B \Rightarrow L^T(B^{-1})^T = L^{-1}B \Rightarrow (B^{-1}L)^T = (B^{-1}L)^{-1}
\]
Now since $L$ and $B$ are lower triangular, $L^{-1}$ and $B^{-1}$ is lower triangular, And thus $B^{-1}L$ and $(B^{-1}L)^{-1}$ is lower triangular Now since
\[
(B^{-1}L)^T = (B^{-1}L)^{-1}
\]
we have
\[(B^{-1}L)^T = (B^{-1}L)^{-1} = D\]
for some diagonal matrix $D$. Hence
\[L^{-1}BL^T(B^{-1})^T = D^2 = (B^{-1}L)(B^{-1}L)^{-1} = I\]
since $B^{-1}L$ is symmetric. Thus $D$ has ±1 on it’s diagonal. But since all entries are positive, we have $D = I$. Therefore $B = L$, and the factorization is unique.

Jan 1997 Advanced Calculus

Problem 1

Let $x = 1/s$, then $dx = (-1/s^2)ds$ and we have

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1} s^\alpha \sin(1/s)ds = \lim_{\epsilon \to 0} \int_{1/\epsilon}^{1} -\sin x \frac{dx}{x^{\alpha-2}} = \lim_{\epsilon \to 0} \int_{1}^{1/\epsilon} \sin x \frac{dx}{x^{\alpha-2}} = \int_{1}^{\infty} \sin x \frac{dx}{x^{\alpha-2}}$$

Now for $\alpha > 3$ then clearly the integral converges absolutely, which implies that it converges. For $\alpha \leq 2$, notice

$$\lim_{x \to \infty} \frac{\sin x}{x^{\alpha-2}} \neq 0$$

hence the integral does not exist. Finally for $2 < \alpha \leq 3$, we have

$$= \int_{1}^{\infty} \sin x \frac{dx}{x^p}$$

where $0 < p \leq 1$. Using integration by parts, we have $u = 1/x^p$, $du = -px^{-p-1}$, $v = -\cos x$ and $dv = \sin x dx$. Thus

$$= -\frac{\cos x}{x^p}\bigg|_{1}^{\infty} - p \int_{1}^{\infty} \frac{\cos x}{x^{p+1}} dx < \infty$$

Hence the integral exists for $\alpha > 2$.

Problem 2

Let $u = a \cdot r = a_1x + a_2y + a_3z$, $v = b \cdot r = b_1x + b_2y + b_3z$, and $w = c \cdot r = c_1x + c_2y + c_3z$. Then

$$du = a_1dx + a_2dy + a_3dz$$

$$dv = b_1dx + b_2dy + b_3dz$$

$$dw = c_1dx + c_2dy + c_3dz$$

Hence
\[ dudvdw = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \, dxdydz \]

Hence

\[ \int \int \int_E (a \cdots r)(b \cdot r)(c \cdot r) dxdydz = \frac{1}{|(a \cdot b \times c)|} \int_0^\gamma \int_0^\beta \int_0^\alpha uvwdudvdw = \frac{(\alpha \beta \gamma)^2}{8|(a \cdot b \times c)|} \]

**Problem 3.a**

We have \( f(x) = \sum_{n=0}^\infty a_n x^n \) for all real \( x \). \( f(0) = 1 \Rightarrow a_0 = 1 \). Notice

\[ 0 = x^2 f''(x) + xf'(x) + (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + 20a_5 x^5 + 30a_6 x^6 + \cdots) \]

\[ + (a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + 5a_5 x^5 + 6a_6 x^6 + \cdots) + (x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots) \]

\[ = a_1 x + (1 + 2a_2 + 2a_2)x^2 + (a_1 + 3a_3 + 6a_3)x^3 + \cdots \]

Hence \( a_1 = 0 \), and \( a_2 = -1/4 \).

**Problem 3.b**

By above we can see that

\[ a_{n-2} + na_n + n(n-1)a_n = 0 \]

for \( n \) even, and \( a_n = 0 \) for \( n \) odd. Thus

\[ a_n = \begin{cases} 0 & n \text{ is odd} \\ \frac{-a_{n-2}}{n} & n \text{ is even} \end{cases} \]

which implies

\[ a_n = \begin{cases} 0 & n \text{ is odd} \\ \frac{(i)^n}{n^2(n-2)^2 \cdots} & n \text{ is even} \end{cases} \]

where \( a_0 = 1 \)

**Problem 3.c**

By letting

\[ g(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots \]

and solving the equation given and applying part (b), we can see that it equals zero.
Problem 4.a

Notice for \( x > 0 \), we have by L'Hopital's rule,

\[
\lim_{t \to 0} \frac{e^{-xt} - e^{-t}}{t} = \frac{-xe^{-xt} + e^{-t}}{1} = -x + 1
\]

Thus it suffices to show that for

\[
\phi(x) = \int_0^1 \frac{e^{-xt} - e^{-t}}{t} dt + \int_1^\infty \frac{e^{-xt} - e^{-t}}{t} dt
\]

that

\[
\int_1^\infty \frac{e^{-xt} - e^{-t}}{t} dt < \infty
\]

Notice by using integration by parts, we have \( u = 1/t, \ du = -1/t^2, \ v = e^{-xt} / -x + e^{-t} \) and \( dv = e^{-xt} - e^{-t} \) implies

\[
= \frac{1}{t} \left( \frac{e^{-xt}}{-x} + e^{-t} \right) \bigg|_1^\infty + \int_1^\infty \frac{e^{-xt}}{-x} + e^{-t} \frac{1}{t^2} dt
\]

and

\[
\int_1^\infty \frac{e^{-xt}}{-x} + e^{-t} \frac{1}{t^2} dt
\]

converges absolutely and so it converges.

Problem 4.b

Notice

\[
\phi'(x) = \int_0^\infty -e^{-xt} dt = -\frac{1}{x}
\]

Problem 4.c

Finally by part (b), we potentially have

\[
\phi(x) = -\log x + C
\]

Notice

\[
\lim_{R \to \infty} \int_R^\infty \left| \frac{e^{-xt} - e^{-t}}{t} \right| dt \leq \lim_{R \to \infty} \frac{1}{R} \int_R^\infty e^{-xt} + e^{-t} dt = \lim_{R \to \infty} \frac{1}{R} \left( \frac{e^{-Rx}}{x} + e^{-R} \right) \to 0
\]

Hence

\[
\lim_{R \to \infty} \int_R^\infty \frac{e^{-xt} - e^{-t}}{t} dt = 0
\]
uniformly for $x \in (0, \infty)$. Thus

$$\phi(1) = \int_0^\infty 0 \, dt = 0 = -\log(1) + C$$

which implies $C = 0$. Therefore $\phi(x) = -\log x$.

**Problem 5**

Notice we have

$$\lim_{t \to 0} \int_{-t}^{t} g(x) \frac{1}{t} \left(1 - \frac{x^2}{t^2}\right) \, dx = \lim_{t \to 0} \frac{1}{t^3} \int_{-t}^{t} g(x)(t^2 - x^2) \, dx = \lim_{t \to 0} \frac{1}{t^3} \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2t}{n} g(-t + k2t/n)(t^2 - (-t + k2t/n)^2)$$

$$= \lim_{t \to 0} \frac{1}{t^3} \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2t}{n} g(-t + 2kt/n)(k4t^2/n - k^24t^2/n^2) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2}{n} g(0) \left(\frac{4k}{n} - \frac{k^24}{n^2}\right)$$

$$= 8g(0) \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{k}{n^2} - \frac{k^2}{n^3}\right) = 8g(0) \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} kn - k^2$$

$$= 8g(0) \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)(2n+1)}{6}\right) = 8g(0) \lim_{n \to \infty} \frac{1}{n^3} \left(\frac{n^3 - n}{6}\right) = \frac{4}{3}g(0)$$

**Jan 1997 Complex Variables**

**Problem 1**

Let $\gamma$ be the semi-circle in the upper half plane with radius $R$. Then let $f(z) = \frac{e^{iz}}{(z+ia)^2(z-ia)^2}$. Then by the Residue Theorem, we have

$$\int_{\gamma} f(z) \, dz = \lim_{z \to -ia} 2\pi i \frac{ie^{iz}(z + ia)^2 - 2(z + ia)e^{iz}}{(z + ia)^4} = \frac{e^{-a\pi}}{4a^3}(2a + 2)$$

Now notice on the outer circle $z = Re^{i\theta}$, we have

$$\left|\int_{C_R} f(z) \, dz\right| \leq \frac{\pi Re^{-r \sin \theta}}{(R^2 - a)^2} \to 0$$

as $R \to \infty$. Thus as $R \to \infty$

$$\int_{-\infty}^{\infty} f(z) \, dz = \frac{e^{-a\pi}}{4a^3}(2a + 2)$$
Problem 2.a

Let \( f(z) = z^n e^A - e^z \). Then on \(|z| = 1\) we have

\[
|z^n e^A - e^z - z^n e^A| = |e^z| = e^{|z|} = e < e^A = |z^n e^A|
\]

Hence \( f(z) \) has \( n \) roots inside \(|z| < 1\).

Problem 2.b

To see that the root is real and positive, it suffices to show \( \exists x \in (0, 1) \) such that

\[
x e^A = e^x
\]

Let \( g(x) = xe^A - e^x \). Then \( g \) is clearly continuous and \( g(0) = -1 \) and \( g(1) = e^A - e > 0 \). Thus by the intermediate value theorem, \( \exists x_0 \in (0, 1) \) such that \( g(x_0) = 0 \). Hence \( x_0 \) is the real positive root such that

\[
x_0 e^A - x_0 = 1
\]

Problem 3.a

For \(|z| > 1\), notice \( \exists M \) such that

\[
\sum_{n=1}^{\infty} \left| \frac{z^n}{(1 + z^n)(1 + z^{n+1})} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{(|z|^n - 1)(|z|^n+1 - 1)} \leq \sum_{n=1}^{\infty} \frac{|z|^n}{(|z|^n - 1)(|z|^n - 1)}
\]

and so

\[
\leq \sum_{n=1}^{\infty} M|z|^n = M \sum_{n=1}^{\infty} \frac{1}{|z|^n} = \frac{M}{1 - (1/|z|)} = \frac{M|z|}{|z| - 1}
\]

Since \(|z| > 1\). Thus it converges absolutely. Now for \(|z| < 1\), notice

\[
\sum_{n=1}^{\infty} \left| \frac{z^n}{(1 + z^n)(1 + z^{n+1})} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{(1 - |z|^n)(1 - |z|^n+1)}
\]

Now \( \exists M \) such that

\[
\leq \sum_{n=1}^{M} \frac{|z|^n}{(1 - |z|^n)(1 - |z|^n+1)} + \sum_{n=M+1}^{\infty} \frac{|z|^n}{(1/2)(1/2)} < \infty
\]
Problem 3.b
By the hint we have
\[
\sum_{n=1}^{\infty} \frac{z^n}{(1 + z^n)(1 + z^{n+1})} = \sum_{n=1}^{\infty} \frac{1}{z-1} \left[ \frac{1}{1 + z^n} - \frac{1}{1 + z^{n+1}} \right] = \frac{1}{z-1} \sum_{n=1}^{\infty} \frac{1}{1 + z^n} - \frac{1}{1 + z^{n+1}}
\]
\[
= \frac{1}{z-1} \left( \frac{1}{1+z} + \frac{1}{1+z^2} + \frac{1}{1+z^3} + \cdots - \frac{1}{1+z^2} - \frac{1}{1+z^3} - \cdots \right) = \frac{1}{z^2 - 1}
\]

Problem 3.c
Same as above but must be careful.

Problem 4.a
Notice that
\[
w = \frac{1}{z} = (x + iy) + \frac{1}{x + iy} = \left( x + \frac{x}{x^2 + y^2} \right) + i \left( y - \frac{y}{x^2 + y^2} \right)
\]
So on the \(x^2 + y^2 = 1\) border, we have \(w = 2x \in [-2,2]\). Also for \(y = 0\), we have \(w = x + 1/x \in (-\infty, -2) \cup (2, \infty)\). Since \(i/2 \to -(3/2)i\), \(w\) maps the region to the LHP.

Problem 4.b
By the Poisson integral formula, we have
\[
\phi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-yG(w)dw}{y^2 + (x-w)^2} = \int_{-2}^{2} \frac{-y}{y^2 + (x-w)^2} dw = \int_{-2}^{2} \frac{-1/y}{1 + \left( \frac{x-w}{y} \right)^2} dw
\]
now by \(u\)-substitution, we have \(u = (x-w)/y\) and \(du - dw/y\). Thus
\[
= \frac{1}{\pi} \int \frac{1}{1+u^2} du = \frac{1}{\pi} \arctan \left( \frac{x-w}{y} \right) \bigg|_{-2}^{2} = \frac{1}{\pi} \arctan \left( \frac{x-2}{y} \right) - \frac{1}{\pi} \arctan \left( \frac{x+2}{y} \right)
\]
So since
\[
w = \left( x + \frac{x}{x^2 + y^2} \right) + i \left( y - \frac{y}{x^2 + y^2} \right)
\]
we have
\[
\phi(x,y) = \frac{1}{\pi} \arctan \left( \frac{x + \frac{x}{x^2 + y^2} - 2}{y - \frac{y}{x^2 + y^2}} \right) - \frac{1}{\pi} \arctan \left( \frac{x + \frac{x}{x^2 + y^2} + 2}{y - \frac{y}{x^2 + y^2}} \right)
\]
Problem 5.a

We know that by the max/min principle, $f$ is analytic and has no zeros inside $|z| \leq 1$ implies that $g(z) = 1/f(z)$ is analytic and again by the min/max principle, $|g(z)| = |1/f(z)| \leq 1$. This implies that $|f(z)| = 1$ in $|z| \leq 1$. Therefore recall that if $|f|, Re(f)$, or $Im(f)$ is constant, then $f$ is constant. Notice if $|f| = 1$, then $f = 1/\overline{f}$ and so $\overline{f}$ is analytic. This implies $g = f + \overline{f}$ is analytic and $Im(g) = 0$. Then by Cauchy Riemann equations, we can see that $g$ is constant and hence $f$ is constant. So $f = e^{i\beta}$.

Problem 5.b

Then let $g(z) = f(z)/z^m$. Then $g$ is analytic in $|z| < 1$ and we can apply part (a) to see that $f(z) = z^m e^{i\beta}$.

Jan 1997 Linear Algebra

Problem 1

The operator for $xd/dx$ in matrix form is the $(n + 1) \times (n + 1)$ matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n+1 \end{pmatrix}$$

And so the eigenvalues are $\lambda \in \{1, 2, \ldots, n, n+1\}$ with eigenvectors $e_i$ for $i \leftarrow 1$ to $n + 1$. Now for the operator $d/dx$ we have the matrix form

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & n \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Thus the eigenvalues are all 0 with eigenvector $\alpha e_1$ where $\alpha \in \mathbb{R}$.

Problem 2.a

Let $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the projection on the $x$ axis and $P_2 = (1/2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ be the projection on the $y = x$ line. Then

$$P_1 P_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = A$$

But $A^2 = 1/2A \neq A$. Hence it is not a projection.
Problem 2.b
For $P_2 P_1$ to be a projection, we must have $(P_2 P_1)^2 = P_2 P_1 P_2 P_1 = P_2 P_1$. This implies we must have either $P_1$ and/or $P_2 = I$, or $M_1 \perp M_2$, or $P_1 = P_2$. Then we would have $P_2 P_1$ be a projection.

Problem 3
We will apply Gram Schmidt on $e_1, e_2, e_3$. Then

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$u_2 = e_2 - (e_2, u_1) u_1$$

Since $(e_2, u_1) = 0$, after normalizing it, we have

$$u_2 = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Finally

$$u_3 = e_3 - (e_3, u_1) u_1 - (e_3, u_2) u_2$$

and since $(e_3, u_1) = 1/\sqrt{2}$ and $(e_3, u_2) = 0$, we have

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

after normalizing it, we have

$$u_3 = \sqrt{\frac{2}{3}} \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

Problem 4
We are given that

$$A \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$A \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$A \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

So after some manipulation, we have

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \\ -1/6 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5/6 \\ 0 \\ 1/6 \end{pmatrix}$$
So

\[ A = \begin{pmatrix}
\frac{1}{2} & \frac{-5}{6} & \frac{1}{6} \\
\frac{1}{3} & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & \frac{1}{3}
\end{pmatrix} \]

Since \( v_1, v_2, v_3 \) are linearly independent, we have \( S(v_1, v_2, v_3) = \mathbb{R}^3 \). So for any \( v = a_1v_1 + a_2v_2 + a_3v_3 \) we have

\[ A^n v = a_1 \left( \frac{1}{2} \right)^n v_2 + a_2 \left( \frac{1}{2} \right)^n + a_3 v_3 \]

so as \( n \to \infty \), we have

\[ A^n v = \begin{pmatrix}
\begin{array}{c}
-a_3 \\
a_3 \\
a_3
\end{array}
\end{pmatrix} = \begin{pmatrix}
\begin{array}{ccc}
0 & 0 & -1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{array}
\end{pmatrix} \begin{pmatrix}
\begin{array}{c}
a_1 \\
a_2 \\
a_3
\end{array}
\end{pmatrix} \]

**Problem 5.a**

**TRUE:** The fact that \( Ax = b \) has no solutions doesn’t really matter. Recall that

\[ R(A) + n(A) = n \]

Notice \( R(A) \leq m < n \) since \( A \) is an \( m \times n \) matrix with \( m \leq n \). Thus \( n(A) = n - R(A) > 0 \). Thus \( Ax = 0 \) has infinitely many solutions.

**Problem 5.b**

**FALSE:** Notice \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), which has no solutions. However

\[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

has only one solution (\( x = 0 \)).

**Problem 5.c**

**FALSE:** Notice that \( b \in \mathbb{R}^m \). So if \( A \) has \( m \) linearly independent column vectors (which is possible since \( m < n \)) say \( (v_1, v_2, \ldots, v_m) \). Then we have

\[ S(v_1, \ldots, v_m) = \mathbb{R}^m \]

So this implies \( b \in S(v_1, \ldots, v_m) \) and so \( \exists \) a solution for \( Ax = b \).

**Problem 5.d**

**TRUE:** If we are given that \( Ax = 0 \) has infinitely many solutions, this implies that \( n(A) > 0 \). Now we have the equation \( R(A) + n(A) = n \), and we also know that \( R(A) < n < m \), this implies \( A \) has at most \( i < n < m \) linearly independent column vectors \( v_1, \ldots, v_i \). Hence since \( i < m \), \( S(v_1, \ldots, v_i) \subseteq R^g \). Thus \( \exists b \in \mathbb{R}^m \), such that \( b \notin S(v_1, \ldots, v_i) \).
Sept 1997 Advanced Calculus

Problem 1

Since $E(u) \leq E(u + \phi)$, Then pick $\phi = \epsilon \phi$. Then let

$$g(\epsilon) = E(u + \epsilon \phi)$$

Then $g'(0) = 0$, since $E$ is smooth and it minimizes at $\epsilon = 0$. Then

$$g'(\epsilon) = \int \int \left( \frac{dv}{dx} + \epsilon \frac{d\phi}{dx} \right)^2 + \left( \frac{dv}{dy} + \epsilon \frac{d\phi}{dy} \right)^2 \ dx \ dy$$

$$= \int \int \left( \frac{dv}{dx} \right)^2 + 2\epsilon \frac{dv}{dx} \frac{d\phi}{dx} + \epsilon^2 \left( \frac{d\phi}{dx} \right)^2 + \left( \frac{dv}{dy} \right)^2 + 2\epsilon \frac{dv}{dy} \frac{d\phi}{dy} + \epsilon^2 \left( \frac{d\phi}{dy} \right)^2 \ dx \ dy$$

Now all are smooth and defined on a compact set. So when we differentiate with respect to $\epsilon$,

$$g'(0) = 2 \int \int \left( \frac{du}{dx} \right) \left( \frac{d\phi}{dx} \right) + \left( \frac{du}{dy} \right) \left( \frac{d\phi}{dy} \right) \ dx \ dy = 2 \int \int \nabla u \cdot \nabla \phi \ dx \ dy = 0$$

Now just like integration by part, notice by the divergence theorem

$$\int_{\delta(D)} \phi \nabla u \cdot dn = \int \int _D \nabla \cdot (\phi \nabla u) \ dx \ dy = \int \int _D (\nabla \phi \cdot \nabla u + \phi \nabla^2 u) \ dx \ dy$$

which implies

$$\int_{\delta(D)} \phi \nabla u \cdot dn - \int \int _D \phi \nabla^2 u \ dx \ dy = \int \int _D (\nabla \phi \cdot \nabla u) \ dx \ dy$$

Since $\phi = 0$ on $\delta(D)$, we have

$$\int \int _D \phi \Delta u \ dx \ dy = 0$$

Since this is true for arbitrary $\phi$ implies $\Delta u = 0$.

Problem 2

Recall that $\cos(mx + nx) = \cos(mx) \cos(nx) - \sin(mx) \sin(nx)$ and $\cos(mx - nx) = \cos(mx) \cos(nx) + \sin(mx) \sin(nx)$. Thus

$$\frac{1}{2} \cos(mx - nx) + \cos(mx + nx) = \cos(mx) \cos(nx)$$

which implies

$$\int \cos(mx) \cos(nx) \ dx = \frac{\sin(mx + nx)}{2(m + n)} + \frac{\sin(mx - nx)}{2(m - n)} + C$$

Also recall that $\cos(2x) = (1/2)(1 + \cos(2x))$. Now
\[ \int_{-1}^{1} \cos^4(x) \cos(nx) dx = \int_{-1}^{1} \frac{1}{4}(1+\cos(2x))^2 \cos(nx) dx = \int_{-1}^{1} \left( \frac{1}{4} + \frac{1}{2} \cos(2x) + \frac{1}{4} \cos^2(2x) \right) \cos(nx) dx \]

\[ = \int_{-1}^{1} \frac{1}{4} \cos(nx) dx + \int_{-1}^{1} \frac{1}{2} \cos(2x) \cos(nx) dx + \frac{1}{4} \int_{-1}^{1} \cos^2(2x) \cos(nx) dx \]

and we can break down

\[ \frac{1}{4} \int_{-1}^{1} \cos^2(2x) \cos(nx) dx = \frac{1}{4} \int_{-1}^{1} \frac{1}{4}(1+\cos(4x)) \cos(nx) dx = \frac{1}{16} \int_{-1}^{1} \cos(nx) dx + \frac{1}{16} \int_{-1}^{1} \cos(4x) \cos(nx) dx \]

By what we have proven above, all integrals will converge to 0 as \( n \to \infty \). Hence

\[ \lim_{n \to \infty} \int_{-1}^{1} \cos^4(x) \cos(nx) dx = 0 \]

**Problem 4**

Notice that \((1+u^2/n)^n \nrightarrow e^{u^2}\). Thus by applying the alternating series theorem, we series converges pointwise. However it does not converge absolutely since

\[ \sum \frac{1}{\sqrt{n(1+u^2/n)^n}} \geq \sum \frac{1}{\sqrt{n}e^{u^2}} \to \infty \]

Now to show that it converges uniformly we will apply Abel’s Test: let

\[ \sum \frac{1}{\sqrt{n(1+u^2/n)^n}} = \sum a_n b_n(u) \]

where \( b_n(u) = \frac{(-1)^n}{n^{1/4}(1+u^2/n)^n} \) and \( a_n = 1/n^{1/4} \). Then \( \sum b_n(u) \) converges by the alternating series theorem, and \( a_n \searrow 0 \). Hence it converges. Notice we really couldn’t apply Weierstrass \( M \)-test. Now recall Theorem 21 in Buck: Let \( \sum u_n(x) \) converge to \( F(x) \) for \( x \in [a,b] \), let \( u'_n(x) \) exist and be continuous in \( x \in [a,b] \) and let \( \sum u'_n(x) \) converge uniformly on \( [a,b] \). Then

\[ \sum u'_n(x) = F'(x) \]

So we need to differentiate each term and see if it converges uniformly. Thus

\[ \frac{d}{du} \left( \frac{(-1)^n \left( 1 + \frac{u^2}{n} \right)^{-n}}{\sqrt{n}} \right) = \frac{(-1)^{n+1}2u}{\sqrt{n}(1+u^2/n)^{n+1}} \]

Again we will use Abel’s theorem since

\[ \sum \frac{(-1)^{n+1}2u}{(1+u^2/n)^{n+1/4}} < \infty \]

by the alternating series theorem, and \( 1/n^{1/4} \searrow 0 \). Thus it converges uniformly and so \( \phi \) is differentiable on \( \mathbb{R} \).
Problem 5

So notice we can parameterize $c_1(t)$ in polar coordinates by

$$r_1(t) = t = \frac{\theta}{\pi}$$

and

$$r_2(t) = t + 1 = \frac{\theta}{\pi} + 1$$

for $\theta : 0 \rightarrow 4\pi$. Let $L : (x_1, y_1) - (x_2, y_2)$ be the line segment from $(x_1, y_1)$ to $(x_2, y_2)$. So graphically, $\Omega$ is the spiral curve with thickness 1 that starts at $L : (-2, 0) - (-1, 0)$ and spirals downward counterclockwise to the line $L : (0, -3/2) - (0, -5/2)$. Then as we continue in the counterclockwise direction, we go to the line $L : (2, 0) - (3, 0)$, then up to $L : (0, 5/2) - (0, 7/2)$, then over to $L : (-4, 0) - (-3, 0)$, down to $L : (0, -7/2) - (0, -9/2)$ then finally up to $L : (4, 0) - (5, 0)$.

The area of $\Omega$ we can compute easily since the equation for polar integration is

$$\int_{\theta}^{4\pi/\pi} 1/2((r_1(\theta))^2 - (r_2(\theta))^2) d\theta$$

so we have

$$A(\Omega) = \int_{0}^{4\pi/\pi} \left(\frac{\theta}{\pi} + 1\right)^2 - \left(\frac{\theta}{\pi}\right)^2 d\theta = \int_{0}^{4\pi/\pi} 1/2 \left(\frac{2\theta}{\pi} + 1\right) d\theta = 10\pi$$

Sept 1997 Complex Analysis

Problem 1

First notice for $\alpha \in \mathbb{C}$,

$$\left|\frac{z - \alpha}{1 - \alpha z}\right| = 1$$

This can be seen by letting $z = x + iy$ and $\alpha = a + ib$ and showing $|z - \alpha| = |1 - \alpha z|$. So let $g(z) = f(Rz)$. Then $g$ is analytic in $|z| < 1$ and continuous for $|z| \leq 1$. So $g$ has zeros at $z_1/R, z_2/R, \ldots, z_N/R$ with

$$0 < \left|\frac{z_i}{R}\right| < 1$$

Then let

$$h(z) = \prod_{i=1}^{N} \frac{g(z)}{\left|\frac{z - z_i}{1 - \alpha z}\right|}$$

has no zeros inside the unit circle and $|h(z)| = 1$ on $|z| = 1$. Thus $|h(z)| = 1$ on $|z| \leq 1$ implies that $h(z)$ is constant on $|z| \leq 1$. So $h(z) = e^{i\beta}$. So
\[ g(z) = \prod_{i=1}^{N} \left( \frac{z - z_i}{1 - \frac{N}{R} z} \right) e^{i\beta} \]

Therefore

\[ f(z) = \prod_{i=1}^{N} \left( \frac{z - z_i}{1 - \frac{z}{z_i} R} \right) e^{i\beta} \]

**Problem 2**

We first map \( D = \{ z : |z| < 1, 0 < \arg(z) < 2\pi/3 \} \) to the upper-half unit semi-circle by \( w_1 = z^{3/2} \).

Then we map that to the first quadrant by

\[ w_2 = i \left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right) \]

then map the first quadrant to the UHP by

\[ w_3 = - \left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 \]

Finally we map the UHP to the interior of the unit circle by

\[ w_f = \frac{- \left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 - i}{- \left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 + i} \]

**Problem 3.a**

Let \( \gamma \) be the rectangular contour from \(-N\) to \(N\) to \(N + i2\pi\) to \(-N + i2\pi\) to \(-N\). Then let

\[ f(z) = \frac{e^{az}}{1 + e^z} \]

Then by the residue theorem, we have

\[ \int_{\gamma} f(z) dz = 2\pi i \text{Res}(f, i) = 2\pi i \frac{e^{ai\pi}}{e^{i\pi}} = \frac{2\pi i}{e^{i\pi(1-\epsilon)}} \]

now on the contour where \( z = N + iy \) for \( y : 0 \to 2\pi \), we have

\[ \left| i \int_{0}^{2\pi} \frac{e^{aN+ayi}}{1 + e^{N+iy}} dy \right| \leq 2\pi \frac{e^{aN}}{e^{N} - 1} \to 0 \]

since \( a < 1 \). Likewise on the contour where \( z = -N + iy \) for \( y : 2\pi \to 0 \), we have

\[ \left| i \int_{2\pi}^{0} \frac{e^{-aN+ayi}}{1 + e^{-N+iy}} dy \right| \leq 2\pi \frac{e^{-aN}}{e^{-N} - 1} \to 0 \]
since $a > 0$. Thus as $N \to \infty$, we have
\[
\int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^{z}} \, dz + \int_{-\infty}^{\infty} \frac{e^{a(x+2\pi i)}}{1 + e^{x+2\pi i}} \, dx = (1 - e^{a2\pi i}) \int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^{z}} \, dz = \frac{2\pi i}{e^{i\pi(1-a)}}
\]
therefore
\[
\int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^{z}} \, dz = \frac{2\pi i}{e^{i\pi(1-a)}} \left( 1 - e^{a2\pi i} \right)
\]

**Problem 3.b**

The problem must be a typo, because it does not converge. Now let’s assume the problem is to solve
\[
\int_{0}^{\infty} \frac{x^b \log x}{1 + x^2} \, dx
\]
Then the $\gamma$ be the key hold contour, and
\[
f(z) = \frac{z^b \log z}{1 + z^2}
\]
where we define $\log z = \ln |z| + i \arg z$ where $0 < \arg z < 2\pi$. Hence we are going to cut out the nonnegative real axis. Then by the Residue Theorem, we have
\[
\int_{\gamma} f(z) \, dz = 2\pi i \text{Res}(f, i) + 2\pi i \text{Res}(f, -i) = \pi e^{i\pi b/2} i\pi/2 - \pi e^{b3\pi i/2} i3\pi/2
\]
Now notice on $C_R$, we have
\[
\left| \int_{C_R} f(z) \, dz \right| \leq 2\pi R \frac{R^b \log R}{R^2 - 1} \to 0
\]
as $R \to \infty$ and
\[
\left| \int_{C_0} f(z) \, dz \right| \leq 2\pi e \frac{\log \epsilon}{\epsilon^2 - 1} \to 0
\]
Thus for $z = re^{i\delta}$ as $\delta \to 0$, we have
\[
\lim_{\delta \to 0} \int_{0}^{\infty} \frac{r^b e^{bi\delta} (\log r + i\delta)e^{i\delta}}{1 + r^2 e^{2i\delta}} \, dr = \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dr
\]
and on $z = re^{i(2\pi - \delta)}$, we have
\[
\lim_{\delta \to 0} \int_{-\infty}^{0} \frac{r^b e^{ib(2\pi - \delta)} (\log r + i(2\pi - \delta))e^{i\delta}}{1 + r^2 e^{4i\pi + 2i\delta}} \, dr = \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dt + \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dr - \int_{0}^{\infty} \frac{r^b e^{i2\pi} i2\pi}{1 + r^2} \, dr
\]
By a similar argument we can see that
Thus

\[
\int_0^\infty \frac{r^b e^{ib2\pi i2\pi}}{1 + r^2} dr = \frac{\pi e^{bi\pi/2} - \pi e^{bi3\pi/2}}{1 - e^{i2\pi b}}
\]

\[
\int_0^\infty \frac{x^b \log x}{1 + x^2} dx = \frac{\pi e^{bi\pi/2} - \pi e^{bi3\pi/2} + \left(\frac{i\pi^2 x^{b/2}}{2} - \frac{i\pi^2 e^{3b/2}}{2}\right)(1 - e^{i2\pi b})}{\pi e^{bi\pi/2} - \pi e^{bi3\pi/2}}
\]

Sept 1997 Linear Algebra

Problem 1.a

Using Gaussian elimination, we have

\[
\begin{array}{ccc|c}
3 & 3 & 2 & 0 \\
3 & 6 & 3 & 1 \\
3 & 0 & 1 & 1 \\
\hline
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 1 \\
0 & -3 & -1 & 1 \\
\hline
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 1 \\
0 & 0 & 0 & 2 \\
\end{array}
\]

Hence we have 0 = 2, which implies there are no solutions to the system.

Problem 1.b

Using Gaussian elimination, we have

\[
\begin{array}{ccc|c}
3 & 3 & 2 & 0 \\
3 & 6 & 3 & 0 \\
3 & 0 & 1 & 0 \\
\hline
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 0 \\
0 & -3 & -1 & 0 \\
\hline
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Hence we have \( z = t \) and \( x = y = -t/3 \) for \( t \in \mathbb{R} \).

Problem 2.a

we have \( V = S((1, 0, 1)^T, (1, 1, 0)^T) \). Then let \( u_1 = (1/\sqrt{2})(1, 0, 1)^T \) and
\[ u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - ((1,1,0)^T, (1/\sqrt{2})(1,0,1)^T) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} 1/\sqrt{2} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} \]

**Problem 2.b**

\[ V^\perp = S((1,-1,-1)^T) \]

**Problem 2.c**

\[ V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \]

So \( P = V(V^TV)^{-1}V^T \)

Now

\[ P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \]

So \( P(3,0,0)^T = (2,1,1)^T \). Hence \( \sqrt{9 - 6} = \sqrt{3} \) is the distance from \( V \).

**Problem 2.d**

The perpendicular matrix of \( v \) is \( u \) where

\[ u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \]

Hence

\[ A = I - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} (1,-1,-1) = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \]

**Problem 3.a**

Notice

\[ \det(A - \lambda I) = \det \begin{pmatrix} s + 1 - \lambda & 1 - t \\ -1 - t & s - 1 - \lambda \end{pmatrix} = \lambda^2 - \lambda 2s + s^2 - 1 + 1 - t^2 \]

using the quadratic equation, we have the two roots

\[ \lambda_1 = s + t \quad \text{and} \quad \lambda_2 = s - t \]

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Problem 3.b
Remember that $A$ is diagonalizable if it has two linearly independent eigenvectors. Since different eigenvalues correspond to linearly independent eigenvectors, we need $s + t = s - t$ (since we don’t want to diagonalize $A$). Hence $t = 0$. Then we have

$$A = \begin{pmatrix} s + 1 & 1 \\ -1 & s - 1 \end{pmatrix}$$

Now the corresponding eigenvector(s) is

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus there is only 1 linearly independent eigenvector $v = (1, -1)$ that satisfies above. So $s \in R$ and $t = 0$ suffices.

Problem 3.c
If $t \neq 0$ then $A$ is diagonalizable. Hence

$$A = M \begin{pmatrix} s + t & 0 \\ 0 & s - t \end{pmatrix} M^{-1}$$

where $M = (v_1, v_2)$, and $v_1, v_2$ are the two linearly independent eigenvectors for $s + t$ and $s - t$ respectively. Now

$$A^k = M \begin{pmatrix} s + t & 0 \\ 0 & s - t \end{pmatrix}^k M^{-1} = M \begin{pmatrix} (s + t)^k & 0 \\ 0 & (s - t)^k \end{pmatrix} M^{-1}$$

Hence $\lim_k A^k$ exists when $|s + t| < 1$ and $|s - t| < 1$. Now if $t = 0$, then

$$A = \begin{pmatrix} s + 1 & 1 \\ -1 & s - 1 \end{pmatrix}$$

and $A$ is not diagonalizable. So the Jordan Form is

$$A = \begin{pmatrix} 1 & x \\ -1 & y \end{pmatrix} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & x \\ -1 & y \end{pmatrix}^{-1}$$

To solve for $v_2 = (x, y)$, we have

$$Av_2 = v_1 + sv_2 \Rightarrow (A - sI)v_2 = v_1$$

which implies

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence we have $v_2 = (0, 1)^T$. So
\[
A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}
\]
which implies
\[
A^k = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s^k & k s^{k-1} \\ 0 & s^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}
\]
Hence \(|s-t| < 1\) and \(|s+t| < 1\) suffices. Note that for any square matrix \(A\), if \(|\lambda_i| < 1\) for all \(i\), then \(A^k\) converges.

**Problem 5.a**

We know that \(\text{Rank}(A) \leq n < m\) and \(\text{Rank}(B) \leq n < m\). Also

\[
\text{Rank}(AB) \leq \min(\text{Rank}(A), \text{Rank}(B)) \leq n < m
\]
Also since

\[
\text{Rank}(AB) + n(AB) = m
\]
this implies \(n(AB) \neq 0\). Hence \(AB\) is singular. Note for rectangular \(A\), \(A^T A\) is nonsingular if the columns of \(A\) are linearly independent. Here we are in the opposite case.

**Problem 5.b**

Assume

\[
\text{Rank}(A + B) > \text{Rank}(A) + \text{Rank}(B)
\]
Now let \((a_1, ..., a_n)\) be the columns of \(A\) and \((b_1, ..., b_n)\) be the columns of \(B\). Let \(\text{Rank}(A) = i\) and \(\text{Rank}(B) = j\). WLOG let \(a_1, ..., a_i\) be the linearly independent column vectors of \(A\) and \(b_1, ..., b_j\) be the linearly independent column vectors of \(B\). Now if \(\text{Rank}(A + B) > \text{Rank}(A) + \text{Rank}(B)\), then \(\exists \) column \(x \in A + B\), such that \(s \not\in S(a_1, ..., a_i, b_1, ..., b_j)\). But then we have

\[
x \in S(a_{i+1}, ..., a_n, b_{j+1}, ..., b_n) \subset S(a_1, ..., a_i, b_1, ..., b_j)
\]
and so we have a contradiction.

**Problem 5.c**

We have \(A\) and \(B\) \(m \times n\) matrices, and

\[
\text{Rank}(A + B) + n(A + B) = n
\]
By above we know that \(\text{Rank}(A + B) \leq \text{Rank}(A) + \text{Rank}(B)\). Thus

\[
n(A+B) = n-\text{Rank}(A+B) \geq n-\text{Rank}(A)-\text{Rank}(B) = n-(n-\text{Rank}(A))-(n-\text{Rank}(B)) = n(A)+n(B)-n
\]
Jan 1998 Advanced Calculus

Problem 1

So we have \( g(\vec{x}) = C = x_1 x_2 \cdots x_n \) and 
\[
f(\vec{x}) = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_n}{x_n}
\]
and so by the Lagrange Multiplier theorem, we have \( \nabla f = \lambda \nabla g \) which implies 
\[
\left( \frac{-a_1}{x_1}, \frac{-a_2}{x_2}, \ldots, \frac{-a_n}{x_n} \right) = \lambda \left( x_2 x_3 \cdots x_n, x_1 x_3 x_4 \cdots x_n, \ldots, x_1 x_2 \cdots x_{n-1} \right)
\]
Which implies 
\[
\frac{-a_1}{x_1 C} = \frac{-a_2}{x_2 C} = \cdots = \frac{-a_n}{x_n C}
\]
which implies 
\[
\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_n}{x_n}
\]
So let \( x_1 = t \). Then \( x_2 = (a_2/a_1)t, x_3 = (a_3/a_1)t, \ldots, x_n = (a_n/a_1)t \). Since \( x_1 x_2 \cdots x_n = C \), we have 
\[
t^n \frac{(a_2 a_3 \cdots a_n)}{a_1^{n-1}} = C
\]
which implies 
\[
t = \frac{C_1/a_1}{(a_1 a_2 \cdots a_n)^{1/n}} = x_1
\]
\[
x_2 = \frac{C_1/a_2}{(a_1 a_2 \cdots a_n)^{1/n}}
\]
\[
\vdots
\]
\[
x_n = \frac{C_1/a_n}{(a_1 a_2 \cdots a_n)^{1/n}}
\]
So then the extrema is 
\[
f(\vec{x}) = \frac{n (a_1 a_2 \cdots a_n)^{1/n}}{C^{1/n}}
\]
Now this is clearly the minimum, since for \( \vec{b} = (b_1, b_2, \ldots, b_n) \) where \( b_i = C^{1/n} \forall i \), we have \( \vec{b} \in S \) and 
\[
f(\vec{b}) = \frac{a_1 + a_2 + \cdots + a_n}{C^{1/n}}
\]
and since $a_i > 0$ for all $i$, we know that the geometric mean of $a_i$ is less than or equal to the arithmetic mean. Hence

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

So

$$f(\mathbf{a}) = \frac{n(a_1 a_2 \cdots a_n)^{1/n}}{C^{1/n}} \leq \frac{n}{C^{1/n}} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_2 + \cdots + a_n}{C^{1/n}} = f(\mathbf{b})$$

So the extrema is indeed the minimum.

**Problem 2**

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x) dx \right|$$

$$= \left| \left( \frac{f(0)}{n} - \int_0^{1/n} f(x) dx \right) + \left( \frac{f(1/n)}{n} - \int_{1/n}^{2/n} f(x) dx \right) + \cdots + \left( \frac{f((n-1)/n)}{n} - \int_{(n-1)/n}^1 f(x) dx \right) \right|$$

by triangle inequality, we have

$$\leq \left| \frac{f(0)}{n} - \int_0^{1/n} f(x) dx \right| + \left| \frac{f(1/n)}{n} - \int_{1/n}^{2/n} f(x) dx \right| + \cdots + \left| \frac{f((n-1)/n)}{n} - \int_{(n-1)/n}^1 f(x) dx \right|$$

Now let $i$ be the maximum of partials. Then notice

$$\left| \frac{f(i/n)}{n} - \int_{i/n}^{(i+1)/n} f(x) dx \right| \leq \left( \frac{M}{n} \right) \frac{1}{n} \frac{1}{2}$$

Thus

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x) dx \right| \leq n \left( \frac{M}{n} \right) \frac{1}{n} \frac{1}{2} = \frac{M}{2n}$$

**Problem 3**

Notice that the center of the circle is at $(t/3,t/3,t/3)$. Now the distance to the origin is $t/\sqrt{3}$. Since the sphere has radius 1, the distance from the center of $S$ to the edge is

$$\sqrt{1 - \frac{t^2}{3}}$$

Now to take advantage of $x^2 + y^2 + z^2 = \rho^2$, let’s define $\rho$ in terms of $r$ where $r$ is the radial distance of $S$. Again by geometry, we have on $S$
\[ x^2 + y^2 + z^2 = r^2 + \frac{t^2}{3} \]

Hence

\[ \int \int_S (1 - (x^2 + y^2 + z^2))dA = \int_0^{2\pi} \int_0^{\sqrt{1-t^2/3}} (1 - r^2 - \frac{t^2}{3})rdrd\theta = \frac{\pi}{18}(3 - t^2)^2 \]

**Problem 4.a**

Notice that

\[ (1 - \cos 1/x) \sim \frac{1}{x^2} \]

indeed

\[ \lim_{x \to \infty} \frac{1 - \cos(1/x)}{x} = \lim_{u \to 0} \frac{1 - \cos u}{u^2} = \lim_{u \to 0} \frac{\sin u}{2u} = \lim_{u \to 0} \frac{\cos u}{2} = \frac{1}{2} \leq 1 \]

So for large enough \( N \) we have for all \( n > N \)

\[ 1 - \cos(1/x) \leq \frac{1}{x^2} \]

Hence

\[ \int_1^\infty x(1 - \cos(1/x))^{\beta} \, dx \leq \int_1^N x(1 - \cos(1/x))^{\beta} \, dx + \int_N^\infty x \left( \frac{1}{x^2} \right)^{\beta} \, dx = K \int_N^\infty \frac{1}{x^{2\beta-1}} \, dx \]

Which converges for \( \beta > 1 \). Now notice

\[ \lim_{x \to \infty} \frac{1 - \cos(1/x)}{(1/4)(1/x^2)} = \lim_{u \to 0} \frac{1 - \cos u}{(1/4)u^2} = \lim_{u \to 0} \frac{\sin u}{(1/2)u} = \lim_{u \to 0} \frac{\cos u}{1/2} = 2 \geq 1 \]

So for large enough \( x \) we have

\[ 1 - \cos(1/x) \geq \frac{1}{4} \left( \frac{1}{x^2} \right) \]

Hence

\[ \int_1^\infty x(1 - \cos(1/x))^{\beta} \, dx \geq \int_1^N x(1 - \cos(1/x))^{\beta} \, dx + \frac{1}{4} \int_N^\infty \frac{1}{x^{2\beta-1}} \, dx \]

which diverges for \( \beta \leq 1 \).
Problem 4.b

Notice

\[ \left| \frac{x^j}{j^2(1 + x^j)} \right| \leq \frac{1}{j^2} = M_j \]

and \( \sum M_j < \infty \). SO by the Weierstrass M-test, we have

\[ \sum_{j=1}^{\infty} \frac{x^j}{j^2(1 + x^j)} \]

converges uniformly for \( x \geq 0 \). Now for \( |x| \leq a < 1 \), we have

\[ \left| \frac{x^j}{j^2(1 + x^j)} \right| \leq \frac{|x|^j}{j^2(|x|^j - 1)} \leq \frac{a}{j^2(1 - a)} = M_j \]

\( \sum M_j < \infty \) and by the Weierstrass M-test, it converges uniformly.

Problem 5.a

since \( 0 \leq 2|u| \), we have

\[ 1 + u^2 \leq 1 + 2|u| + u^2 \Rightarrow 1 + u^2 \leq (1 + |u|)^2 \Rightarrow \sqrt{1 + u^2} \leq 1 + |u|^2 \]

Jan 1998 Complex Variables

Problem 2

Let \( \gamma \) be the contour with outer radius \( R \) and inner radius \( \epsilon \) in the upper half plane. Also let \( f(z) = \frac{1}{(1+z^2)^{\frac{1}{2}}} \). Then by the residue theorem, we have

\[ \int_{\gamma} f(z) \, dz = 2\pi i \frac{1}{2i\sqrt{i}} = \pi \sqrt{i} = \frac{\pi}{\sqrt{2}} (1 - i) \]

Notice on the outer radius

\[ \left| \int_{|z|=R} f(z) \, dz \right| \leq \pi R \frac{1}{\sqrt{R}(R^2 - 1)} \to 0 \]

as \( R \to \infty \). Likewise on the inner radius we have

\[ \left| \int_{|z|=\epsilon} f(z) \, dz \right| \leq \pi \frac{\sqrt{\epsilon}}{1 - \epsilon^2} \to 0 \]

as \( \epsilon \to 0 \). Hence as \( R \to \infty \) and \( \epsilon \to 0 \), we have

\[ \int_{-\infty}^{0} f(z) \, dz + \int_{0}^{\infty} f(z) \, dz = (1 - i) \int_{0}^{\infty} \frac{dz}{(1 + z^2)^{\frac{1}{2}}} = \frac{\pi \sqrt{2}}{\sqrt{2}} (1 - i) \]

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Hence

\[ \int_{0}^{\infty} \frac{dx}{(1+x^2)\sqrt{x}} = \frac{\pi}{\sqrt{2}} \]

**Problem 3.a**

Let

\[ g(z) = \frac{(z-z_1)(z-z_2) \cdots (z-z_m)f(z)}{z} \]

then \( g \) is an entire function. So it has a taylor series about \( z = 0 \)

\[ g(z) = a_0 + a_1z + a_2z^2 + \cdots \]

Also we know that

\[ \lim_{z\to0} |g(1/z)| < \infty \]

Hence \( g(z) = a_0 \). Therefore

\[ f(z) = \frac{a_0z}{(z-z_1)\cdots(z-z_m)} \]

**Problem 3.b**

By part (a), \( \text{deg}(P) = 1 \) and \( \text{deg}(Q) = m \).

**Problem 4**

We start by defining

\[ (g(z))^{2/3} = e^{\frac{2}{3} \log(g(z))} = e^{\frac{2}{3}(\ln(|g(z)|)+i \arg(g(z)))} \]

where

\[ -\frac{3\pi}{2} < \arg(g(z)) < \frac{\pi}{2} \]

Hence now \( \Phi \) is analytic in the UHP. So notice by Schwartz-Christoffel this is an equilateral triangle. We have

\[ \Phi(-\infty) = 0 \]

and

\[ \Phi(0) = \int_{-\infty}^{0} \frac{dx}{(x)\sqrt{2/(1-x)^{2/3}}} = \int_{0}^{\infty} x^{-2/3}(1+x)^{-2/3} dx > 0 \]

So let
\[ C = \int_{0}^{\infty} x^{-2/3}(1 + x)^{-2/3} \, dx \]

So \((-\infty, 0] \mapsto [0, C].\) Then for \(x \in (0, 1),\) we have
\[ \Phi'(x) = (-x)^{-2/3}(1 - x)^{-2/3} \]
and
\[ \text{arg}(\Phi'(x)) = (-\pi)(-2/3) + 0 = 2\pi/3 \]

And for \(x > 1\) we have
\[ \text{arg}(\Phi(x)) = (-\pi)(-2/3) + (-\pi)(-2/3) = 4\pi/3 \]

So we have an equilateral triangle with vertices \((0, C, C/2 + iC\sqrt{3}/2).\) By the orientation of the mapping, the UHP maps to the interior of the triangle.

**Problem 5**

We will use Rouche’s Theorem. Notice for \(|z| = 2n,\) we have
\[ \left| P_n(z) - \frac{z^n}{n!} \right| = \left| P_{n-1} \right| \leq \sum_{k=0}^{n-1} \frac{(2n)^k}{k!} < \frac{(2n)^n}{n!} = \left| \frac{z^n}{n!} \right| \]

Hence there are \(n\) zeros inside \(|z| < 2n.\) By the fundamental theorem of algebra, this means that all the zeros are inside the circle. Now to see the strict inequality notice
\[ 1 + 2 + 2^2 + \cdots + 2^{n-1} < 2^n \]

Hence
\[ \frac{n^n}{n!} \left(1 + 2 + 2^2 + \cdots + 2^{n-1}\right) < \frac{2^n n^n}{n!} \]

which implies
\[ \frac{n^n}{n!} + \frac{2n^n}{n!} + \cdots + \frac{2^{n-1}n^n}{n!} < \frac{2^n n^n}{n!} \]

and
\[ \frac{n^n}{n!} \geq 1 \quad \frac{2n^n}{n!} \geq 2n \quad \cdots \quad \frac{2^{n-1}n^n}{n!} \geq \frac{2^{n-1}n^{n-1}}{(n-1)!} \]

So
\[ \sum_{k=0}^{n-1} \frac{(2n)^k}{k!} < \frac{(2n)^n}{n!} \]
Jan 1998 Linear Algebra

Problem 1

Notice we have

\[ \det(A - \lambda I) = \det \begin{pmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{pmatrix} = (3 - \lambda)^2 - 4 = (\lambda - 5)(\lambda - 1) \]

Hence \( \lambda_1 = 5 \) and \( \lambda_2 = 1 \). Notice

\[ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} \]

Hence we have \( 3x + 2y = 5x \), which implies \( x = y \). Thus our eigenvector \( u_1 = (1, 1) \). Also notice

\[ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \]

implies \( 3x + 2y = x \), and thus \( x = -y \). Therefore our second eigenvector \( u_2 = (1, -1) \). Likewise for matrix \( B \) we have

\[ \det \begin{pmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{pmatrix} = (4 - \lambda)^2 - 1 = (\lambda - 3)(\lambda - 5) \]

Hence for our matrix \( B \), we have eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 5 \). Notice

\[ \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \]

Hence we have \( 4x + y = 3x \), which implies \( x = -y \). Thus our eigenvector \( u_1 = (1, -1) \). Also notice

\[ \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} \]

implies \( 4x + y = 5x \), and thus \( x = y \). Therefore our second eigenvector \( u_2 = (1, 1) \). Now notice

\[ x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Hence

\[ A^N x = A^N \left( \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{5^{N+1}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left( \frac{5^{N+1} + 1}{2} \right) \left( \frac{5^{N^2 - 1}}{2} \right) \]

and

\[ B^N x = B^N \left( \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{5^{N+1}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3^N}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left( \frac{5^{N+1} + 3^N}{2} \right) \left( \frac{5^{N^2 - 3^N}}{2} \right) \]

Therefore

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\[ \|A^N x\| = \frac{\sqrt{2 \cdot 5^{2N+2} + 2}}{2} \]

and

\[ \|B^N x\| = \frac{\sqrt{2 \cdot 5^{2N+2} + 2 \cdot 3^{2N}}}{2} \]

Finally

\[ \lim_{n \to \infty} \frac{\|A^N x\|}{\|B^N x\|} = 1 \]

**Problem 1.b**

By part (a), we can see that \(\|B^N x\|\) is greater than \(\|A^N x\|\) for large \(N\).

**Problem 2.a**

Notice that \(\det(A - \lambda I) = (4 - \lambda)(5 - \lambda)^2\). Thus we know that there are 2 linearly independent eigenvectors corresponding to 4 and 5. Notice

\[
(A - 5I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Hence we have \(-x - 2z = 0\) and \(2x + 4z = 0\). So notice we have two linearly independent eigenvectors \(v_1\) and \(v_2\) with eigenvalues of 5

\[
v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]

Thus \(A\) is diagonalizable and

\[
A = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 & 1 \\ 0 & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -\sqrt{5} & 0 & -2\sqrt{5} \\ 0 & 0 & -\sqrt{5} \\ 0 & 0 & 1 \end{pmatrix} = SDS^{-1}
\]

**Problem 2.b**

Notice we have

\[ \det(B - \lambda I) = (5 - \lambda)^2(4 - \lambda) \]

By inspection we can see that \(B\) is not diagonalizable, and hence the Jordan matrix is

\[
J = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\]
Hence for $M = (v_1, v_2, v_3)$, $v_1$ and $v_3$ are eigenvectors. Now for $\lambda = 5$, we have

$$B - 5I = \begin{pmatrix} -1 & 0 & -6 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies $-x - 6z = 0$ and $6x + 4z = 0$, which implies $v_1 = (0, 1, 0)^T$. Now for $\lambda = 4$ we have

$$\begin{pmatrix} 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which implies $-6z = 0$, $6x + y + 4z = 0$, and $z = 0$. Thus $v = 1/\sqrt{37}(-1, 6, 0)^T$. Now notice $Av_2 = v_1 + 5v_2$. Hence $(A - 5I)v_2 = v_1$,

$$\begin{pmatrix} -1 & 0 & -6 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence $x = 3/16$, $y = -1/32$, and $z = -1/32$. Hence

$$A = MJM^{-1} = \begin{pmatrix} 0 & 3_{16} & - \frac{1}{\sqrt{37}} \\ 1 & \frac{1}{\sqrt{37}} & \frac{5}{\sqrt{37}} \\ 0 & \frac{2}{\sqrt{37}} & 0 \end{pmatrix} \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 6 & 1 & 35 \\ 0 & 0 & -32 \\ -\sqrt{37} & 0 & -6\sqrt{37} \end{pmatrix}$$

**Problem 3.a**

$$\begin{pmatrix} a & b & c & d \\ 1/2 & e & f & g \\ 1/2 & 1/2 & h & i \\ 1/2 & 1/2 & 1/2 & j \end{pmatrix} \begin{pmatrix} a & 1/2 & 1/2 & 1/2 \\ b & e & 1/2 & 1/2 \\ c & f & h & 1/2 \\ d & g & i & j \end{pmatrix}$$

Clearly we must have $j = 1/2$. Then

$$1/2h^2 + i^2 = 1 \quad \text{and} \quad 1/2 + (1/2)h + (1/2)i = 0$$

Hence we must have $h = -1/2$ and $i = -1/2$. Then

$$1/4 + (1/2)e + (1/2)f + (1/2)g = 0 \quad \text{and} \quad 1/4 + (1/2)e - (1/2)f - (1/2)g = 0 \quad \text{and} \quad 1/4 + e^2 + f^2 + g^2 = 1$$

which implies we have $e = -1/2$, $f = 1/2$ and $g = -1/2$. Finally

$$(1/2)a + (1/2)b + (1/2)c + (1/2)d = 0 \quad \text{and} \quad (1/2)a + (1/2)b - (1/2)c - (1/2)d = 0$$

$$(1/2)a - (1/2)b + (1/2)c - (1/2)d = 0 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 = 1$$

which implies that $a = d = -1/2$ and $b = c = 1/2$. 

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Problem 3.b
By part (a) we can see that it is unique.

Problem 5.a
Clearly if \( \exists |\lambda_i| < 1 \) with eigenvector \( v_i \), then
\[
\frac{||A^N x||}{||x||} = \frac{||(\lambda_i)^N x||}{||x||} = |\lambda_i|^N < c_1
\]
for large enough \( N \), and thus we have a contradiction. Now if \( \exists |\lambda_j| > 1 \) with eigenvector \( v_j \) such that
\[
\frac{||A^N v_j||}{||v_j||} = \frac{||(\lambda_j)^N v_j||}{||v_j||} = |\lambda_j|^N > c_2
\]
for large enough \( N \), which implies we have a contradiction. Hence \( |\lambda_i| = 1 \) for all \( i \).

Problem 5.b
Assume \( A \) has the Jordan form \( A = MJM^{-1} \), and assume that \( J \) has a Jordan block (i.e. \( J \) is not diagonal). The \( \exists x_1, x_2 \) such that \( x_1 \neq x_2 \) and
\[
Ax_1 = \lambda_1 x_1 \quad \text{and} \quad Ax_2 = x_1 + \lambda_1 x_2
\]
Thus
\[
A^N x_2 = N \lambda_1^{N-1} x_1 + \lambda_1^N x_2
\]
Thus
\[
\frac{||A^N x_2||}{||x_2||} = \frac{||N \lambda_1^{N-1} x_1 + x_2||}{||x_2||} \to \infty
\]
as \( N \to \infty \). Hence we have a contradiction which implies that \( J \) has no Jordan blocks. Thus \( A = MJM^{-1} \) and \( J \) is unitary by part (a).

Sept 1998 Advanced Calculus

Problem 1.a
Notice that \( \frac{n^2+1}{n^3 \log n} \searrow 0 \) for \( n \geq 2 \). Hence by the alternating series theorem,
\[
\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^3 \log n}
\]
converges. However it does not converge absolutely. Notice
\[
\sum_{n=1}^{\infty} \frac{n^2+1}{n^3 \log n} \leq 2 \sum_{n=1}^{\infty} \frac{n^2}{n^3 \log n} = 2 \sum_{n=1}^{\infty} \frac{1}{n \log n} \to \infty
\]
Problem 1.b

be letting \( u = 1/x \) and \( du = -x^{-2} \), we have

\[
\int_0^\infty \frac{x^2 e^{1/x}}{1 + x^4} \, dx = \int_0^\infty \frac{-u^{-4} e^u}{1 + u^{-4}} \, du = \int_{u=0}^{\infty} \frac{e^u}{u^4 + 1} \, du
\]

we know that \( \int_{u=0}^{\infty} \frac{e^u}{u^4 + 1} \, du \) diverges since \( \frac{e^u}{u^4 + 1} \to \infty \) as \( u \to \infty \). Hence it diverges.

Problem 2

We have

\[
\frac{df}{dx} = \frac{dG}{du} \frac{du}{dx}
\]

\[
\frac{d^2 f}{dx^2} = \frac{d^2 G}{du^2} \left( \frac{du}{dx} \right)^2 + \frac{d^2 u}{dx^2} \frac{dG}{du}
\]

and

\[
\frac{df}{dy} = \frac{dG}{du} \frac{du}{dy}
\]

\[
\frac{d^2 f}{dy^2} = \frac{d^2 G}{du^2} \left( \frac{du}{dy} \right)^2 + \frac{d^2 u}{dy^2} \frac{dG}{du}
\]

So

\[
\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{d^2 G}{du^2} (4u) + 2 \frac{dG}{du} = H(u)
\]

Problem 3.a

Let

\[
L(m) = \prod_{n=2}^{m} \left[ (1 + a_n)e^{-a_n + a_n^2/2} \right]
\]

then

\[
\log(L(m)) = \sum_{n=2}^{\infty} \left( \log(1 + a_n) - a_n + a_n^2/2 \right)
\]

Now we are given that \( \sum |a_n|^3 < \infty \). Notice that \( a_n \to 0 \). So WLOG, we can assume that \( |a_n| < 1 \) for all \( n \), since otherwise we can just look at the tail. So recall the Taylor expansion

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\]

Let \( E(x) = -x^4/4 + \cdots \). Then

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So for $x < \delta$, $E(x) < x^3$. So
\[
\log(L(m)) = \sum_{n=2}^{m} a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} + E(a_n) - a_n + \frac{a_n^2}{2} = \sum_{n=2}^{m} \frac{a_n^3}{3} + E(a_n)
\]
So now we $N$ such that $n > N$ we have $a_n < \delta$. Then
\[
\leq \left| \sum_{n=2}^{N} \frac{a_n^3}{3} + E_n(a_n) \right| + \left| \sum_{N+1}^{m} \frac{a_n^3}{3} + a_n^3 \right|
\]
which converges since $\sum |a_n|^3 < \infty$.

**Problem 3.b**

Again let’s take the log.
\[
L(m) = \prod_{n=2}^{m} \left[ 1 + \frac{(-1)^n}{\sqrt{n}} \right]
\]
Hence
\[
\log(L(m)) = \sum_{n=2}^{\infty} \log \left( 1 + \frac{(-1)^n}{\sqrt{n}} \right) = \sum \left( \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^{2n}}{2n\sqrt{n}} + E \left( \frac{(-1)^n}{\sqrt{n}} \right) \right)
\]
Notice that the first, third, and fourth term converges, but the second term diverges. Hence
\[
\lim_{m} \log(L(m)) = -\infty,
\]
which implies $\lim_{m} L(m) = 0$.

**Problem 4**

Notice for $y = \sqrt{nx}$, we have
\[
\lim_{n \to \infty} \sqrt{n} \int_{-1}^{1} (1 - x^2)^n dx = \lim_{n \to \infty} \int_{-\sqrt{n}}^{\sqrt{n}} \left( 1 - \frac{y^2}{n} \right)^n dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} \chi([-\sqrt{n}, \sqrt{n}]) \left( 1 - \frac{y^2}{n} \right)^n dy
\]
Now I claim that I can apply the dominated converges theorem, since for all $n$
\[
\left| \chi([-\sqrt{n}, \sqrt{n}]) \left( 1 - \frac{y^2}{n} \right)^n \right| \leq e^{-y^2}
\]
This can be shown explicitly by noticing
\[
\log(1 - x) \leq -x
\]
Thus by DCT
\[
\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}
\]
Problem 5.a
Not true. Let \( f_n(x) = \sin(nx)/n \). Then pointwise \( f_n(x) \to 0 \). But \( f'_n(x) = \cos(nx) \neq 0 \).

Sept 1998 Complex Variables

Problem 1
Let \( \gamma \) be the unit circle. Then for \( z \in \gamma \), we have by Rouche’s Theorem
\[
|z^8 - 4z^5 + z^2 - 1 + 4z^5| = |z^8 + z^2 - 1| \leq |z^8| + |z^2| + 1 < 4 = 4z^5
\]
Hence there are 5 zeros with modulus less than one.

Problem 2
Let \( f(z) = (z^2 - z^4)/(1 - z^6) \), and let \( \gamma \) be the contour in the upper half of the plane with radius \( R \). Then by the residue theorem, we have
\[
\int_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{3} \text{Res}(f, z_k)
\]
where \( z_k \) are the three poles in the upper half of the plane. Notice on the outer radius
\[
\left| \int_{|z|=R} f(z) \, dz \right| \leq \frac{2\pi R(R^2 - R^4)}{R^6 - 1} \to 0
\]
as \( R \to \infty \). Thus as \( R \to \infty \)
\[
\int_{-\infty}^{\infty} f(z) \, dz = 2\pi i \left(0 + \frac{e^{i2\pi/3} - e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i4\pi/3} - e^{i8\pi/3}}{6e^{i10\pi/3}}\right)
\]

Problem 3.a
So we can define the square root with a branch cut on the negative real axis. Now for \( z \in [-\pi/2, \pi/2] \)
\( f(z) = \sqrt{\cos z} \) is analytic inside the circle \( |z| = \pi/2 \). This can be seen by integrating \( f(z) \) around the circle and see that the integral is zero. Hence By Morera’s Theorem, the function \( f \) is analytic inside the circle. Hence it has a Taylor expansion at \( z = 0 \)
\[
f(z) = \sqrt{\cos z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots
\]
Now we know that \( f(z)f(z) = \cos z \). Hence
\[
(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \pm \cdots
\]
Hence we know that \( a_0 = 1 \), and \( a_1 = 0 \), \( a_2 = -1/4 \), \( a_3 = 0 \), and \( a_4 = -1/96 \). Hence
\[
\sqrt{\cos z} = 1 - \frac{1}{4} z^2 - \frac{1}{96} z^4 \cdots
\]
**Problem 3.b**
Clearly the radius of convergence is $\pi/2$, since that’s where we have our branch point.

**Problem 4**

$$w = z^{\pi/\alpha} - iIm(z_0) + (t - Re(z_0))$$

**Problem 5**

We have \(f\) is entire with zeros at \(z_1, z_2, ..., z_n\) with multiplicity \(m_1, m_2, ..., m_n\). Recall that \(f\) has a zero at \(z_1\) with multiplicity \(m_1\) if \(f^{(m_1)}(z_1) \neq 0\) and \(f^{(i)}(z_1) = 0\) for \(i < m_1\). So then for

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_1)^{m_1}} & \text{if } z \in \mathbb{C} \setminus z_1 \\ a_{m_1} & \text{if } z = z_1 \end{cases}$$

Now notice that \(g\) is continuous and entire since

$$\lim_{z \to z_1} g(z) = \lim_{z \to z_1} \frac{f(z)}{(z-z_1)^{m_1}} = \frac{m_1!a_{m_1}}{m_1!} = a_{m_1}$$

where \(a_{m_1} \neq 0\). Also \(g(z) = 0\) if and only if \(f(z) = 0\) with the same multiplicity. By repeating the same argument, we have

$$f(z) = \prod_{i=1}^{n} (z - z_i)^{m_i} h(z)$$

where \(h(z)\) is entire and \(h(z) \neq 0\) for all \(z \in \mathbb{C}\). Hence \(1/h(z)\) is entire and \(h'(z)\) is entire. Then \(h'(z)/h(z)\) is entire. Now we define

$$g(z) = \int_0^z \frac{h'(\zeta)}{h(\zeta)} d\zeta$$

which is entire with \(g'(z) = h'(z)/h(z)\). Thus

$$\left(h(z)e^{-g(z)}\right)' = h'(z)e^{-g(z)} - g'(z)h(z)e^{-g(z)} = 0$$

which implies

$$h(z)e^{-g(z)} = c \Rightarrow h(z) = e^{g_2(z)}$$

where \(g_2(z)\) is entire. This completes the proof.
Problem 1

We have \( A + uv^T \) singular if and only if \( A^{-1}(A + uv^T) \) is singular. So notice \( A^{-1}(A + uv^T) = I + A^{-1}uv^T \) is singular implies that there exists non-zero vector \( x_1 \) such that

\[
(I + A^{-1}uv^T)x_1 = 0
\]

Hence

\[
x_1 + A^{-1}uv^Tx_1 = x_1 + (v^Tx_1)A^{-1}u = 0
\]

which implies \( A^{-1}u = -x_1/(v^Tx_1) \). So notice for \( x_2 = x_1/(v^Tx_1) \), \( x_2 \) is also in the null space. Therefore

\[
(I + A^{-1}uv^T)x_2 = 0 = x_2 + (v^Tx_2)A^{-1}u = 0
\]

and we must have \( A^{-1}u = -x_2 \). Therefore \( x_2 = -A^{-1}u \) and

\[
(I + A^{-1}uv^T)(-A^{-1}u) = 0 \Rightarrow -A^{-1}u - A^{-1}uv^T A^{-1}u = -1
\]

Problem 2

Recall that if two square matrices \( A \) and \( B \) are invertible implies that \( AB \) is invertible where \((AB)^{-1} = B^{-1}A^{-1} \). Hence \( A(A + B)^{-1}B \) and \( B(A + B)^{-1}A \) are both invertible and notice

\[
A(A + B)^{-1}B = B(A + B)^{-1}A \Leftrightarrow B^{-1}(A + B)A^{-1} = A^{-1}(A + B)B^{-1} \Leftrightarrow B^{-1} + A^{-1} = B^{-1} + A^{-1}
\]

which is indeed true.

Problem 3.a

Notice that

\[
(Fv, Fw) = \sum_{k=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \overline{v}_p w_q \overline{F}_{k,p} F_{k,q}
\]

Now notice that

\[
\sum_{k=1}^{N} \overline{F}_{k,p} F_{k,q} = \sum_{k=1}^{N} \frac{1}{N} e^{i2\pi k(p-q)/N} = 0
\]

for \( p \neq q \) and

\[
\sum_{k=1}^{N} \overline{F}_{k,p} F_{k,q} = \sum_{k=1}^{N} \frac{1}{N} e^{i2\pi k(p-q)/N} = 1
\]
for $p = q$. Thus

$$(Fv, Fw) = \sum_{k=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \tau_p \tau_q F_{k,p} F_{k,q} = \sum_{p=1}^{N} \tau_p w_p = (v, w)$$

**Problem 3.b**

By part (a) we can see that the inverse of $F$ is $F^H$.

**Problem 4.a**

Elimination subtracts 2 times row 1 from row 2, likewise 2 from 3, 3 from 4, 4 from 5. Hence

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = LL^T$$

**Problem 4.b**

Notice we have $\det(A) = \det(L) \det(L^T) = 1$

**Problem 4.c**

$A$ is indeed positive semi-definite. Recall that when a symmetric matrix has one of these four properties, it has all:

**Property 1:** All $n$ eigenvalues are positive

**Property 2:** All $n$ upper left determinants are positive

**Property 3:** All $n$ pivots are positive

**Property 4:** $x^T A x$ is positive except at $x = 0$. Then matrix $A$ is positive definite.

Hence as we seen above, all pivots of $A$ were 1’s, and thus positive definite. Or the long way is let $x = (x_1, x_2, x_3, x_4, x_5)$. Then we have

$$x^T A x = x_1(x_1 + 2x_2) + x_2(2x_1 + 5x_2 + 2x_3) + x_3(2x_2 + 5x_3 + 2x_4) + x_4(2x_3 + 5x_4 + 2x_5) + x_5(2x_4 + 5x_5)$$

$$= (x_1 + 2x_2)^2 + (x_2 + 2x_3)^2 + (x_3 + 2x_4)^2 + (x_4 + 2x_5)^2 + x_5^2$$

Hence it is clearly positive unless $x_i = 0$ for all $i$. Thus $A$ is positive-definite.

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Problem 5.a

Notice that \( \begin{pmatrix} 1 \\ 5 \end{pmatrix} \) is an eigenvector with eigenvalue of 2. Hence

\[
M^4 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 2^4 \begin{pmatrix} 1 \\ 5 \end{pmatrix}
\]

Problem 5.b

Notice that the eigenvectors span \( \mathbb{R}^2 \). Hence for any \( v \in \mathbb{R}^2 \) we have

\[
v = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix}
\]

Hence

\[
M^n \begin{pmatrix} a \\ b \end{pmatrix} = c_1 \left( \frac{1}{3} \right)^n \begin{pmatrix} a \\ b \end{pmatrix} + c_2 2^n \begin{pmatrix} a \\ b \end{pmatrix}
\]

Therefore

\[
\lim_{n \to \infty} M^n \begin{pmatrix} a \\ b \end{pmatrix} < \infty
\]

if \( c_2 = 0 \). Hence all the vectors in the form \((c_1, c_1)\) for \( c_2 \in \mathbb{R} \) the limit exists.

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Problem 1.a

Notice

\[
\sum_{k=n+1}^{m} u_k v_k = \sum_{k=n+1}^{m} (U_k - U_{k-1}) v_k
\]

\[
= (U_{n+1} - U_n) v_{n+1} + (U_{n+2} - U_{n+1}) v_{n+2} + (U_{n+3} - U_{n+2}) v_{n+3} + \cdots + (U_{m-1} - U_{m-2}) v_{m-1} + (U_m - U_{m-1}) v_m
\]

\[
= U_m v_m - U_n v_{n+1} + \sum_{k=n+1}^{m-1} U_k (v_k - v_{k+1}) = U_m v_m - U_n v_{n+1} - \sum_{k=n+1}^{m-1} U_k (v_{k+1} - v_k)
\]
Problem 1.b

To show that \( \sum a_n(x)b_n(x) \) converges uniformly on \( I \), it suffices to show that we can bound the tail \( |\sum_{N}^{\infty} a_n(x)b_n(x)| < \epsilon \) for all \( \epsilon > 0 \). Now let

\[ s_n(x) = a_1(x) + a_2(x) + \cdots + a_n(x) \]

then \( a_n(x) = s_n(x) - s_{n-1}(x) \). Since \( b_n(x) \downarrow 0 \) uniformly, and

\[ \sum |b_n(x) - b_{n-1}(x)| \] converges uniformly, implies \( \exists N \) such that \( \forall n \geq N \),

\[ |b_n(x)| < \frac{\epsilon}{3M} \quad \text{and} \quad \sum_{n=N+1}^{\infty} |b_n(x) - b_{n-1}(x)| < \frac{\epsilon}{3M} \]

So

\[
\left| \sum_{n=N+1}^{m} a_n(x)b_n(x) \right| = \left| b_m(x)s_m(x) - b_N(x)s_{N+1}(x) - \sum_{k=N+1}^{m-1} s_k(x)(b_{k+1}(x) - b_k(x)) \right|
\]

\[
\leq |b_M(x)s_m(x)| + |b_n(x)s_{N+1}(x)| + \left| \sum_{k=N+1}^{m-1} s_k(x)(b_{k+1}(x) - b_k(x)) \right|
\]

\[
\leq \frac{\epsilon}{3M}M + \frac{\epsilon}{3M}M + M \sum_{k=N+1}^{m-1} |b_{k+1}(x) - b_k(x)| \leq \epsilon
\]

Since \( |s_n(x)| < M \) for all \( n \). Hence this completes the proof. This is the proof of Abel’s Test:

1: The series \( \sum_{n=1}^{\infty} a_n(x) \) converges uniformly on \( X \)

2: For every \( x \in X \), the sequence \( b_n(x) \) is monotonic

3: There exists \( K \in \mathbb{R} \) such that \( |b_n(x)| \leq K \) for every \( n \in \mathbb{N} \) and every \( x \in X \)

Then

\[ \sum_{n=1}^{\infty} a_n(x)b_n(x) \]
converges uniformly on \( X \).
Problem 2.a

Notice

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k/n}{1 + (k/n)^2} = \int_{0}^{1} \frac{x}{1 + x^2} \, dx = \frac{1}{2} \ln 2
\]

Problem 3

Going right to left, notice by performing \(u\) substitution with \(u = \sqrt{y}\)

\[
\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{y^{-1/2}}{ae^y - 1} \, dy = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{2du}{ae^{u^2} - 1} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-u^2}}{1 - \frac{1}{a} e^{-u^2}} \, du
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2} \sum_{k=0}^{\infty} \frac{e^{-ku^2}}{a^k} \, du = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \int_{0}^{\infty} e^{-(k+1)u^2} \, du
\]

Now notice

\[
\int_{0}^{\infty} e^{-(k+1)u^2} \, du = \frac{\sqrt{\pi}}{2(k+1)^{3/2}}
\]

Hence

\[
= \sum_{n=1}^{\infty} a^{-n} n^{-1/2}
\]

Problem 5.a

Let \(dl = (dl_1, dl_2, dl_3)\) and \(F = (F_1, F_2, F_3)\). Then

\[
dl \times F = \begin{vmatrix}
    dl_1 \\
    F_1 \\
    F_2 \\
    dl_3 \\
    F_3
\end{vmatrix} = (dl_2 F_3 - dl_3 F_2, dl_3 F_1 - dl_1 F_3, dl_1 F_2 - dl_2 F_1)
\]

So

\[
\int_{\delta \Sigma} dl \times F = \left( \int_{\delta \Sigma} (0, F_3, -F_2) \cdot \, dl, \int_{\delta \Sigma} (-F_3, 0, F_1) \cdot \, dl, \int_{\delta \Sigma} (F_2, -F_1, 0) \cdot \, dl \right)
\]

By stokes theorem we have

\[
= \left( \int_{\Sigma} \nabla \times (0, F_3, -F_2) \cdot \, dS, \int_{\Sigma} \nabla \times (-F_3, 0, F_1) \cdot \, dS, \int_{\Sigma} \nabla \times (F_2, -F_1, 0) \cdot \, dS \right)
\]

Now for the other side let \(dS = (dS_1, dS_2, dS_3)\) we have

\[
dS \times \nabla = \begin{vmatrix}
    \frac{dS_1}{dx} \\
    \frac{dS_2}{dy} \\
    \frac{dS_3}{dz}
\end{vmatrix} = \left( \frac{dS_2}{dz} \frac{d}{dy} - \frac{dS_3}{dx} \frac{d}{dx} - \frac{dS_1}{dz} \frac{d}{dy} - \frac{dS_1}{dx} \frac{d}{dz} - \frac{dS_2}{dy} \frac{d}{dz} - \frac{dS_3}{dx} \frac{d}{dx} \right)
\]

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So \((dS \times \nabla) \times F = \)
\[
\begin{vmatrix}
  dS_2 \frac{\partial}{\partial x} - dS_3 \frac{\partial}{\partial y} & dS_3 \frac{\partial}{\partial x} - dS_1 \frac{\partial}{\partial y} & dS_1 \frac{\partial}{\partial y} - dS_2 \frac{\partial}{\partial x} \\
  F_1 & F_2 & F_3 
\end{vmatrix}
\]
which equals
\[
= \left( dS_3 \frac{dF_3}{dx} - dS_1 \frac{dF_3}{dz} - dS_1 \frac{dF_2}{dy} + dS_2 \frac{dF_2}{dx} , dS_1 \frac{dF_1}{dy} - dS_2 \frac{dF_1}{dx} - dS_2 \frac{dF_3}{dz} + dS_3 \frac{dF_3}{dy} ,
\right.
\]
\[
\left. dS_2 \frac{dF_2}{dz} - dS_3 \frac{dF_2}{dy} - dS_3 \frac{dF_1}{dx} + dS_1 \frac{dF_1}{dz} \right)
\]
So we have
\[
\int_{\Sigma} (dS \times \nabla) \times F = \int_{\delta \Sigma} dl \times F
\]

**Problem 5.b**

By Stokes Theorem, we have
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} \nabla \times (u \nabla v) \cdot dS
\]
Now recall that \(\nabla \times (f v) = (\nabla f) \times v + f (\nabla \times v)\). I.e. the \(\text{curl}(fv) = \text{grad}(f) \times v + f \text{curl}(v)\). (page 1174 Salas). Hence
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} \nabla \times (u \nabla v) \cdot dS = \int_{\Sigma} \left[(\nabla u) \times (\nabla v) + u(\nabla \times (\nabla v))\right] \cdot dS
\]
finally recall that the curl of a gradient is zero, i.e. \(\nabla \times (\nabla f) = 0\). Thus
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} \left[(\nabla u) \times (\nabla v) + u(\nabla \times (\nabla v))\right] \cdot dS = \int_{\Sigma} (\nabla u \times \nabla v) \cdot dS
\]

**Jan 1999 Complex Analysis**

**Problem 1.a**

Let \(g(z) = z^n - f(z)\). Then on the circle \(|z| = 2\), we have
\[
|z^n f(z) - z^n| = |f(z)| < 2^n = |z^n|
\]
which implies \(z^n - f(z)\) has \(n\) roots inside the circle \(|z| = 2\). Also notice on the unit circle, \(|z| = 1\), we have
\[
|z^n - f(z) - z^n| = |f(z)| < 1 = |z^n|
\]
which implies that all \(n\) roots are inside the unit circle according to Rouche’s Theorem. Hence \(z^n - f(z)\) does not vanish inside the annulus \(1 \leq |z| \leq 2\).
Problem 1.b
We can only conclude that there are $n$ zeros inside the unit circle counting multiplicity. It is possible that there are other zeros outside the unit circle. Let $f(z) = (1/2)z^{n+1}$. Then it satisfies all of the above. However there is a zero at $z = 2$.

Problem 2
We first map $D$ to the strip $0 < \text{Im}(z) < 1/2$ by $w = (z - 1)/z$. Then we map it to $0 < \text{Im}(v) < \pi$ by $v = 2\pi w$. Finally we map that strip to the upper half plane $U$ by $t = e^{\theta}$. Hence the mapping is $t = e^{2\pi(z-1)/z}$

Problem 3.a
$\Re(z) = N + 1/2$, implies $z = N + 1/2 + iy$. Hence

$$g(z) = \frac{\cos(\pi(N + 1/2) + iy)}{\sin(\pi(N + 1/2) + iy)} = -i \tanh(\pi y)$$

therefore $|g(z)| \leq 1$. Likewise

$$h(z) = \frac{1}{\sin(\pi z)} = \frac{1}{\sin(\pi(N + 1/2) + iy\pi)} = \frac{1}{\sin(\pi(N + 1/2)) \cosh(y\pi)}$$

no $|h(z)| \leq 1$. Therefore both $g$ and $h$ are uniformly bounded.

Problem 3.b

$$\frac{1}{2\pi i} \int \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz = \sum_i \text{Res}(f, z_i) = \text{Res}(f, i) + \text{Res}(f, -i) + \sum_{k=-N}^{N} \text{Res}(f, k)$$

So

$$\text{Res}(f, i) = \frac{\cos(\pi i)}{\sin(\pi i)(2i)} = \frac{1}{2} \left( e^{-\pi} + e^{\pi} \right)$$

and

$$\text{Res}(f, -i) = \frac{\cos(-\pi i)}{\sin(-\pi i)(-2i)} = \frac{1}{2} \left( e^{-\pi} + e^{\pi} \right)$$

and

$$\text{Res}(f, k) = \frac{\cos(k\pi)}{\pi \cos(k\pi)(z^2 + 1) + 2z \sin(\pi z)} = \frac{1}{\pi(k^2 + 1)}$$

Thus

$$\frac{1}{2\pi i} \int \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz = \frac{1}{\pi} \sum_{n=-N}^{n} \frac{1}{n^2 + 1} + \frac{e^{\pi} + e^{-\pi}}{e^{-\pi} - e^{\pi}}$$
Likewise
\[
\frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} dz = \text{Res}(f, i) + \text{Res}(f, -i) + \sum_{n=-N}^{N} \text{Res}(f, n)
\]
So
\[
\text{Res}(f, i) = \frac{1}{\sin(\pi i)(2\pi)} = \frac{1}{e^{-\pi} - e^{\pi}}
\]
and
\[
\text{Res}(f, -i) = \frac{1}{\sin(-\pi i)(-2\pi)} = \frac{1}{e^{-\pi} - e^{\pi}}
\]
and
\[
\text{Res}(f, n) = \frac{1}{\pi \cos(n\pi)(n^2 + 1)} = \frac{(-1)^n}{\pi(n^2 + 1)}
\]
\[
\frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} dz = \frac{2}{e^{-\pi} - e^{\pi}} + \frac{1}{\pi} \sum_{n=-N}^{N} \frac{(-1)^n}{n^2 + 1}
\]

Problem 3.c
So
\[
\lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{C_N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} dz \right| = \lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{|z|=N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} dz \right| \leq \frac{2\pi R}{2\pi} \frac{M}{R^2 - 1} \to 0
\]
as \(R \to \infty\). Thus
\[
\lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} dz = 0 = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} + \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}
\]
so
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{\pi} \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}}
\]
Likewise
\[
\lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} dz \right| = \lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{1}{\sin(\pi z)(z^2 + 1)} dz \right| \leq \frac{2\pi R}{2\pi} \frac{1}{M(R^2 - 1)} \to 0
\]
as \(R \to \infty\). Thus
\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1} = \frac{2\pi}{e^{\pi} - e^{-\pi}}
\]
Problem 5.a

Let \( I, II, III, IV \) be the first, second, third, and fourth quadrant respectively. Then we know that \( if(z) \) is analytic in \( I \). Also we know that \( if(z) \) is purely real on the real axis, which implies we can apply the reflection principle: \( \overline{if(z)} \) is analytic in \( IV \). Then notice that by Morera’s theorem, for

\[
F_1(z) = \begin{cases} 
  if(z) & z \in I \\
  \overline{if(z)} & z \in IV
\end{cases}
\]

\( F_1(z) \) is analytic in \( I \cup IV \). Now notice \( -if(-z) \) is analytic in \( III \). We can apply the same reflection principle and Morera’s Theorem, to show that

\[
F_2(z) = \begin{cases} 
  -if(-z) & z \in III \\
  \overline{-if(-z)} & z \in II
\end{cases}
\]

Now I claim that \( F_1 \cup F_2 \) is analytic in \( |z| > 0 \). it suffices to show that

\[
\lim_{\epsilon \to 0} i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} -i(f(-\epsilon + iy))
\]

which reduces to

\[
\lim_{\epsilon \to 0} i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} i(f(\epsilon + iy))
\]

But remember \( f \) is purely real on the imaginary axis, and hence the equality holds. We also need to show

\[
\lim_{\epsilon \to 0} -i(f(-(-\epsilon - iy))) = \lim_{\epsilon \to 0} -i(f(\epsilon + iy))
\]

Which implies

\[
\lim_{\epsilon \to 0} -i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} -i(f(\epsilon + iy))
\]

and since \( f \) is purely real on the imaginary axis, the equality holds. This completes the extension for \( F = F_1 \cup F_2 \).

Problem 5.b

By construction above, \( F = F_1 \cup F_2 \) is an odd function.

Problem 5.c

Since \( |f(z)| \leq |z^{-4}| + |z^{-2}| + 1 \) and \( f \) is an analytic function in \( C \setminus \{0\} \), \( f \) has a Laurent expansion at \( z = 0 \)

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n z^n
\]

Since \( |f(z)| \leq |z^{-4}| + |z^{-2}| + 1 \), and \( f \) is odd, we have
\[ f(z) = az^{-3} + bz^{-1} \]

where \( |a + b| \leq 3 \).

**Problem 5.d**

Now if \( f \) is purely imaginary on the positive real and imaginary axis, the \( if(z) \) is purely real on the positive real and imaginary axis. Hence by the reflection principle and Morera’s theorem, we can extend \( f \) as an analytic function where

\[
f(z) = \begin{cases} 
if(z) & z \in I \\
-if(z) & z \in IV
\end{cases}
\]

Then by rotating it, we can define it as an analytic function, in the upper half plane

\[
f(z) = \begin{cases} 
if(iz) & z \in II \\
-if(-iz) & z \in I
\end{cases}
\]

Then by the reflection principle and Morera’s theorem, we can extend this into a function that is analytic in \( \mathbb{C} \setminus \{0\} \)

\[
f(z) = \begin{cases} 
-if(-iz) & z \in I \\
if(iz) & z \in II \\
-if(-iz) & z \in III \\
if(iz) & z \in IV
\end{cases}
\]

Hence \( f \) is analytic in \( \mathbb{C} \setminus \{0\} \). Also notice that \( f \) is EVEN! So likewise \( f \) has a Laurent expansion about \( z = 0 \), and by part (c) we have

\[ f(z) = az^{-4} + bz^{-2} + c \]

where \( |a + b + c| \leq 3 \).

**Jan 1999 Linear Algebra**

**Problem 1.1**

\[
M = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Problem 1.2

\[ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Problem 1.3

\[ M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

where \( MA \) performs the operation.

Problem 1.4

\[ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

where \( AM \) performs the operation.

Problem 2

Let’s first calculate the eigenvalues. Notice

\[
det \begin{pmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)((2 - \lambda)(1 - \lambda) - 1) = \lambda(1 - \lambda)(\lambda - 3)
\]

Hence the eigenvalues of \( A \) are 0, 1, and 3. Thus \( A \) is diagonalizable

\[ A = MDM = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} M^{-1} \]

Hence

\[ \lim_{n \to \infty} a^n A^n = M \begin{pmatrix} 0 & a^n \\ 0 & 3^n a^n \end{pmatrix} M^{-1} \]

Thus we must have \( a < 1/3 \).
Problem 3.a

Notice

\[ AP = (I - 2P)P = P - 2P^2 = P - 2P = -P \]

Then notice

\[ A^2 = A(I - 2P) = A + 2P + I \]

Hence \( A^{27} = A \).

Problem 3.b

So \( \det(A^2) = \det(I) = 1 \) implies that \( \det(A) = \pm 1 \). I think the answer is

\[ \det(A) = (-1)^{n-k-1} \]

Problem 3.c

Recall that a square matrix is orthogonal if \( A^T A = I \). Hence notice

\[ (P^T)^2 = (P^2)^T = P^T \]

So


Now I claim that \( 2P^T - 2P + 4P^T P = 0 \). Indeed since notice

\[ 4P^T P = 2P^T + 2P \Rightarrow 4P^T P = 2P^T P + 2P \Rightarrow 4P^T P = 2P^T P + 2P^T P \]

Hence \( A \) is indeed orthogonal.

Problem 4.a

Notice that

\[ UMV^T = 0 \Rightarrow (U^T U)M(V^T V) = 0 \]

and since \( U \) and \( V \) are at full rank, \( U^T U \) and \( V^T V \) are invertible, which implies \( M = 0 \).

Problem 4.b

So

\[ GG^{-1} = (I - UV^T)(I - UWV^T) = I - UWV^T - UV^T + UWV^T UV^T = I \]

implies that

\[ V^T UW = I + W \Rightarrow WV^T U = W + I \Rightarrow (V^T U - I)W = I \]
and $W(V^T U - I) = I$

hence $W = (V^T U - I)^{-1}$.

**Problem 5.a**

We want $U \in \mathbb{R}^{n \times (n-k)}$ such that the columns space the null space of $A$. Now these vectors span the null space if and only if it is orthogonal to all the row vectors of $A$. Now we will use Gram Schmidt. Let $a_i$ be the $i$th row of $A$ and $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$. Then we will perform Gram Schmidt on the vectors $\{a_1, ..., a_k, e_1, ..., e_n\}$. Since $A$ is full rank, we know that the first $k$ vectors will be orthogonalized. Now when we continue with the rest of the vectors $e_1, ..., e_n$, we will throw away $k$ zero vectors. So let $u_1, ..., u_n$ be the output of the process. Then $S(u_1, ..., u_k) = S(a_1, ..., a_k)$ and $\{u_{k+1}, ..., u_n\}$ are all orthogonal to the row space. Hence

$$U = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

---

### Sept 1999 Advanced Calculus

**Problem 1.a**

Notice we have

$$\frac{\sqrt{n+1} - \sqrt{n-1} \sqrt{n+1 + \sqrt{n-1}}}{n} = \frac{(n+1) - (n-1)}{n\sqrt{n+1 + \sqrt{n-1}}} = \frac{2}{n\sqrt{n+1 + \sqrt{n-1}}} = \frac{2}{n(\sqrt{n+1 + \sqrt{n-1}})} = \frac{2}{n^2\sqrt{n}} = \frac{2}{n^{3/2}}$$

Hence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n} \leq \sum_{n=1}^{\infty} \frac{2}{n^{3/2}} < \infty$$

Hence the sum converges.

**Problem 1.b**

Recall that if $\lim_{n} n^p u_n = A$, then

(i) $\sum u_n$ converges if $p > 1$ and $A$ is finite.

(ii) $\sum u_n$ diverges if $p \leq 1$ and $A \neq 0$ ($A$ may be infinite)

Therefore apply L'Hospital’s rule, we have

$$\lim_{n \to \infty} n^2 \sin^2(1/n) = \lim_{n \to \infty} \left(\frac{\sin(1/n)}{1/n}\right)^2 = \lim_{n \to \infty} \frac{\sin^2(1/n)}{(1/n)^2} = \lim_{n \to \infty} n \sin(1/n) \cos(1/n)$$

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\[
\lim_{n \to \infty} \frac{\cos(1/n)(-n^{-2}) \cos(1/n) - \sin(1/n)(-n^{-2}) \sin(1/n)}{-n^{-2}} = \lim_{n \to \infty} \cos^2(1/n) - \sin^2(1/n) = 1
\]

Hence \(\sum_{n=1}^{\infty} \sin^2(1/n)\) converges.

**Problem 1.c**

Since \(\lim_{n} \sqrt[n]{n} \to 1\), \(\exists M\) such that \(\sqrt[n]{n} \leq M\) for all \(n\). Hence

\[
\sum_{n=2}^{\infty} \frac{1}{n \sqrt[n]{n} \log n} \geq \sum_{n=2}^{\infty} \frac{1}{n M \log n} \to \infty
\]

Hence the summation diverges.

**Problem 2.a**

By using integration by parts, we set

\[
u = x^{1/2} \quad v = -\cos x \quad du = (-1/2)x^{-3/2}dx \quad dv = \sin x dx
\]

and we have

\[
\int_{1}^{\infty} \frac{\sin x}{\sqrt{x}} dx = \left[ -\frac{\cos x}{x^{1/2}} \right]_{1}^{\infty} - \frac{1}{2} \int_{1}^{\infty} \frac{\cos x}{x^{3/2}} dx
\]

and we know that \(\int_{1}^{\infty} \frac{\cos x}{x^{3/2}} dx\) converges, and hence the original integral converges.

**Problem 2.b**

Notice

\[
\int_{0}^{\infty} \frac{|\sin x|}{\sqrt{x}} dx \geq \sum_{k=0}^{\infty} \int_{k\pi + \pi/6}^{(k+1)\pi/6} \frac{|\sin x|}{\sqrt{x}} dx \geq \sum_{k=0}^{\infty} \frac{1}{2} \frac{(5\pi/6 - \pi/6)}{\sqrt{k\pi + 5\pi/6}} = \sum_{k=0}^{\infty} \frac{\pi}{\sqrt{\pi(k + 5/6)}} \to \infty
\]

or recognize for \(x \geq 0\), we have \(\sin x \leq x\). Hence

\[
\sum \sin^2(1/n) \leq \sum \left( \frac{1}{n} \right)^n < \infty
\]

**Problem 3.a**

Let

\[
f_N(x) = \sum_{n=1}^{N} x^n (1 - x^n)
\]
Then clearly $f_N$ converges pointwise since $f(0) = f(1) = 0$ and for $x \in (0, 1)$.

$$\lim_{N \to \infty} f_N(x) = \sum_{n=0}^{\infty} x^n (1 - x^n) \leq \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} < \infty$$

However it does not converge uniformly since $f$ is not continuous at $x = 1$. Indeed since $f(1) = 0$, but $\forall \epsilon > 0$, we have

$$\sum_{n=1}^{\infty} (1 - \epsilon)^n (1 - (1 - \epsilon)^n) = \sum_{n=1}^{\infty} (1 - \epsilon)^n - \sum_{n=1}^{\infty} (1 - \epsilon)^{2n} = \frac{1}{1 - (1 - \epsilon)} - \frac{1}{1 - (1 - \epsilon)^2} = \frac{1 - \epsilon}{2\epsilon - \epsilon^2} > 1$$

for small enough $\epsilon$.

**Problem 3.b**

Notice

$$\sum_{n=0}^{\infty} (1 - x)^n x^n (1 - x^n) = (1 - x)^\alpha \sum_{n=0}^{\infty} x^n (1 - x^n) = (1 - x)^\alpha \frac{1}{1 - x} \frac{x}{1 + x} = (1 - x)^{\alpha - 1} \frac{x}{1 + x}$$

Which is not continuous at $x = 1$ if $\alpha < 1$. Now if $\alpha \geq 1$, it is continuous for $x \in [0, 1]$. Since $[0, 1]$ is compact in each function is continuous implies that it converges uniformly.

**Problem 4.a**

Remember the exterior derivative from the workshop

$$\int_{dE} w = \int_E dw$$

But anyways, by Greens Theorem we have

$$\text{Area}(E) = \int \int_E xdy = \frac{1}{2} \int \int_E 1 - (-1)dx dy = \frac{1}{2} \int_{dE} xdy - ydx$$

**Problem 4.b**

By the divergence theorem, we have for $v = (1/2)(x, y, z)$

$$\text{Vol}(E) = \int \int \int_V xdydz = \int \int \int_V \nabla \cdot v dx dydz = \int \int_{dE} v \cdot ndS$$

$$= \frac{1}{3} \int \int_{dE} (x, y, z) \cdot ndS = \frac{1}{3} \int \int_{dE} xdydz + ydzdx + zdxdy$$
Problem 4.c

The best way to solve this is

\[
\int_{x=0}^{1} \int_{y=0}^{-1} z \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{-1} (1 - x - y) \, dy \, dx = \int_{x=0}^{1} \left( \frac{1}{2} x^2 - x + \frac{1}{2} \right) \, dx = \frac{1}{6}
\]

Problem 5

We have \( f = x^2 + y^2 + z^2 \) and \( g_1 = x + y - z = 0 \) and \( g_2 = x^2/4 + y^2/5 + z^2/25 = 1 \). Thus

\[
dT = \begin{pmatrix}
2x & 2y & 2z \\
1 & 1 & -1 \\
x/2 & 2y/5 & 2z/25
\end{pmatrix}
\]

Now we must have \( \det(dT) = 0 \). Hence this implies

\[
16yz = 21xz + 5xy
\]

Now by \( g_1 \) we have \( z = x + y \), and so

\[
16y(x + y) = 21x(x + y) + 5xy \Rightarrow 0 = (7x + 8y)(3x - 2y)
\]

So one set of possible solutions is when \( x = -8y/7 \). Then by using \( g_2 \) we have

\[
\frac{16y^2}{49} + \frac{y^2}{5} + \frac{y^2}{49(25)} = 1
\]

which implies

\[
y = \pm \frac{35}{\sqrt{646}}
\]

Hence our first set of solutions is

\[
x = -\frac{8}{7} \left( \frac{35}{\sqrt{646}} \right) , \quad y = \frac{35}{\sqrt{646}} , \quad z = -\frac{1}{7} \frac{35}{\sqrt{646}}
\]

and our second is

\[
x = \frac{8}{7} \left( \frac{35}{\sqrt{646}} \right) , \quad y = -\frac{35}{\sqrt{646}} , \quad z = \frac{1}{7} \frac{35}{\sqrt{646}}
\]

Now if \( x = 2y/3 \) we have

\[
\frac{4y^2}{9(4)} + \frac{y^2}{5} + \frac{(2y/3 + y)^2}{25} = 1
\]

which implies

\[
y = \pm \sqrt{\frac{45}{19}}
\]
So our third set of solutions is

\[ x = \frac{2}{3} \sqrt[3]{\frac{45}{19}}, y = \frac{45}{19}, z = \frac{5}{3} \sqrt[3]{\frac{45}{19}} \]

and our fourth set is

\[ x = \frac{-2}{3} \sqrt[3]{\frac{45}{19}}, y = \frac{-45}{19}, z = \frac{5}{3} \sqrt[3]{\frac{-45}{19}} \]

And by direct calculation, we can see that the first and second set of solutions is minimum, and the third and fourth is the max.

**Sept 1999 Complex Variables**

**Problem 1**

Remember from the workshop that when we integrate through a simple pole at \( z_0 \), then the integral over \( C_\epsilon \) is just

\[ \frac{1}{2} 2\pi i \text{Res}(f, z_0) \]

Very useful. But anyways, let \( \gamma \) be the contour of two upper semi-circles with radius \( \epsilon \) and \( R \). Then let \( f(z) = \frac{e^{iz}}{z} \), and we have

\[ \int_\gamma f(z)dz = 0 \]

Now notice

\[ \int_\gamma f(z)dz = \int_{-\epsilon}^{-R} \frac{e^{iz}}{z}dz + \int_{C_\epsilon} \frac{e^{iz}}{z}dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz = 0 \]

Thus

\[ \int_\gamma f(z)dz = \int_{\epsilon}^{R} \frac{e^{iz} - e^{-iz}}{z}dz + \int_{C_\epsilon} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz = 0 \]

which implies

\[ \int_{\epsilon}^{R} \frac{e^{iz} - e^{-iz}}{z}dz = -\int_{C_\epsilon} \frac{e^{iz}}{z}dz - \int_{C_R} \frac{e^{iz}}{z}dz \]

On \( C_R \) we have \( z = Re^{i\theta} \), and we have

\[ \left| \int_{C_R} \frac{e^{iz}}{z}dz \right| = \int_{0}^{\pi} \frac{e^{iR(\cos \theta + \sin \theta)}}{R} Re^{i\theta}d\theta \leq \int_{0}^{\pi} e^{-R\sin \theta}d\theta = 2 \int_{0}^{\pi/2} e^{-R\sin \theta}d\theta \]

Now recall that \( \sin \theta \geq 2\theta/\pi \) for \( 0 \leq \theta \leq \pi/2 \). Hence

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\[
2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq 2 \int_0^{\pi/2} e^{-R \theta / \pi} d\theta = \frac{\pi}{R} (1 - e^{-R}) \to 0
\]
as \( R \to \infty \). Now notice that on \( C_\epsilon \) we have \( z = \epsilon e^{i \theta} \) and

\[
\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z} dz = \lim_{\epsilon \to 0} \int_0^\pi \frac{e^{i\epsilon e^{i \theta}}}{\epsilon e^{i \theta}} i \epsilon e^{i \theta} d\theta = \lim_{\epsilon \to 0} \int_0^\pi ie^{i \epsilon e^{i \theta}} d\theta = -i \pi
\]

There as \( \epsilon \to 0 \) and \( R \to \infty \) we have

\[
\int_\epsilon^\infty \frac{e^{iz} - e^{-iz}}{z} dz = 2i \int_0^\infty \frac{\sin z}{z} dz = i \pi
\]

Hence we have

\[
\int_0^\infty \frac{\sin z}{z} dz = \frac{\pi}{2}
\]

Since \( \frac{\sin z}{z} \) is an even function, we have

\[
\int_{-\infty}^\infty \frac{\sin z}{z} dz = \pi
\]
as long as \( \sin x/x = 1 \) when \( x = 0 \).

**Problem 2.a**

Yes a function exists and it is NOT unique. Notice

\[
F(z) = \frac{(z - 1)(z - 2)(z - 3)}{(z - 4)(z - 5)} e^{g(z)}
\]

where \( g \) is an entire function. So

\[
F(0) = 1 = \left( \frac{-3}{10} \right) e^{g(0)} \quad \Rightarrow \quad -\frac{10}{3} e^{x+iy}
\]

Hence \( x = \ln(10/3) \) and \( y = (2n + 1)\pi \) for \( n \in \mathbb{Z} \). But it’s not unique since we can have

\[
F(z) = \frac{(z - 1)(z - 2)(z - 3)}{(z - 4)(z - 5)} e^{\ln(10/3)+i\pi+x}
\]
or

\[
F(z) = \frac{(z - 1)(z - 2)(z - 3)}{(z - 4)(z - 5)} e^{\ln(10/3)+i\pi}
\]

would work.
Problem 2.b
Now in order to have polynomial growth at $\infty$, we need $e^{g(z)}$ to have polynomial growth at $\infty$. Notice
\[ g(x) \leq M \log |z| \]
by Cauchy estimates and since $g$ is entire, implies that $g$ must be constant. Hence $g$ must be constant and we must have
\[ F(z) = \frac{(z - 1)(z - 2)(z - 3) - 10}{(z - 4)(z - 5)} \frac{3}{3} \]

Problem 2.c
If $F$ is bounded at $\infty$, then we must have $e^{g(z)}$ at least bounded. But then we must have
\[ F(z) = \frac{(z - 1)(z - 2)(z - 3) - 10}{(z - 4)(z - 5)} \frac{3}{3} \]
Hence no function exists since the other part goes to infinity.

Problem 3
Now we map the first quadrant to the UHP by $w = z^2$. Then we map the UHP to the unit disk by linear transformation
\[ w_2 = \frac{aw + b}{cw + d} = \frac{a'w + b'}{w + d'} \]
Now since we must satisfy $0 \mapsto i$ and $\infty \mapsto -i$, we have $a' = -i$ and $b' = id'$. Hence let $b' = -1$ and $d' = i$. Then we have
\[ w_2 = \frac{-iw - 1}{w + i} = \frac{-iz^2 - 1}{z^2 + i} \]
Now the mapping is not unique, since we could have let $b' = -2$ and $d' = 2i$. Then
\[ w_2 = \frac{-iw - 2}{w + 2i} = \frac{-iz^2 - 2}{z^2 + 2i} \]
which does the same conformal mapping.

Problem 4
Lets define the branch cut for $\sqrt{\cdot}$ on the negative real axis. Notice for $z \in (-\pi, 0) \subset R$, $\sin(z)$ and $z$ are both negative. Hence we can define
\[ f(z) = \sqrt{z} \sin z \]
to be $f(0) = 0$ and continuous in $|z| < \pi$. Then we just will just fill in values for $f$ on the negative real axis to make $f$ continuous. This is possible since both $z$ and $\sin z$ are negative real values from
\((-\pi, 0)\). Then notice the integral around the contour \(|z| = \pi\) is equivalent to the integral around two semicircles in the upper and lower half planes, only \(\epsilon\) above the real line. Hence as \(\epsilon \to 0\), the integrals are equivalent since \(f\) is continuous on the real line. So by Cauchy, we know that the integral around both semicircles is 0, and so the integral of \(f\) on \(|z| = \pi\) is zero. Therefore my Morera’s Theorem, \(f\) is analytic in \(|z| < \pi\).

Now for the second part, I claim that the radius of convergence is \(\pi\). By above we know it is at least \(\pi\). Notice if \(z > \pi\), on the real line, \(z\) is positive and \(\sin z\) is negative, and so \(f\) is not continuous as we cross the \(x\)-axis. Therefore it is not analytic and the radius of convergence is \(\pi\).

**Problem 5**

For \(y > 0\),

\[
f(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \sum_j e^{ia_j x + ia_j t} dt = \frac{y}{\pi} \sum_j c_j e^{ia_j x} \int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt
\]

now notice for

\[
\int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt
\]

Let \(\gamma\) be the contour of the semi-circle with radius \(R\) in the upper-half plane. Then

\[
\left| \int_{C_R} \frac{e^{ia_j z}}{z^2 + y^2} dz \right| \leq \pi R \max \left\{ e^{-R \arcsin \theta} \right\} \leq \pi R \max \left\{ e^{-a_j R \theta / \pi} \right\} = \frac{\pi R}{R^2 - y^2} \to 0
\]

as Hence by the residue theorem

\[
\int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt = 2\pi i \frac{e^{-a_j y}}{-2iy} = \frac{\pi}{y} e^{-a_j y}
\]

Hence let

\[
g(x, y) = \sum_j c_j e^{ia_j x - a_j y} = \sum_j c_j e^{ia_j z}
\]

Which everyone knows is analytic in the upper and lower half planes. We also know that the imaginary and real part of an analytic function is harmonic. Therefore

\[
u(x, y) = \text{Im}(g(x, y))
\]

is indeed harmonic in the upper and lower half planes. Also notice that the function is entire, which implies that in general the function defined by \(u(x, y)\) for \(y \neq 0\) and by \(f(x)\) for \(y = 0\) is harmonic.

**Sept 1999 Linear Algebra**

**Problem 1.a**

So we have

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\[ x^2 \frac{d^2}{dx^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2 \cdot 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \end{pmatrix} \]

and

\[ -bx \frac{d}{dx} = -b \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix} \]

\[ -c \frac{d}{dx} = -c \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \]

Hence

\[ A = \begin{pmatrix} 0 & -c & 0 & 0 & \cdots & 0 \\ 0 & -b & -2c & 0 & \cdots & 0 \\ 0 & 0 & 2 - 2b & -3c & \cdots & 0 \\ 0 & 0 & 0 & 2 \cdot 3 - 3b & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) - nb \end{pmatrix} \]

therefore the eigenvalues are \( \{0, -b, 2 - 2b, 2 \cdot 3 - 3b, \ldots, (n-1)n - nb\} \).

**Problem 1.b**

We know that we can diagonalize \( A \) if \( A \) has all distinct eigenvalues. Hence if \( c \in \mathbb{R} \) and \( b \neq 0, 1, 2, \ldots, n - 1 \), we can diagonalize \( A \). Now if \( b \in \{0, 1, \ldots, n - 1\} \), then \( A \) has \( n - 1 \) distinct eigenvalues and \( \lambda_i = 0 \) twice. So if \( c = 0 \), then the null space of \( A \) has dimension 2 and so the eigenvectors for \( \lambda_i = 0 \) are linearly independent. Therefore we can diagonalize \( A \). Now if \( c \neq 0 \) notice

\[ R(A) = n \Rightarrow n(A) = 1 \]

Hence there is only one linearly independent eigenvector of \( \lambda_i = 0 \). Hence it is not diagonalizable.
Problem 2.a

They are \( p(x) = (x - 2)(x - 1)^2 \) and \( p(x) = (x - 2)^2(x - 1) \).

Problem 2.b

Notice \( A^2 - 3A + 2I = 0 \) implies that for all \( x \in \mathbb{R}^3 \), we have

\[
(A - 2I)(A - I)x = 0
\]

So the only possible eigenvalues are 2 and 1. for \( \lambda = 1 \) we cannot have a Jordan Block of size > 1. indeed notice

\[
Av_2 = v_1 + v_2 \Rightarrow A^2v_2 = v_1 + Av_2 \Rightarrow -v_1 = 0
\]

and hence we have a contradiction. Likewise for \( \lambda = 2 \) we cannot have a Jordan Block of size > 1. Indeed that would imply

\[
Av_2 = v_1 + 2v_2 \Rightarrow A^2v_2 = Av_1 + 2Av_2 = 4v_1 + 4v_2 \Rightarrow v_1 = 0
\]

Thus the only Jordan forms we can have are

\[
J_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad J_2 = \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}
\]

\[
J_3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad J_4 = \begin{pmatrix} 2 & 2 \\ & 2 \end{pmatrix}
\]

Problem 3

By Gaussian elimination we have

\[
\begin{array}{ccc|cc}
1 & b & c & 1 + b - c \\
b & 1 & b & b \\
c & b & c & 1 - b + c \\
\hline
1 & b & c & 1 + b - c \\
0 & -b^2/c + 1 & 0 & -b/c - b^2/c + 2b \\
0 & 0 & 0 & -2b + 2c
\end{array}
\]

So we have no solutions if \( b \neq c \), an infinite number of solutions if \( b = c \). So notice we cannot have a unique solution to our system.
Problem 4

Let’s first solve for the eigenvalues. Notice

\[
\det(A - I\lambda) = \det \begin{pmatrix} -1 - \lambda & 1 & 1 \\ 1 & -1 - \lambda & 1 \\ 1 & 1 & -1 - \lambda \end{pmatrix} = -\lambda^3 - 3\lambda^2 + 4
\]

Then we can see that \(\lambda = 1\) is a solution. Thus by factoring, we have

\[
-\lambda^3 - 3\lambda^2 + 4 = -(x - 1)(x + 2)^2
\]

Now notice that the eigenvectors of \(\lambda = -2\) are linearly independent since \((-1, 0, 1)^T\) and \((-1, 1, 0)^T\) would work. Hence \(A\) is diagonalizable

\[
A = MDM^{-1} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -2 & \& 2 \\ \& -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^{-1}
\]

which implies for all integer \(n\)

\[
A^n = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -2^n & \& 2^n \\ \& -2^n & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^{-1}
\]

and \(v_1, v_2\) are eigenvectors corresponding to eigenvalue \(-2\) and \(v_3\) is the eigenvector corresponding to 1.

Jan 2000 Advanced Calculus

Problem 1.a

So we have

\[
f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{1 + n^2 x}
\]

for \(x > 0\), notice

\[
f(x) \leq \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{n^2 x} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 x} + \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
\]

by the comparison test. We can also see that when \(x = 0\), the series diverges. Now let’s look at when \(x < 0\). Clearly if \(\sqrt{\frac{1}{|x|}} \in \mathbb{N}\), then the series diverges. Thus let \(S = \{-1, -1/2, -1/4, -1/9, -1/16/...\} \cup \{0\}\. Thus for \(x < 0\) and \(x \not\in S\), we have

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{1 + n^2 x} \leq \sum_{n=1}^{\infty} \frac{\sqrt{n} + |x|}{n^2 |x|} - 1 \leq M \sum_{n=1}^{\infty} \frac{\sqrt{n} + |x|}{n^2 |x|} < \infty
\]

by above. Hence it converges absolutely which implies that it converges. Hence \(f(x)\) converges for \(x \in \mathbb{R} \setminus S\).
Problem 1.b
On \( x \in \mathbb{R} \setminus S \), we need to show for all \( \epsilon > 0 \) \( \exists \delta \) such that if \( (x - x_0) < \delta \) which implies
\[
|f(x) - f(x_0)| < \epsilon
\]
Indeed notice
\[
|f(x) - f(x_0)| = \left| \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{1 + n^2x} - \sum_{n=1}^{\infty} \frac{\sqrt{n} + x_0}{1 + n^2x_0} \right|
= \left| \sum_{n=1}^{\infty} \frac{(x - x_0) + n^{2.5}(x_0 - x)}{(1 + n^2x)(1 + n^2x_0)} \right| = \delta \left| \sum_{n=1}^{\infty} \frac{n^{2.5} - 1}{(1 + n^2x)(1 + n^2x_0)} \right|
\]
So let \( \delta = \epsilon/M \). Then we have \( |f(x) - f(x_0)| < \epsilon \). So \( f \) is indeed continuous.

Problem 2.a
Notice we have
\[
df = \frac{df}{dx} = \frac{(3x^2y - y^3)(x^2 + y^2) - 2x(x^3y - xy^3)}{(x^2 + y^2)^2} = \frac{y((x^2 + y^2)^2 + 2y^2(x^2 - y^2))}{(x^2 + y^2)^2}
\]
Hence it is continuous everywhere except possibly at \((0,0)\). So let’s check using polar coordinates.
\[
= \frac{r \sin \theta (r^4 + 2r^2 \sin^2 \theta (r^2 \cos 2\theta))}{r^4} = r (1 + 2 \sin^2 \theta \cos 2\theta) r \to 0
\]
as \( r \to 0 \). Hence regardless of what direction (regardless of \( \theta \)), \( \lim_{(x,y) \to (0,0)} df/dx = 0 \). Hence it is continuous. Similar argument for \( df/dy \).

Problem 2.b
Now we know that \( \frac{d^2}{dxdy} = \frac{d^2}{dydx} \). So we have
\[
d \frac{y((x^2 + y^2)^2 + 2y^2(x^2 - y^2))}{(x^2 + y^2)^3} = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3}
\]
Hence it exists at every point in \( \mathbb{R}^2 \), except possibly at \((0,0)\). Notice when we approach \((0,0)\) on the \( x \)-axis, we have \( x \to 0 \) and \( y = 0 \). Hence
\[
\lim_{stackrel{x \to 0}{y=0}} \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3} = \lim_{x \to 0} \frac{x^6}{x^6} = \lim_{x \to 0} 1 = 1
\]
However when we approach \((0,0)\) on the \( y = x \) line, we have
\[
\lim_{stackrel{x \to 0}{y=x}} \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3} = \lim_{x \to 0} \frac{16x^6 - 16x^6}{8x^6} = 0
\]
Hence the limit does not exist at \((0,0)\).
Problem 2.c

Clearly
\[
\frac{d^2 f}{dydx}(0,0) = \frac{d^2 f}{dxdy}(0,0) = 0
\]

Problem 3.a

Notice
\[
f(x) = 2 \sin(\pi x) = 4 \sin(\pi x/2) \cos(\pi x/2) = -4 \sin(\pi x/2) \sin(\pi(x-1)/2)
\]

So let
\[
P_n(x) = -4 \left( \frac{(\pi x/2)}{3!} - \frac{(\pi x/2)^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi x/2)^{2n+1}}{(2n+1)!} \right) \times \left( \frac{(\pi/2(x-1))}{3!} - \frac{(\pi/2(x-1))^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi/2(x-1))^{2n+1}}{(2n+1)!} \right)
\]

Then we have \( P_n(0) = P_n(1) = 0 \) and \( P_n \to f \) uniformly since
\[
\left( \frac{(\pi x/2)}{3!} - \frac{(\pi x/2)^3}{5!} + \cdots + (-1)^n \frac{(\pi x/2)^{2n+1}}{(2n+1)!} \right) \to \sin(\pi x/2)
\]
\[
\left( \frac{(\pi/2(x-1))}{3!} - \frac{(\pi/2(x-1))^3}{5!} + \cdots + (-1)^n \frac{(\pi/2(x-1))^{2n+1}}{(2n+1)!} \right) \to \sin(\pi(x-1)/2)
\]

Problem 4.a

Notice by the divergence theorem, we have
\[
\int \int_S x \cdot ndS = \int \int \int \nabla \cdot x dV = \int \int \int_\Omega 3dV = 3 \text{(Area of ellipsoid)}
\]
So what is the area of the ellipsoid. Recall that the area of an ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
is \(\pi a^2 b^2\). This can be shown by integrating the ellipse using polar coordinates \( r = a \cos \theta + b \sin \theta \).
So for the ellipsoid, each \( z \) value gives us a slice of the ellipsoid, which is an ellipse
\[
\frac{x^2}{4(1-z^2/25)} + \frac{y^2}{9(1-z^2/25)} = 1
\]
which has an area of \(\pi/2(13-13z^2/25)\). So the area of the ellipsoid is
\[
\int_{z=-5}^{5} \frac{\pi}{2}(13-13z^2/25)dz = \pi \left( 13 \cdot 5 - \frac{13 \cdot 125}{75} \right) = \frac{130\pi}{3}
\]

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Problem 4.b
So we have

\[ P(X) = (x, y, z) \cdot n(X) \]

where

\[ n(X) = \frac{(2x/4, 2y/9, 2z/25)}{\sqrt{\frac{4x^2}{16} + \frac{4y^2}{9} + \frac{4z^2}{25}}} \]

So

\[ P(X) = \frac{2}{\sqrt{\frac{4x^2}{16} + \frac{4y^2}{9} + \frac{4z^2}{25}}} \]

Notice the denominator is just the distance. So Max point is \((2, 0, 0)\) and Min point is \((0, 0, 5)\).

\[ P_{\text{max}} = 2 \quad P_{\text{min}} = \frac{10}{2} \]

Jan 2000 Complex Variables

Problem 1
Since

\[ \sinh^2 z = \left( \frac{e^z - e^{-z}}{2} \right)^2 = \frac{e^{2z} - 2 + e^{-2z}}{4} = \frac{1}{2} \left( \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \cdots \right) \]

we have

\[ \oint_C \frac{\cosh z}{z \sinh^2 z} \, dz = \oint_C \frac{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}{\frac{z^3}{2!} + \frac{2z^5}{4!} + \frac{2z^7}{6!} + \cdots} \, dz \]

Hence the only pole is at \(z = 0\). By the residue theorem, we have

\[ \oint_C \frac{\cosh z}{z \sinh^2 z} \, dz = 2\pi i \text{Res} (f, 0) \]

So notice

\[ \frac{\cosh z}{z \sinh^2 z} = 2 \frac{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}{z^3 \left( \frac{2^2}{2!} + \frac{2^4z^2}{4!} + \frac{2^6z^4}{6!} + \cdots \right)} = a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + \cdots \]

which implies

\[ 2 \left( 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) = z^3 \left( \frac{2^2}{2!} + \frac{2^4z^2}{4!} + \frac{2^6z^4}{6!} + \cdots \right) \left( a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + \cdots \right) \]

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so we have $a_{-3} = 1, a_{-2} = 0, a_{-1} = 1/6$. Thus

$$\oint_C \cosh z z \sinh^2 z \, dz = 2\pi i (1/6) = \frac{\pi i}{3}$$

**Problem 2.a**

Using Rouche’s Theorem, we have for $|z| = 1$

$$|z^5 - 12z^2 + 14 - 14| = |z^5 - 12z^2| \leq |z|^5 + |12z^2| = 13 < 14 = |14|$$

Thus there are no zeros inside the unit circle.

**Problem 2.b**

Using Rouche’s Theorem, we have for $|z| = 2$

$$|z^5 - 12z^2 + 14 + 12z^2| = |z^5 + 14| \leq |z|^5 + 14 = 46 < 48 = |12z^2|$$

Hence there are two zeros inside the annulus $1 \leq |z| < 2$.

**Problem 2.c**

Using Rouche’s Theorem, we have for $|z| = 5/2$

$$|z^5 - 12z^2 + 14 - z^5| = |-12z^2 + 14| \leq |12z^2| + 14 = 89 < 97.6563 = |z^5|$$

Thus there are 3 zeros inside the annulus $2 \leq |z| < 5/2$.

**Problem 3.a**

We have $z = x + iy$ where $x^2 + y^2 = 1$. Thus

$$|g(z)| = \frac{|(x - a) + i(y - b)|}{|1 - ax - by + i(bx - ay)|} = \frac{1 - 2ax - 2yb + a^2 + b^2}{1 - 2ax - 2yb + a^2 + b^2} = 1$$

and

$$g'(z) = \frac{(1 - cz) + c(z - c)}{(1 - cz)^2} = 1 - |c|^2$$

**Problem 3.b**

The answer is

$$f(z) = \prod_{i=1}^N \left( \frac{z - z_i}{1 - \overline{z}_i z} \right) e^{h(z)}$$

where $h(z)$ is analytic on $|z| \leq 1$. Notice
\[ |\text{Re}(h(z))| = 0 \]
on \(|z| = 1\). By the Max Modulus theorem for the real part, we know that \(|\text{Re}(h(z))| \leq 0\) inside \(|z| \leq 0\). Thus \(\text{Re}(h(z)) = 0\) inside \(|z| \leq 1\). Hence since \(h\) is analytic inside \(|z| \leq 1\) and \(\text{Re}(h(z))\) is constant, implies that \(h\) is constant inside \(|z| \leq 1\). Therefore

\[
f(z) = \left( \prod_{i=1}^{N} \frac{z - z_i}{1 - \frac{z_i}{z}} \right) e^{i\alpha}
\]
for \(\alpha \in \mathbb{R}\).

**Problem 3.c**

\(f\) is analytic on \(|z| \leq 1\). So we have

\[
f(z) = \left( \frac{z - (1/2 + i/2)}{1 - (1/2 - i/2)z} \right)^3 e^{i\alpha}
\]
which implies

\[
f'(0) = -3ie^{i\alpha} = 3
\]
and so \(e^{i\alpha} = i\). Hence

\[
f(z) = i \left( \frac{z - (1/2 + i/2)}{1 - (1/2 - i/2)z} \right)^3
\]
and so it is unique.

**Problem 4**

Here we can see that \(w\) maps \(D\) to the lower half plane, where \(A' = \sqrt{2}/2, B' = -\sqrt{2}/2, C' = \infty\). The Branch cut \(C\) to \(\infty\) is \(\text{Im}(w) = 0\) and \(\text{Re}(w) \geq 1\) since \(z = x + iy\) and \(y = 0\), \(x \geq 1\). we have

\[
\frac{x}{\sqrt{x^2 - 1}}
\]
which is a real number since \(x \geq 1\) and ranges from 1 to \(\infty\). Now the segment \(OC\) maps to \(\text{Re}(w) = 0\) and \(\text{Im}(w) \leq 0\) since \(z = x + iy\) and \(y = 0\) and \(0 \leq x \leq 1\). Thus

\[
w = \frac{x}{\sqrt{x^2 - 1}}
\]
is a purely imaginary number since \(x \leq 1\). It ranges from \(-\infty\) to 0.
Problem 5

The roots for the denominator are \( z = r \pm \sqrt{r^2 - 1} \). Hence the radius of convergence is \( r - \sqrt{r^2 - 1} \) (or the other root, doesn’t matter). Now clearly the function is analytic at \( z = 0 \). Hence \( \exists \) an \( \epsilon > 0 \) such that for all \( z \) such that \( |z| < \epsilon \), \( f(z) \) is continuous. Hence \( f(z) \) has a Taylor series, and

\[
f(z) = 1 + A_1(r)z + A_2(r)z^2 + \cdots
\]

Now by the Cauchy integral formula and the Residue Theorem, we have

\[
A_n(r) = \frac{1}{2\pi i} \int_{|z| = \epsilon} \frac{f(\zeta)}{z^{n+1}} d\zeta = \frac{1}{n!} \frac{d^n}{dz^n} \left[ (1 - 2rz + z^2)^{-1/2} \right]_{z=0}
\]

Hence we can see that

\[
A_n(r) = \frac{1}{n!} \prod_{k=1}^{n} (2k - 1) 2^n r^n + O(r^{n-1})
\]

Hence \( A_n(r) \) are polynomials of degree \( n \).

Jan 2000 Linear Algebra

Problem 1

For \( p(x) = a_0 + a_1x + a_2x^2 \), we have

\[
L(p) = \begin{pmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \\ a_0 + 3a_1 + 9a_2 \end{pmatrix}
\]

Hence the kernel is the trivial space. The image of \( L \) is \( S((1, 1, 1, 1), (0, 1, 2, 3), (0, 1, 4, 9)) \). Finally in order to find \( \text{im}(L)^\perp \), we need to solve the equation

\[
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0
\]

Hence we have \( x_4 = t, x_3 = -3t, x_2 = 3t, \) and \( x_1 = -t \). So \( \text{im}(L)^\perp = S((-1, 3, -3, 1)) \).

Problem 2

Let \( f = p(x)e^{3x} \), where \( p \) is a polynomial of degree at \( n \). Then

\[
L(f) = p''(x)e^{3x}
\]

Hence the basis of the kernel is \( \{e^{3x}, xe^{3x}\} \) and the basis for the image is \( \{e^{3x}, xe^{3x}, xe^{3x}, x^2e^{3x}, \ldots, x^{n-2}e^{3x}\} \). Our transformation is essentially \( D^2 \)
\( D^2 = \begin{pmatrix}
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & n(n-1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \)

So the eigenvalues of this transformation is 0. Now for \( n = 3 \), we have

\[
L = \begin{pmatrix}
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

**Problem 3**

By Gaussian elimination, we have

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & a \\
\hline
1 & 2 & 3 \\
0 & -3 & -6 \\
0 & -6 & -21 + a \\
\hline
1 & 2 & 3 \\
0 & 3 & 6 \\
0 & 0 & a - 9 \\
\hline
10 & 11 & b \\
-29 & \end{array}
\]

Hence the system has a unique solutions when \( a \neq 9 \) and \( b \in \mathbb{R} \). The system has no solutions when \( a = 9 \) and \( b \neq 12 \). Finally the system has infinite number of solutions when \( a = 9 \) and \( b = 12 \).

**Problem 4.a**

We know that \( A \) has \( m \) distinct eigenvalues, which implies that \( A \) is diagonalizable. Hence

\[
A = MDM^{-1} = \begin{pmatrix}
\lambda_1 & & & \\
& \lambda_2 & & \\
& & \ddots & \\
& & & \lambda_m
\end{pmatrix} M^{-1}
\]

Hence

\[
\lim_{n \to \infty} A^n v = v \quad \lim_{n \to \infty} A^n = v \quad \lim_{n \to \infty} M \begin{pmatrix}
1 \\
1
\end{pmatrix} = \begin{pmatrix}
1 \\
1
\end{pmatrix} M^{-1}
\]

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which converges since \( r_i < 1 \) for all \( i \).

**Problem 4.b**

We have

\[
    w = M \begin{pmatrix} 1 & 0 \\ & \ddots \\ & & 0 \end{pmatrix} M^{-1}
\]

So

\[
    \lim_{n \to \infty} n^{10}|A^n v - w| = \lim_{n \to \infty} n^{10} \left( M \begin{pmatrix} 0 & \lambda^n_2 \\ & \ddots \\ & & \lambda^n_n \end{pmatrix} M^{-1} \right) = \lim_{n \to \infty} M \begin{pmatrix} 0 & n^{10} \lambda^n_2 \\ & \ddots \\ & & n^{10} \lambda^n_n \end{pmatrix} M^{-1} = 0
\]

**Problem 5**

First I claim that the eigenvalues of \( B \) must be \( \pm \sqrt{\lambda_i} \). Indeed since if \( \lambda \neq \pm \sqrt{\lambda_i} \forall i \), then

\[
    B v = \lambda v \Rightarrow B^2 v = A v = \lambda^2 v
\]

which implies that \( \lambda^2 \) is an eigenvalue of \( A \), and hence we have a contradiction. Thus the eigenvalues of \( B \) are \( \{ \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_n \} \). So \( B \) is diagonalizable since all eigenvalues are distinct.

\[
    B = M D^{1/2} M^{-1}
\]

and hence

\[
    A = B^2 = M D M^{-1}
\]

which implies that \( A \) and \( B \) share the same eigenvectors. Thus there are \( 2^n \) possible \( B' \)s.

**Sept 2000 Advanced Calculus**

**Problem 1**

Notice we have

\[
    \sum_{n=2}^{\infty} \frac{1}{(2n-1)(2n-1)} = \sum_{n=2}^{\infty} \frac{(1/2)}{2n-1} - \frac{(1/2)}{2n+1} = \frac{1}{2} \left( \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{7} + \frac{1}{7} + \cdots \right) = \frac{1}{6}
\]

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Problem 2.a

We want to show
\[ R_n = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) \, dt \]

let
\[ g(t) = f(t) + f'(t)(x-t) + \frac{f''(t)(x-t)^2}{2!} + \cdots + \frac{f^{(n)}(t)(x-t)^n}{n!} \]

Then \( g(x) = f(x) \) and
\[ g(a) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} \]

Thus
\[ R_n = g(x) - g(a) = \int_a^x g' \]

notice
\[ g'(t) = f'(t) - f'(t) + f''(t)(x-t) - f''(t)(x-t) \pm \cdots + \frac{f^{(n-1)}(t)(x-t)^n}{n!} = \frac{f^{(n-1)}(t)(x-t)^n}{n!} \]

Thus
\[ R_n = \int_a^x \frac{f^{(n-1)}(t)(x-t)^n}{n!} \, dt \]

Problem 2.b

It suffices to show that
\[ \lim_{n \to \infty} e^{\xi x^{n+1}} (n+1)! = 0 \]

Indeed by Stirling’s approximation, we have
\[ n! = n^n e^{-n} \sqrt{n e^n} \]

where \( 1 \leq c_n \leq 1 - (1/2) \log 2 \). So
\[ \lim_{n \to \infty} \frac{e^{\xi x^{n+1}}}{(n+1)!} = \lim_{n \to \infty} \frac{e^{\xi x^{n+1}} e^{n+1}}{(n+1)^{n+1} \sqrt{n+1} e^{c_n}} = \lim_{n \to \infty} \frac{e^{k_n^{n+1}}}{(n+1)^{n+1} \sqrt{n+1}} \to 0 \]
**Problem 3.a**

Notice we have

\[
f'_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0
\]

and

\[
f'_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0
\]

Hence \(f'_x(0,0)\) and \(f'_y(0,0)\) exists.

**Problem 3.b**

Notice as we approach \((0,0)\) along the \(y = x\) line, we have

\[
\lim_{x \to 0} f(x,y) = \lim_{x \to 0} \frac{x^2}{2x^2} = \lim_{x \to 0} \frac{1}{2} = 1/2 \neq 0
\]

Hence it is not continuous at \((0,0)\).

**Problem 3.c**

Notice

\[
f(x,y) = f(r,\theta) = \begin{cases} 
(1/2) \sin(2\theta) & r \neq 0 \\
0 & r = 0 
\end{cases}
\]

thus the set of limit points at \((0,0)\) is \((1/2) \sin(2\theta) \in [-1/2,1/2]\).

**Problem 5.a**

Recall Greens Theorem: Given that \(P\) and \(Q\) are continuous in a Jordan region \(\Omega\) with boundary \(C\), we have

\[
\iint_{\Omega} Q_x(x,y) - P_y(x,y)dxdy = \oint_C P(x,y)dx + Q(x,y)dy
\]

I claim that

\[
\iint_{\Omega} \frac{P}{Q} dxdy = - \oint_C P(x,y)dx
\]

indeed since by the Fundamental Theorem of Calculus we have

\[
\int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} dQ(x,y)dy \right) dx = \int_a^b (P(x, \phi_2(x)) - P(x, \phi_1(x)))dx
\]

Also lets parameterize \(C\) by \(C_1 : (x, \phi_1(x))\) and \(C_2 : (x, \phi_2(x))\) for \(a \leq x \leq b\). Thus

\[ -82 \]
\[ \int_C P(x,y)dx = \int_a^b P(x,\phi_1(x))dx + \int_b^a P(x,\phi_2(x))dx \]

\[ = \int_a^b P(x,\phi_1(x)) - P(x,\phi_2(x))dx = - \int \int_{\Omega} \frac{dP}{dy}dxdy \]

Likewise I claim that

\[ \int \int_{\Omega} dQ dxdy = \oint_C Q(x,y)dy \]

then by the Fundamental Theorem of Calculus, we have

\[ \int \int_{\Omega} dQ dxdy = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} \frac{dQ}{dx}dx \right) dy = \int_c^d Q(\psi_2(y),y) - Q(\psi_1(y),y)dy \]

also notice we have

\[ \oint_C Q(x,y)dy = \int_c^d Q(\psi_1(y),y) + \int_c^d Q(\psi_2(y),y)dy = \int_c^d Q(\psi_2(y),y) - Q(\psi_1(y),y)dy = \int \int_{\Omega} \frac{dQ}{dx}dxdy \]

This completes the proof of Green’s Theorem.

**Problem 5.b**

By Green’s theorem, we have

\[ \oint_C (2xy - x^2)dx + (x + y^2)dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - 2x)dydx = \int_0^1 \sqrt{x} - 2x^{3/2} - x^2 + 2x^3dx = \frac{1}{30} \]

Now by direct calculation, we have \( C = C_1 \cup C_2 \), where \( C_1 : (t,t^2) \) for \( t : 0 \rightarrow 1 \), and \( C_2 : (t^2,t) \) for \( t : 1 \rightarrow 0 \). Thus we have

\[ \oint_C (2xy - x^2)dx + (x + y^2)dy = \int_{C_1} (2xy - x^2)dx + (x + y^2)dy + \int_{C_2} (2xy - x^2)dx + (x + y^2)dy \]

\[ = \int_0^1 (2t^3 - t^2) + t + t^4dt + \int_0^0 (2t^3 - t^4)2t + 2t^2dt = \frac{7}{6} + \frac{-17}{15} = \frac{1}{30} \]

Thus we have verified Green’s Theorem.
Sept 2000 Complex Variables

Problem 1.a

Since \( f \) is entire, it has a Taylor series about \( z = 0 \),

\[
f(z) = a_0 + a_1 z + a_2 z^2 + \cdots
\]

So we have \( f(0) = a_0 \) and let \( h(z) = 1/(f(z) - a_0) \). Then

\[
h(z) = \frac{1}{a_1 z + a_2 z^2 + \cdots} = \frac{1}{z(a_1 + a_2 z + a_3 z^2 + \cdots)}
\]

notice that

\[
\frac{1}{a_1 + a_2 z + a_3 z^2 + \cdots}
\]

is analytic about \( z = 0 \). Thus it has a Taylor series about \( z = 0 \)

\[
\frac{1}{a_1 + a_2 z + a_3 z^2 + \cdots} = b_0 + b_1 z + b_2 z^2 + \cdots
\]

So

\[
h(z) = \frac{b_0}{z} + b_1 + b_2 z + b_3 z^2 + \cdots
\]

Now since \( f \) is entire and one-to-one, by the open mapping theorem, for \( |z| < 1 \) \( \exists \delta > 0 \) such that

\[
\{w : |w - a_0| < \delta\} \subset \{f(z) : |z| < 1\}
\]

Since \( f \) is one to one, \( \forall z \) such that \( |z| > 1 \), we have

\[
|f(z) - a_0| \geq \delta
\]

which implies

\[
\lim_{z \to \infty} \frac{1}{|f(z) - a_0|} = |h(z)| \leq \frac{1}{\delta} < \infty
\]

Hence \( h \) has a removable singularity at \( \infty \). This implies that \( b_1 = b_2 = b_3 = \cdots = 0 \). Since \( \lim f(z) = \infty, b_1 = 0 \). Hence

\[
\frac{1}{f(z) - a_0} = \frac{b_0}{z}
\]

which implies that

\[
f(z) = a_1 z + a_0
\]

and \( a_1 \neq 0 \) since \( f \) is one to one.
**Problem 1.b**

Let $f$ be such a function. Then let $T$ be a linear fractional transformation such that $\infty \mapsto \infty$. Then

$$T(f(z)) = \frac{af(z) + b}{cf(z) + d}$$

where $T(f(z))$ is one to one and onto of $\mathbb{C} \mapsto \mathbb{C}$. Hence by part (a), we have

$$T(f(z)) = \frac{af(z) + b}{cf(z) + d} = pq + q$$

Thus

$$f(z) = \frac{(pd)z + (qd - b)}{(-pc)z + (a - qc)} = \frac{a'z + b'}{c'z + d'}$$

**Problem 2.a**

(Ahlfors) Recall that all lines in the complex plane can be represented as $a + bt$ where $a, b \in \mathbb{C}$ and $t \in \mathbb{R}$. Also the left half of the line is all $z$ such that

$$\text{Im} \left( \frac{z - a}{b} \right) < 0$$

and the right half of the line is all $z$ such that

$$\text{Im} \left( \frac{z - a}{b} \right) > 0$$

Now if $a + bt$ is a horizontal line, then it’s trivial. Now let

$$P(z) = A(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \cdots (z - \alpha_n)$$

Then we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \cdots + \frac{1}{z - \alpha_n}$$

So suppose the that half plane $H$ is defined as the part of the plane where $\text{Im} \left( \frac{z - a}{b} \right) < 0$. Now we know that for all $k, \alpha_k$ is in $H$. Suppose that $z \not\in H$, where $P'(z) = 0$. Then we have

$$\text{Im} \left( \frac{z - \alpha_k}{b} \right) = \text{Im} \left( \frac{z - a}{b} \right) - \text{Im} \left( \frac{\alpha_k - a}{b} \right) > 0$$

Also recall that imaginary parts of reciprocal numbers have opposite sign. Therefore, under the same assumption, $\text{Im} \left( \frac{b}{z - \alpha_k} \right) < 0$. Since this is true for all $k$, we have

$$\text{Im} \left( \frac{z - \alpha_k}{b} \right) = \sum_{k=1}^{n} \text{Im} \left( \frac{\alpha_k - a}{b} \right) < 0$$

hence $P'(z) \neq 0$, and we have a contradiction. This concludes the proof.
Problem 3.a
Clearly $f$ is one to one since if
\[ f(z_1) = f(z_2) \Rightarrow z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2} \Rightarrow z_1 - z_2 = \frac{z_1 - z_2}{z_1 z_2} \]
Now if $z_1 \neq z_2$, then we have
\[ z_1 z_2 = 1 \]
However this is impossible since both $|z_1|$ and $|z_2|$ is greater than 1. Hence $f$ is one-to-one. Now $f$ is analytic in $E$ and for all $z \in E$, we have
\[ f'(z) = \frac{1}{2} - \frac{1}{2z^2} \neq 0 \]
Hence $f$ is also a conformal mapping.

Problem 3.b
Now I claim that $f$ maps $E$ to the entire complex plane except the real interval $[-1, 1]$. For $z = x + iy \in E$
\[ f(z) = \frac{1}{2} \left( (x + iy) + \frac{1}{x + iy} \right) = \frac{1}{2} \left( \frac{x(r^2 + 1)}{r^2} + iy \frac{r^2 - 1}{r^2} \right) \]
if $f(z) \in [-1, 1]$, then either $r^2 = 1$, which is not possible in $E$, or $y = 0$. But then $(1/2)(x+1/x) \in [-1, 1]$. But that implies $x = 1$ which is not possible. Now to show that $f$ is onto, notice the preimage
\[ w = \frac{1}{2} \left( z + \frac{1}{z} \right) \Rightarrow z^2 - 2wz + 1 = 0 \]
Hence
\[ z = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1} \]
Hence if $w + \sqrt{w^2 - 1} = re^{i\theta}$, then $w - \sqrt{w^2 - 1} = (1/r)e^{-i\theta}$. Hence one of these is outside the unit circle since $r \neq 1$ (otherwise that would imply $z = \pm 1$).

Problem 3.c
The residue at $\infty$ is the residue of $f(1/z)$ at 0. Hence the residue at $\infty$ for $f$ is $1/2$. 

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Problem 5.a

Let \( f(z) = \log z/(1 + z^2) \), and let \( \gamma \) be the contour of two semi-circles with radius \( R \) and \( \epsilon \) in the upper-half plane. Then by the Residue Theorem, we have

\[
\int_{\gamma} f(z) \, dz = 2\pi i \frac{\log i}{2i} = i \frac{\pi^2}{2}
\]

Now notice on \( C_R \) we have \( z = Re^{i\theta} \). Hence

\[
\left| \int_{C_R} \frac{\log z}{1 + z^2} \, dz \right| \leq \pi R \frac{\log(Re^{i\theta})}{R^2 - 1} \leq \pi R \frac{\sqrt{\log^2 R + \pi^2}}{R^2 - 1} \to 0
\]
as \( R \to \infty \). Now notice that on \( C_\epsilon \) we have \( z = \epsilon e^{i\theta} \). Hence

\[
\left| \int_{C_\epsilon} \frac{\log z}{1 + z^2} \, dz \right| \leq \frac{\pi \epsilon \sqrt{\log^2 \epsilon + \pi^2}}{1 - \epsilon^2} \to 0
\]
as \( \epsilon \to 0 \). So as \( \epsilon \to 0 \) and \( R \to \infty \), we have

\[
\int_0^\infty \frac{\log z}{1 + z^2} \, dz + \int_0^\infty \frac{\log z}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

which implies

\[
\int_0^\infty \frac{\log(-z)}{1 + z^2} \, dz + \int_0^\infty \frac{\log(z)}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

which implies

\[
2 \int_0^\infty \frac{\log z}{1 + z^2} \, dz + i \int_0^\infty \frac{\pi}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

Thus

\[
\int_0^\infty \frac{\log z}{1 + z^2} \, dz = 0
\]

Problem 5.b

We have two cases. Case 1: Assume that \( w \) is a purely imaginary number, \( w = ib \) where \( b \in \mathbb{R} \). Then we have

\[
\int_{\infty}^{-\infty} e^{-2\pi i tw} e^{-t^2/2} \, dt = \int_{\infty}^{-\infty} e^{(t-2\pi b)^2/2 + 4\pi^2 b^2/2} = e^{\frac{4\pi^2 b^2}{2}} \int_{-\infty}^{\infty} e^{-(t-2\pi b)^2/2} \, dt
\]

by applying \( u \)-substitution, we can see that

\[
\int_{\infty}^{-\infty} e^{-2\pi i tw} e^{-t^2/2} \, dt = e^{\frac{4\pi^2 b^2}{2}} \sqrt{2\pi}
\]
CASE 2: Now let’s assume \( w = a + bi \) where \( b \neq 0 \). Let \( f(z) = e^{-z^2/2} \) and \( \gamma \) be the contour from \(-R\) to \( R\) on the real axis, then from \( R\) to \( R + i2\pi w\), and to \(-R + i2\pi w\), and final back to \(-R\). By the Residue theorem we have

\[
\int_{\gamma} f(z) \, dz = 0
\]

So we have

\[
\int_{-R}^{R} e^{-z^2/2} \, dz + \int_{0}^{2\pi} e^{-(R+iwz)^2/2} iwdz + \int_{R}^{-R} e^{(-z+i2\pi w)^2/2} \, dz + \int_{0}^{2\pi} e^{-(R+iwz)^2/2} iwdz = 0
\]

Notice that

\[
\left| \int_{0}^{2\pi} e^{-(R+iwz)^2/2} \, dz \right| \leq e^{-R^2/2} \int_{0}^{2\pi} e^{Rbz + a^2 z^2/2 - b^2 z^2/2} \, dz \leq e^{-R^2/2} \frac{e^{Rb^2 + a^2 z^2/2}}{2\pi} \to 0
\]
as \( R \to \infty \). Likewise

\[
\int_{2\pi}^{0} e^{-(R+iwz)^2/2} \, dz \to 0
\]
as \( R \to \infty \). So as \( R \to \infty \) we have

\[
\int_{-R}^{R} e^{-z^2/2} \, dz + \int_{R}^{-R} e^{(-z+i2\pi w)^2/2} \, dz = \int_{-R}^{R} e^{-z^2/2} \, dz + \int_{-\infty}^{-\infty} e^{-z^2/2 - 2\pi izw} \, dz = 0
\]

and hence

\[
\int_{-\infty}^{-\infty} e^{-z^2/2 - 2\pi izw} \, dz = \sqrt{2\pi} e^{-2\pi^2 w^2}
\]

Now let notice it does converge absolutely for all \( w \in \mathbb{C} \).

\[
\int_{-\infty}^{\infty} \left| e^{-2\pi itw - t^2/2} \right| \, dt = \int_{-\infty}^{\infty} e^{-2\pi tw} e^{-t^2/2} \, dt = \int_{-\infty}^{\infty} e^{-(t+w\pi b)^2/2 + 4\pi^2 b^2} \, dt = e^{4\pi^2 b^2} \sqrt{2\pi}
\]

Sept 2000 Linear Algebra

Problem 1.a

We have

\[
D = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & k \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
Problem 1.b

We will let \( v_i = x^i \) be the bases for \( i \leftarrow 0 \) to \( k \). So notice what \( L \) does on the bases. For \( j \leftarrow 0 \) to \( k \), we have

\[
Lx^j = \sum_{p=0}^{n} \frac{1}{p!} D^p x^j
\]

Now we know that \( D^p x^j = j(j-1)(j-2) \cdots (j-p+1) x^{j-p} \) if \( p \leq j \) and \( D^p x^j = 0 \) else. Thus

\[
Lx^j = \sum_{p=0}^{j} \frac{j!}{p!(j-p)!} x^{j-p} = \sum_{p=0}^{j} \left( \begin{array}{c} j \\ p \end{array} \right) x^{j-p} = (1 + x)^j
\]

Hence \( \text{dim}(\ker(L - T)) = k + 1 \).

Problem 2.a

Notice for

\[ V_1 = \{ Bx : x \in \mathbb{R}^n \} \]

we have \( \text{dim}(V_1) + n(V_1) = n \) where \( n \geq 0 \). Hence

\[
\text{dim}(V_1) = \text{Rank}(B) \leq n
\]

Now let \( V_2 = \{ Ax : x \in V_1 \} \). Then \( \text{dim}(V_2) + n(V_2) = \text{dim}(V_1) = \text{Rank}(B) \). Also \( \text{Rank}(AB) = \text{dim}(V_2) \). SO we have

\[
\text{Rank}(A) = \text{Rank}(AB) \leq \text{Rank}(B)
\]

Problem 2.b

It suffices to show that \( \text{Rank}(AB) \leq \text{Rank}(A) \). Suppose \( \text{Rank}(AB) > A \). Then \( \exists x \in \mathbb{R}^n \) such that for

\[
A = \left( \begin{array}{c|c|c} \| & \cdots & \| \\ c_1 & \cdots & c_p \end{array} \right)
\]

Then \( ABx \notin S(c_1,\ldots,c_p) \). But

\[
ABx = \left( \begin{array}{c|c|c} \| & \cdots & \| \\ c_1 & \cdots & c_p \end{array} \right) Bx = \left( \begin{array}{c|c|c} \| & \cdots & \| \\ c_1 & \cdots & c_p \end{array} \right) \left( \begin{array}{c} \vdots \\ a_1 \\ \vdots \\ a_p \end{array} \right) \in S(a_1,\ldots,a_p)
\]

Hence we have a contradiction and thus \( \text{Rank}(AB) \leq \text{Rank}(A) \).
Problem 3.a

If $A$ satisfies $(A - I)(A - 2I)^2 = 0$, then $(A - I)(A - 2I)^2 x = 0 \quad \forall x \in \mathbb{R}^n$

If $Ax = \lambda x$, then the above is $(\lambda - 1)(\lambda - 2)^2 x$. So the only possible eigenvalues are 1 and 2. So $A - \lambda I$ just bumps down the generalized eigenvectors. So $A$ cannot have a Jordan Block of size $> 1$ for $\lambda = 1$ and cannot have a Jordan Block of size $> 2$ for $\lambda = 2$. So the 9 possible Jordan Blocks are

$$
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 1 \\
2 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
1 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix}
$$

remember that 2 matrices are similar if they have the same Jordan Block. So describing the number of conjugacy class is equivalent to describing the number of Jordan blocks.

Problem 4

We have

$$
\det(A - \lambda I) = \begin{pmatrix}
1 - \lambda & 2 \\
4 & 1 - \lambda
\end{pmatrix} = \lambda^2 - 2\lambda - 7 = 0
$$

Then we have $\lambda_1 = 1 + 2\sqrt{2}$ and $\lambda_2 = 1 - 2\sqrt{2}$. Then we find our eigenvectors $v_1 = (1, \sqrt{2})^T$ and $v_2 = (1, -\sqrt{2})^T$. Now notice that $S(v_1, v_2) = \mathbb{R}^2$. Hence

$$
A^n v = c_1 A^n \left( \begin{array}{c}
1 \\
\sqrt{2}
\end{array} \right) + c_2 A^n \left( \begin{array}{c}
1 \\
-\sqrt{2}
\end{array} \right) = c_1 \left( \begin{array}{c}
1 + 2\sqrt{2} \\
4
\end{array} \right)^n \left( \begin{array}{c}
1 \\
\sqrt{2}
\end{array} \right) + c_2 \left( \begin{array}{c}
1 - 2\sqrt{2} \\
4
\end{array} \right)^n \left( \begin{array}{c}
1 \\
-\sqrt{2}
\end{array} \right)
$$

so when $c_1 = 0$, then series converges. Therefore $v = c_2 (1, -\sqrt{2})^T$ for it to converge.

Problem 5.a

Let $A = (v_1, v_2, ..., v_n)$. Then $M = A^T A$ and

$$
x^T M x = x^T A^T A x = (Ax)^T A x = |Ax|^2 \geq 0
$$

Problem 5.b

We know that $A^T A$ is invertible if and only if the columns of $A$ are linearly independent.
Problem 5.c

Notice when we perform Gram Schmidt of $v_i$, we have

$$e_n = v_1 - \frac{(v_1, v_n)}{(v_1, v_1)} v_1 - \frac{(v_2, v_n)}{(v_2, v_2)} v_2 - \cdots - \frac{(v_{n-1}, v_n)}{(v_{n-1}, v_{n-1})} v_{n-1}$$

So

$$\begin{pmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} | & \cdots & | \\ v_1 & \cdots & v_{n-1} & e_n \end{pmatrix} \begin{pmatrix} 1 & X \\ \vdots & \ddots & \vdots & \vdots \\ X & X & 1 \end{pmatrix}$$

So

$$A^T A = \begin{pmatrix} 1 \\ \vdots \\ X & X & 1 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ - & \vdots & - \\ - & e_n & - \end{pmatrix} \begin{pmatrix} | & \cdots & | \\ v_1 & \cdots & v_{n-1} & e_n \end{pmatrix} \begin{pmatrix} 1 & X \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & e_n^T e_n \end{pmatrix}$$

So $\det(M_n) = |e_n|^2 \det(M_{n-1})$. 
Jan 2001 Advanced Calculus

Problem 1.a

Since $f$ is continuous and periodic with period $p$, we have $\forall x \in \mathbb{R}$, $f(x + p) = f(x)$ and $f$ is bounded. For any given $c > 0$, we want to show that $\exists x_0 \in \mathbb{R}$, such that

$$f(x_0 + c) = f(x_0)$$

now since $f(x_0 + c + np) = f(x_0 + c)$ for all $n \in \mathbb{Z}$, it suffices to show that

$$f(x_0 + c) = f(x_0)$$

for $c \in [0, p]$. Let $g(t) = f(t + c) - f(t)$. Assume for all $t_0 \in R$, $g(t_0) \neq 0$ which implies $g(t) > 0$ or $g(t) < 0$ for all $t$. WLOG assume $g(t) > 0$ for all $t$. Then

$$f(t + c) > f(t)$$

for all $t$. Then $f$ is unbounded since $f(t + c) > f(t)$ and so we have a contradiction. Hence $\exists t_0$ such that $g(t_0) = 0$ and so $f(t_0 + c) = f(t_0)$.

Problem 1.b

By part (a), we know that $\exists x_0$ such that $f(x_0 + c) = f(x_0)$. Hence for $c = 1/5$ and $p = 1$, for all integers $n$, $f(x_0 + c + n) = f(x + 0)$ since $f$ is periodic. Let $x_1 = x_0 + n$ such that $x_1 \in [0, 1]$. Then $f(x_1 + c) = f(x_1)$. Now if $x_1 \in [0, 4/5]$, there’s nothing to prove. Hence if $x_0 \in (4/5, 1)$, then

$$g(x_1) = f(x_1 + c) - f(x_1) = 0 = g(x_1 + 1/5 - 1)$$

hence this completes the proof since $x_1 + 1/5 - 1 \in [0, 4/5]$.

Problem 2.a

We want to show for suitable $A$ and $B$ we have

$$A\sqrt{n\log n} \leq \sqrt[3]{3}\sqrt[4]{4}\sqrt[5]{5} \cdots \sqrt[n]{n} \leq B\sqrt{n\log n}$$

It suffices to show that

$$\log(A) + \frac{1}{2} \log^2 n \leq \sum_{k=3}^{n} \frac{\log k}{k} \leq \log(B) + \frac{1}{2} \log^2 n$$
By a crude Riemann sum estimation, we have

$$\int_3^n \log \frac{x}{x} \, dx \leq \int_3^{n+1} \log \frac{x}{x} \, dx \leq \sum_{k=3}^n \frac{\log k}{k} \leq \int_2^n \log \frac{x}{x} \, dx$$

which implies

$$-\frac{1}{2} \log^2(3) + \frac{1}{2} \log^2 n \leq \sum_{k=3}^n \frac{\log k}{k} \leq -\frac{1}{2} \log^2(2) + \frac{1}{2} \log^2 n$$

**Problem 2.b**

By part (a) we can see that

$$A \leq \frac{\sqrt[3]{3} \sqrt[4]{4} \sqrt[5]{5} \cdots \sqrt[n]{n}}{\sqrt[n]{n\log n}} \leq B$$

which implies

$$\log(A) \leq \sum_{k=3}^n \frac{\log k}{k} - \frac{1}{2} \log^2 n \leq \log(B)$$

It suffices to show that for

$$a_n = \sum_{k=3}^n \frac{\log k}{k} - \frac{1}{2} \log^2 n$$

is a monotonically decreasing function with a lower bound. This implies that it does indeed converge. So notice

$$a_{n+1} - a_n = \frac{\log(n+1)}{n+1} - \frac{1}{2} \log^2(n+1) + \frac{1}{2} \log^2 n$$

$$= \frac{\log(n+1)}{n+1} + \frac{1}{2} (\log n - \log(n+1)) (\log(n^2 + n)) = \frac{\log(n+1)}{n+1} + \frac{1}{2} \left( \log \left( \frac{n}{n+1} \right) \right) (\log(n^2 + n))$$

and notice that \(\log(n/(n+1)) < 0\) and \(\log(n^2 + n) > \log(n+1)/(n+1)\). Since \(a_{n+1} - a_n < 0\) for all \(n\). This completes the proof.

**Problem 3.a**

Since \(f \in C[0,1] f\) and \(f'\) is bounded on \([0,1]\). Also we can see that \(g\) is continuous and periodic with period 1. Let

$$G(x) = \int_0^x g(w) \, dw$$

Then \(G(x)\) is a bounded function on \([0,1]\). So let \(u = f(x)\, du = f'(x), dv = \int g(nx) \, dx\) and \(v = G(nx)/n\). Then by integration by parts we have
\[
\lim_{n \to \infty} f(x) g(nx) dx = \lim_{n \to \infty} \frac{f(x) G(nx)}{n} \left[ 1 - \int_0^1 G(nx) f'(x) dx \right]
\]

So by the Dominated Convergence Theorem, we have

\[
\lim_{n \to \infty} \left[ \int_0^1 f(x) g(nx) dx \right] = 0
\]

**Problem 3.b**

\( f \) is continuous on \([0, 1]\) can be uniformly approximated by a polynomial \( p(x) \) by the Weierstrass-Stone Theorem. Hence

\[
\lim_{n \to \infty} \left| \int_0^1 f(x) g(nx) dx \right| \leq \lim_{n \to \infty} \int_0^1 |p(x) g(nx)| dx + \int_0^1 |e g(nx)| dx \to 0
\]

by part (a) and since \( \epsilon \) is arbitrary.

**Problem 4.a**

Since \( f \) is continuous on \([0, 1]\) \( f \) can be approximated uniformly by a polynomial \( P(x) \). Then notice

\[
\lim_{n \to \infty} n \int_0^{1/n} P(t) dt = \lim_{n \to \infty} n \left[ \frac{P_0}{n} + \frac{P_1}{2n^2} + \cdots + \frac{P_k}{(k+1)n^{k+1}} \right] = P_0 = P(0) = f(0) + \epsilon = f(0)
\]

since \( \epsilon \) is arbitrary. So notice

\[
\lim_{n \to \infty} n \int_0^{1/n} (P(t) - \epsilon) dt \leq \lim_{n \to \infty} \int_0^{1/n} f(t) dt \leq \lim_{n \to \infty} (P(t) + \epsilon) dt
\]

implies

\[
\lim_{n \to \infty} n \int_0^{1/n} f(t) dt = f(0)
\]

**Problem 5.a**

We have

\[
\frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx}
\]

and

\[
\frac{d^2 f}{dx^2} = \frac{d^2 f}{dt^2} \left( \frac{dt}{dx} \right)^2 + \frac{d^2 t}{dx^2} \frac{df}{dt} = \frac{d^2 f}{dt^2} 4x^2 + 2 \frac{df}{dt}
\]

Likewise
\[
\frac{d^2 f}{dy^2} = \frac{d^2 f}{dt^2} \left( \frac{dt}{dy} \right)^2 + \frac{d^2 t}{dy^2} \frac{df}{dt} = \frac{d^2 f}{dt^2} 4y^2 + 2 \frac{df}{dt}
\]

Hence

\[
\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = 0 = 4t \frac{d^2 f}{dt^2} + 4 \frac{df}{dt} = 4 \frac{d}{dt} \left( t \frac{df}{dt} \right) = 0
\]

Hence for constant \( c \) we have

\[
t \frac{df}{dt} = c \Rightarrow f = c \log t + b
\]

**Problem 5.b**

Let \( u = x + y \) and \( v = x \). Then

\[
\int_0^1 \int_0^1 1 - xf(x+y) dxdy = \int_0^1 \int_0^u f(u) dud = \int_0^1 uf(u) du
\]

**Jan 2001 Complex Analysis**

**Problem 1**

Since we can define a branch cut for

\[
\sqrt{1 - z^2} = e^{\frac{i}{2} \log(1-z^2)}
\]

where

\[
\log(1 - z^2) = \ln(|1 - z^2|) + i \text{arg}(1 - z^2)
\]

where \( 0 < \text{arg}(1 - z^2) < 2\pi \). Then we have two contours \( \gamma_1 \) and \( \gamma_2 \) in the upper and lower half plane with radius \( R \) and are only \( \epsilon \) away from the interval \([-1, 1]\). Then by the residue theorem, we have for \( f(z) = \frac{1}{\sqrt{1 - z^2}} \)

\[
\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz = 0 = \oint_{C_R} f(z) dz + \int_1^{-1} \frac{1}{\sqrt{1 - z^2}} + \int_{-1}^{1} e^{i}\sqrt{1 - z^2} dz
\]

Hence

\[
\oint_{C_R} f(z) dz = 4 \int_0^1 \frac{1}{\sqrt{1 - z^2}} dz = 2\pi
\]

so it does not change as \( R \to \infty \).
Problem 2

By performing $u$ substitution, we can see that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = 2 \int_0^\infty \sin x^2 dx$$

Thus let $f(z) = e^{iz^2}$, and let $\gamma$ be the curve along $[0, R]$, $Re^{it}$ for $0 \leq t \leq \pi/4$, and $te^{i\pi/4}$ for $0 \leq t \leq R$. Thus by the residue theorem we have

$$\int_\gamma f(z)dz = \int_0^R e^{iz^2}dz + \int_0^{\pi/4} iRe^{i\theta} e^{i(Re^{i\theta})^2}d\theta + \int_0^R e^{i(te^{i\pi/4})^2} e^{i\pi/4}dt$$

Now notice

$$\left| \int_0^{\pi/4} iRe^{i\theta} e^{i(Re^{i\theta})^2}d\theta \right| \leq \int_0^{\pi/4} \left| Re^{i2\theta} e^{-R^2 \sin(2\theta)} \right| d\theta = \int_0^{\pi/4} Re^{-R^2 \sin(2\theta)}d\theta$$

Now recall that for $0 < \theta < \pi/2$ we have $2\theta/\pi \leq \sin \theta$. Thus our situation we have $0 < 2\theta < \pi/2 \Rightarrow 4\theta/\pi \leq \sin 2\theta$. Hence

$$\int_0^{\pi/4} Re^{-R^2 \sin(2\theta)}d\theta = \frac{\pi}{4R} (e^{-R^2} - 1) \to 0$$

as $R \to \infty$. Hence we have

$$\int_0^R e^{iz^2}dz + \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \int_0^R e^{-z^2}dz = 0$$

now recall that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$. Thus as $R \to \infty$ we have

$$\int_0^\infty \cos z^2 + i \sin z^2dz = \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \frac{\sqrt{\pi}}{2}$$

Which implies that

$$\int_0^\infty \sin z^2dz = \frac{\sqrt{2\pi}}{4}$$

Hence

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = \frac{\sqrt{2\pi}}{2}$$
Problem 3

we perform our conformal mappings in the sequence \( z^2 \to (z + 1/z) \to z + 2 \to 1/z \). Then we have

\[
w = \frac{z^2}{(z^2 + 1/z^2) + 2} = \frac{z^2}{(z^4 + 1)^2 + 2z^2}
\]

In order to satisfy the boundary points given, we have

\[
w = \frac{4z^2}{(z^2 + 1)^2}
\]

Problem 4

For \( c > 3 \) by Rouche we have

\[
|z^3 + cz^2 + z + 1 - cz^2| = |z^3 + z + 1| \leq 3 < |cz^2| = |cz^2|
\]

Hence \( z^3 + cz^2 + z + 1 \) has 2 roots inside \( |z| < 1 \). Now for \( c > 1 \) notice

\[
-2x < c + 1 \quad \forall x \in [-1, 1]
\]

due to

\[
-2x(c-1) < (c+1)(c-1) \quad \Rightarrow \quad 2x-2xc < c^2-1 \quad \Rightarrow \quad x^2+y^2+2x+1 < c^2+2xc+x^2+y^2
\]

which implies

\[
(x + 1)^2 + y^2 < (x + c)^2 + y^2 \quad \Rightarrow \quad |z + 1| < |z + c|
\]

Thus by Rouche we have

\[
|z^3 + cz^2 + z + 1 - z^3 - cz^2| = |z + 1| < |z + c| = |z|^2|z + c| = |z|^3 + cz^2|
\]

Thus \( z^3 + cz^2 + z + 1 \) has two roots inside \( |z| < 1 \). Finally if \( c = 1 \) notice

\[
z^3 + z^2 + z + 1 = (z + 1)(z + i)(z - i)
\]

and so there are no roots inside \( |z| < 1 \).

Problem 5

Let \( g(z) = f(1/z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \). Now clearly \( z_0 = re^{i\theta} \) is a zero of \( f \) if and only if \( (1/r)e^{-i\theta} \) is a zero of \( g \). So we will show that all the zeros of \( g \) lie outside the circle with radius \( \sqrt{n} \) for all \( n \). Notice by Rouche we have

\[
|g(z) - e| = \left| \frac{z^{n+1}}{(n+1)!} + \frac{z^{n+2}}{(n+2)!} + \cdots \right| \leq \frac{(\sqrt{n})^{n+1}}{(n+1)!} + \frac{(\sqrt{n})^{n+2}}{(n+1)!} + \cdots < \frac{(\sqrt{n})^n}{n!}
\]

We can see the last inequality since
\[
\frac{n!}{(\sqrt{n})^n} \left( \frac{(\sqrt{n})^{n+1}}{(n+1)!} + \frac{(\sqrt{n})^{n+2}}{(n+2)!} + \cdots \right) = \frac{n}{n+1} + \frac{(\sqrt{n})^2}{(n+1)(n+2)} + \cdots < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1
\]

Now since

\[
\lim_{n \to \infty} \frac{\sqrt{n}}{(n!)^{1/n}} \to 0
\]

We have

\[
\frac{\sqrt{n}}{(n!)^{1/n}} + \frac{1}{\sqrt{n}} \leq 1
\]

for large enough \( n \) Hence

\[
\frac{\sqrt{n}}{(n!)^{1/n}} \leq 1 - \frac{1}{\sqrt{n}} = 1 - \frac{\sqrt{n}}{n}
\]

which implies

\[
\frac{(\sqrt{n})^n}{n!} \leq \left( 1 - \frac{\sqrt{n}}{n} \right)^n \leq e^{-\sqrt{n}}
\]

Thus

\[
|g(z) - e^z| < |e^{-\sqrt{n}}| \leq |e^z|
\]

for \( |z| = \sqrt{n} \). Hence there are no zeros inside the circle \( |z| = \sqrt{n} \) which implies all zeros of \( f \) are inside the circle \( |z| = 1/\sqrt{n} \). SO for any \( r > 0 \), \( \exists N \) such that for \( n > N \) \( f \) has all zeros inside \( |z| = r \).

**Jan 2001 Linear Algebra**

**Problem 1.a**

Let \( x = (a, b, c, d) \), such that \( (x, v) = a + b + 2c + 2d = 0 \). Then

\[
Ax = \begin{pmatrix}
4b + 2c + 6d \\
4a + 4c + 4d \\
2a + 4b + 9c + 8d \\
6a + 4b + 8c + 7d 
\end{pmatrix}
\]

Then

\[
(Ax, v) = (4b + 2c + 6d) + (4a + 4c + 4d) + 2(2a + 4b + 9c + 8d) + 2(6a + 4b + 8c + 7d) = 0
\]
Problem 1.b

First we will show that $W$ is a subspace. If $w_1, w_2 \in W$, then $((w_1 + w_2), v) = (w_1, v) + (w_2, v) = 0$. Also $0 \in W$ since $(0, v) = 0$. Finally for $w \in W$, we have $(\alpha w, v) = \alpha (w, v) = 0$. So $W$ is a subspace. Now notice for $w_1, w_2 \in W$, we have

$$L(w_1 + w_2) = A(w_1 + w_2) = Aw_1 + Aw_2 = L(w_1) + L(w_2) \in W$$

and $L(0) = A(0) = 0$, and for $w \in W$

$$L(\alpha w) = A(\alpha w) = \alpha Aw = \alpha L(w) \in W$$

Hence $L$ is a linear mapping from $W$ to $W$.

Problem 1.c

We need to represent $A$ in the basis of $W$. Now the basis of $W$ is $\{(1, -1, 0, 0), (0, 1, -1, 0), (0, 0, 1, -1)\}$. Notice dim($W$) = 3. Let let $L$ represent the matrix form of $L$ in basis $W$. We can let the first column of $L$ be $(1, -1, 0, 0)^T$. Then

$$L \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = L \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + L \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies

$$L \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 2 \\ -2 \end{pmatrix}$$

Now

$$L \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} L \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2} L \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which implies

$$L \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 5 \\ -4 \end{pmatrix}$$

Finally

-8
\[
L \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = L \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + L \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

which implies
\[
L \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 6 \\ -3 \end{pmatrix}
\]

Hence
\[
L = \begin{pmatrix} 1 & 5 & 4 & 8 \\ -1 & -5 & -6 & 6 \\ 0 & 2 & 5 & 6 \\ 0 & -2 & -4 & -3 \end{pmatrix}
\]

Hence \(Tr(L) = Tr(L) = -2\).

**Problem 2**

If \(A^T = A\), then
\[
(AB)^T = B^T A^T = BA
\]

So let \(D = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), then we have
\[
AB + BA = D + D^T = \begin{pmatrix} 2a & b + c \\ b + c & 2d \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 7 & 12 \end{pmatrix}
\]
this implies \(a = 2, d = 6\) and \(b + c = 7\). So
\[
AB = \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\]

If \(B\) is invertible, then we have
\[
A = \begin{pmatrix} 2 & b \\ c & 6 \end{pmatrix} B^{-1}
\]

So there exists symmetric \(A\) if \(B\) is invertible. Now notice
\[
det(B) = (t + 4)(t - 1) = 0
\]
when \(t = -4, 1\). So now we need to see what happens if \(t = 1\) or \(t = -4\). If \(t = 1\), then we have
\[
AB = \begin{pmatrix} c & f \\ f & g \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & b \\ c & 6 \end{pmatrix}
\]
which implies

\[ e + 2f = 2 \quad 2e + 4f = b \]

\[ f + 2g = c \quad 2f + 4g = 6 \]

which implies \( e + 2f = 2 \) and we don’t get a contradiction. So \( t = 1 \) is ok. Now if \( t = -4 \) we have

\[ AB = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & b \\ c & 6 \end{pmatrix} \]

which implies

\[ -4e + 2f = 2 \quad 2e - f = b \]

\[ -4f + 2g = c \quad 2f - g = 6 \]

which implies \( 0 = 21 \), and hence we have a contradiction. Therefore \( t \neq -4 \) is our solution.

**Problem 3.a**

Find counter example

**Problem 3.b**

No, notice

\[
\lambda_{AB_{\text{max}}} \leq \max_x \frac{x^T A B x}{x^T x} = ||A|| \leq ||A|| ||B|| \lambda_{\text{max}(A)} \lambda_{\text{max}(B)} < 1
\]

**Problem 4**

\( A \) is symmetric and let \( \lambda_1, ..., \lambda_n \) be the eigenvalues of \( A \) which are real since \( A \) is symmetric. Also let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Now

\[
\lambda_2 = \max_{\dim(S) = n-1} \min_{x \in S} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}
\]

WLOG we can assume that \( x \) are unit vectors. Thus \( \langle x, x \rangle = 1 \). Now notice that \( V = \{ x \in F^n : \sum_k x_k = 0 \} \) with dimension \( n - 1 \). This can easily be shown. Hence

\[
\lambda_2 = \max_{\dim(S) = n-1} \min_{x \in S} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \geq \min_{x \in V} \langle x, Ax \rangle \geq 0
\]

Thus \( \lambda_2, ..., \lambda_n \geq 0. \)
Sept 2001 Advanced Calculus

Problem 1a

\[ \sum_{p=1}^{\infty} (-1)^p 2^{1/p} \]

is indeed divergent since

\[ \lim_{p \to \infty} (-1)^p 2^{1/p} \neq 0 \]

Now notice

\[ \sum_{p=1}^{\infty} (-1)^p (1 - 2^{1/p}) = \sum_{p=1}^{\infty} (-1)^{p+1} (2^{1/p} - 1) \]

and notice \((2^{1/p} - 1) \searrow 0\). Thus by the alternating series theorem, \(\sum (-1)^p (1 - 2^{1/p})\) converges.

Problem 1b

Given that \(\sum_{n=1}^{\infty} n a_n\) converges, then notice

\[ \sum_{n=1}^{\infty} n a_n = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_i < \infty \]

Hence \(\sum_{i=j}^{\infty} a_i < \infty\).

Problem 2a

I claim that \(\nabla (\phi_1 \nabla \phi_2) = \nabla \phi_1 \nabla \phi_2 + \phi_1 \nabla^2 \phi_2\). Indeed since

\[ \nabla (\phi_1 \nabla \phi_2) = \nabla \left( \phi_1 \frac{d \phi_2}{dx}, \phi_1 \frac{d \phi_2}{dy}, \phi_1 \frac{d \phi_2}{dz} \right) = \frac{d \phi_1}{dx} \frac{d \phi_2}{dx} + \frac{d^2 \phi_2}{dx^2} \phi_1 + \frac{d \phi_1}{dy} \frac{d \phi_2}{dy} + \frac{d^2 \phi_2}{dy^2} \phi_1 + \frac{d \phi_1}{dz} \frac{d \phi_2}{dz} + \frac{d^2 \phi_2}{dz^2} \phi_1 \]

and notice

\[ \nabla \phi_1 \nabla \phi_2 + \phi_1 \nabla^2 \phi_2 \]

\[ = \nabla \left( \phi_1 \frac{d \phi_2}{dx}, \phi_1 \frac{d \phi_2}{dy}, \phi_1 \frac{d \phi_2}{dz} \right) = \frac{d \phi_1}{dx} \frac{d \phi_2}{dx} + \frac{d^2 \phi_2}{dx^2} \phi_1 + \frac{d \phi_1}{dy} \frac{d \phi_2}{dy} + \frac{d^2 \phi_2}{dy^2} \phi_1 + \frac{d \phi_1}{dz} \frac{d \phi_2}{dz} + \frac{d^2 \phi_2}{dz^2} \phi_1 \]

Thus by the divergence theorem

\[ \int \int \int \nabla \phi_1 \nabla \phi_2 + \phi_1 \nabla^2 \phi_2 dV = \int \int \int \nabla (\phi_1 \nabla \phi_2) dV = \int \int \int n \cdot (\phi_1 \nabla \phi_2) dS \]
Problem 2b

By the divergence theorem, we have
\[ \int \int_S n \cdot (x, y, z) \, dS = \int \int \int \nabla (x, y, z) \, dx \, dy \, dz = \int \int \int 3 \, dx \, dy \, dz = 3V \]
and
\[ \int \int_S x n \cdot (x, y, z) \, dS = \int \int \int \nabla (x^2, xy, xz) \, dx \, dy \, dz = \int \int \int \nabla 4x \, dx \, dy \, dz = 4xV \]

Problem 3

Let
\[ f(x, y, z) = x + y + z \]
\[ h(x, y, z) = x^2 + y^2 + z^2 + 2xz - 1 \]

Then we have
\[ f(\sqrt{2}/2, \sqrt{2}/2, -2) = h(\sqrt{2}/2, \sqrt{2}/2, -2) = 0 \]

and
\[ \frac{d(f, h)}{d(x, y)} = \begin{vmatrix} \frac{df}{dx} & \frac{dh}{dx} \\ \frac{df}{dy} & \frac{dh}{dy} \end{vmatrix} \Big|_{(\sqrt{2}/2, \sqrt{2}/2, -2)} = 2y - 2x - 2z \neq 0 \]

Thus by the implicit function theorem, we can define \( x \) and \( y \) as a function of \( z \) in a neighborhood of \(-\sqrt{2}\). However notice that
\[ \frac{d(f, h)}{d(x, z)} = \begin{vmatrix} \frac{df}{dx} & \frac{dh}{dx} \\ \frac{df}{dz} & \frac{dh}{dz} \end{vmatrix} = 0 \]

for all \( x, y, z \). Thus we cannot define \( x \) and \( y \) as a function of \( y \) in some neighborhood.

Problem 4a

Let \( p \) be some point in \( D \). Since \( D \) is convex, \( \exists \) a line from \( p \) to the origin in \( D \). Also since \( D \) is convex, we can extend that line to a boundary point \( b \) on \( C \) such that the line remains in \( D \). Thus for all points in \( D \), \( x = uf(\phi) \) and \( y = ug(\phi) \) for \( 0 \leq u \leq 1 \) and \( 0 \leq \phi < 2\pi \).

Problem 4b

Recall that the area of a region enclosed by \( C \) is
\[ A = \frac{1}{2} \oint_C -y \, dx + x \, dy \]
Thus
\[ A = \frac{1}{2} \int_{0}^{2\pi} f(\phi)g'(\phi) - f'(\phi)g(\phi) d\phi \]

**Problem 4c**

Thus for calculating the area of the ellipse, we have \( x = a \cos(\phi) \) and \( y = b \sin(\phi) \). Thus

\[ A = \frac{1}{2} \int_{0}^{2\pi} a \cos(\phi)b \cos(\phi) + a \sin(\phi)b \sin(\phi) d\phi = \frac{1}{2} \int_{0}^{2\pi} ab d\phi = \pi ab \]

**Problem 5a**

\[ |f'(x)| = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{y \to x} \frac{|x - y|^2}{|x - y|} = \lim_{y \to x} |x - y| = 0 \]

Thus \( f \) is constant.

**Problem 5b**

We define the function \( g(x) \geq f(x) \) on \([a, b]\), where

\[ g(x) = \frac{f(b) - f(a)}{b - a} x + f(b) - \frac{f(b) - f(a)}{b - a} b \]

Thus

\[ f \left( \frac{a + b}{2} \right) \leq g \left( \frac{a + b}{2} \right) = \frac{a}{2} \left( \frac{f(b) - f(a)}{b - a} \right) + f(b) - \frac{b}{2} \left( \frac{f(b) - f(a)}{b - a} \right) = \frac{f(a) + f(b)}{2} \]

**Aug 2001 Complex Analysis**

**Problem 1.a**

Let

\[ f(z) = e^{iaz} - ae^{iz} - (1 - a) \]

then let \( \gamma \) be the contour of two semi-circles with radius \( \epsilon \) and \( R \). Then notice on the outer semi-circle, \( z = Re^{i\theta} \) and we have

\[ \left| \int_{C_R} f(z) dz \right| \leq \pi R \frac{|e^{\epsilon R \sin \theta}| + |ae^{-R \sin \theta}| + |1 - a|}{R^3} \to 0 \]

as \( R \to \infty \). Also notice

\[ f(z) = \frac{1 + iaz + (iaz)^2/2! + (iaz)^3/3! + \cdots - (a + aiz + a(iz)^2/2! + a(iz)^3/3! + \cdots) - (1 - a)}{z^3} \]
Thus on $C_{\epsilon}$, we have

$$\int_{C_{\epsilon}} f(z)dz = \int_{C_{\epsilon}} f(\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta = -\pi i \frac{a-a^2}{2}$$

as $\epsilon \to 0$. Thus

$$\int_{-R}^{-\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz = \pi i \frac{a-a^2}{2}$$

and notice

$$\int_{-R}^{-\epsilon} \cos(az) - a \cos(z) - (1-a) \frac{z}{z^3} dz + \int_{\epsilon}^{R} \cos(az) - a \cos(z) - (1-a) \frac{z}{z^3} dz$$

$$+ i \left( \int_{-R}^{-\epsilon} \sin(az) - a \sin(z) - (1-a) \frac{z}{z^3} dz + \int_{\epsilon}^{R} \sin(az) - a \sin(z) - (1-a) \frac{z}{z^3} dz \right)$$

Now notice that $\frac{\cos(az) - a \cos(z) - (1-a)}{z^3(z^2+1)}$ is an odd function. Thus the integrals cancel each other out. Also notice that $\frac{\sin(az) - a \sin(z)}{z^3(z^2+1)}$ is an even function. Hence as $R \to \infty$ and $\epsilon \to 0$ we have

$$\int_{0}^{\infty} \frac{\sin(az) - a \sin(z)}{z^3} dz = \pi \frac{a-a^2}{4}$$

**Problem 1.b**

Let

$$f(z) = \frac{e^{iaz} - ae^{iz} - (1-a)}{z^3(z^2+1)}$$

then let $\gamma$ be the contour of two semi-circles with radius $\epsilon$ and $R$. Then notice on the outer semi-circle, $z = Re^{i\theta}$ and we have

$$\left| \int_{C_{\infty}} f(z)dz \right| \leq \pi R \frac{|e^{aR\sin \theta} + |ae^{-R\sin \theta}| + |1-a|}{R^3(R^2+1)} \to 0$$

as $R \to \infty$. Also notice

$$f(z) = \frac{1 + iaz + (iaz)^2/2! + (iaz)^3/3! + \cdots - (a + aiz + a(iz)^2/2! + a(iz)^3/3! + \cdots) - (1-a)}{z^3(z^2+1)}$$

$$= \frac{a-a^2}{2!} \frac{1}{z z^2 + 1} - \frac{i(a^3-a)}{3!} \frac{1}{z^2 + 1} + \frac{(a^4-a)}{4!} \frac{1}{z^2 + 1} + \cdots$$

Thus on $C_{\epsilon}$, we have
\[
\int_{C_\epsilon} f(z)\,dz = \int_{\pi}^{0} f(\epsilon e^{i\theta})i\epsilon e^{i\theta}\,d\theta = -\pi i \frac{a - a^2}{2}
\]

as \(\epsilon \to 0\). Notice we have a pole at \(z = i\), thus

\[
\int_{-\epsilon}^{-\epsilon} f(z)\,dz + \int_{\epsilon}^{R} f(z)\,dz = \pi i \frac{a - a^2}{2} = 2\pi i \frac{e^{-a} - ae^{-1} - (1 - a)}{-i2i} = \pi i \left( e^{-a} - ae^{-1} + 3a/2 - a^2/2 - 1 \right)
\]

and notice

\[
\int_{-\epsilon}^{-\epsilon} \cos(az) - a\cos(z) - (1 - a)z^3 \,dz + \int_{\epsilon}^{R} \cos(az) - a\cos(z) - (1 - a)z^3 \,dz
\]

\[
+ i \left( \int_{-\epsilon}^{-\epsilon} \sin(az) - a\sin(z) - (1 - a)z^3 \,dz + \int_{\epsilon}^{R} \sin(az) - a\sin(z) - (1 - a)z^3 \,dz \right)
\]

Now notice that \(\frac{\cos(az) - a\cos(z) - (1 - a)}{z^3}\) is an odd function. Thus the integrals cancel each other out. Also notice that \(\frac{\sin(az) - a\sin(z)}{z^3}\) is an even function. Hence as \(R \to \infty\) and \(\epsilon \to 0\) we have

\[
\int_{0}^{\infty} \frac{\sin(az) - a\sin(z)}{z^3} \,dz = \frac{\pi}{2} \left( e^{-a} - ae^{-1} + 3a/2 - a^2/2 - 1 \right)
\]

**Problem 2.a**

We know that in order to map \(D\) on to \(D' = \{ z : |z| > 1, 0 < \arg(z) < \pi \}\), we use the mapping \(w = z^2\). Then in order to map \(D'\) to the upper half plane, we use \(Z = \frac{\alpha}{2}(w^2 + 1/w^2)\). Finally mapping the upper half plane onto the unit circle we use the mapping

\[
Z' = \frac{Z - z_0}{Z - \frac{z_0}{\alpha}}
\]

where \(z_0\) is any point in the upper half plane. Thus

\[
w = \frac{\alpha}{2} \left( z^2 + 1/z^2 \right) - z_0 \quad \frac{\alpha}{2} \left( z^2 + 1/z^2 \right) - \frac{z_0}{\alpha}
\]

**Problem 2.b**

We cannot have \(\infty\) map to zero since \(\infty\) is on the boundary of the Domain, and 0 is in the interior of the range. We must have boundary mapping to boundary, and interior mapping to the interior. However we can map \(2\sqrt{i}\) to zero by letting \(\alpha = 2\) and \(z_0 = 15i/4\). I.e.

\[
w = \frac{z^2 + 1/z^2 - 15i/4}{(z^2 + 1/z^2) + 15i/4}
\]
Problem 2.c
The most general mapping would be setting $\alpha = 2$ and $z_0 = i$, hence
$$w = \frac{(z^2 + 1/z^2) - i}{(z^2 + 1/z^2) + i}$$

Problem 4.a
I claim that $f$ is a polynomial of degree at most $m$. Since $f$ is entire, it has a Taylor expansion about $z = 0$. Now using Cauchy’s estimate, we have
$$|f^{(q)}(0)| \leq \frac{q!}{r^q} \max_{|z|=r} |f(z)|$$
Thus for $q > m$, we have
$$|f^{(q)}(0)| \leq \frac{q!}{r^q} |kr^m + L| \to 0$$
as $r \to \infty$. Thus $f$ is a polynomials of degree at most $m$. Thus $n \geq m$.

Problem 4.b
$f$ is a polynomial of degree at most $m$.

Problem 4.c
$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)e^{a_0 + a_1z}$$

Aug 2001 Linear Algebra

Problem 1.a
$$A^n = \frac{1}{3^n} \begin{pmatrix} 2 \cdot 3^{n-1} & -(3^{n-1}) & -(3^{n-1}) \\ -(3^{n-1}) & 2 \cdot 3^{n-1} & -(3^{n-1}) \\ -(3^{n-1}) & -(3^{n-1}) & 2 \cdot 3^{n-1} \end{pmatrix}$$

Problem 1b
Orthonormal Null space is
$$\left\{ \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \right\}$$
Problem 1c

Orthonormal Basis for the Range of $A$ is

$$\left\{ \left( \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right), \left( 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \right\}$$

Problem 1d

It the the projection of a vector onto the surface $x + y + z = 0$. Not sure why.

Problem 2a

$$L = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{pmatrix}$$

Problem 3.a

So if $R(t)$ is orthogonal for all $t$. Then

$$R(t)R^T(t) = I$$

so

$$\frac{d}{dt} \left( R(t)R^T(t) \right) = \frac{dR(t)}{dt}R^T(t) + R(t)\frac{dR^T(t)}{dt} = 0$$

which implies

$$\Omega RR^T + R(\Omega R)^T = 0$$

Hence

$$\Omega + \Omega^T = 0$$

now to prove sufficiency, we have

$$\frac{d}{dt}(R^T R) = \frac{dR^T}{dt}R + R^T\frac{dR}{dt} = (\Omega R)^T R + R^T \Omega R$$

If $\Omega$ is skew symmetric, then we have

$$\frac{d}{dt}(R^T R) = 0$$

which implies that $R^T R$ is constant. Since $R(0)$ is orthogonal, this implies $R(t)$ is orthogonal for all $t$. 

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Problem 3.b

\[ \Omega = \begin{pmatrix} 0 & a(t) & b(t) \\ -a(t) & 0 & c(t) \\ -b(t) & -c(t) & 0 \end{pmatrix} \]

Problem 3.c

So \( x(t) = R(t)x_0 \), then \( x'(t) = \Omega(t)R(t)x_0 = \Omega(t)x(t) \). Then in components we have

\[ x'(t) = \vec{x} \times \begin{pmatrix} c \\ -b \\ a \end{pmatrix} \]

Problem 5a

I claim that \( A = 0 \). Notice if \( \exists a_{i,j} = 0 \). Then by define \( B \) such that \( b_{j,i} = 1 \) and zero everywhere else. Then \( Tr(AB) = a_{i,j} \neq 0 \). Hence we have a contradiction and thus \( A = 0 \).

Problem 5b

If \( Tr(AB) = 0 \) for all \( B \) such that \( B^T = B \), then \( A = -A^T \).

Problem 5c

If \( Tr(AB) = 0 \) for all \( B \) such that \( B^T = B \), then \( A = A^T \).

Jan 2002 Advanced Calculus

Problem 1a

For \( x \leq 1 \) notice

\[ \sum_{n=1}^{\infty} \frac{1}{n^x} \left( 1 + \frac{1}{2^x} + \frac{1}{3^x} + \cdots + \frac{1}{n^x} \right) \geq \sum_{n=1}^{\infty} \frac{1}{n^x} \]

And we know that the right side diverges for \( x \leq 1 \), Thus the summation diverges. Now for \( x > 1 \), we know that \( \sum 1/n^x \leq M \) since it converges. Hence

\[ \sum_{n=1}^{\infty} \frac{1}{n^x} \left( 1 + \frac{1}{2^x} + \frac{1}{3^x} + \cdots + \frac{1}{n^x} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^x} \sum_{n=1}^{\infty} \frac{1}{n^x} \leq M^2 \]

Hence the summation converges for \( x > 1 \).
Problem 1b

Notice for $x < -1/2$, we have $0 < |p| = |(x + 2)/(x - 1)| < 1$. Hence

\[ \lim_{n \to \infty} |\sin(x/n)p^{n-1}| \leq \lim_{n \to \infty} |p^{n-1}| < \infty \]

Thus since it converges absolutely, the summation converges. Now for $x = -1/2$, notice

\[ \sum_{n=1}^{\infty} \sin(x/n)(-1)^{n-1} \]

since $\sin(x/n) \searrow 0$, by the alternating series theorem, the summation converges. Now for $x > -1/2$, we have

\[ \lim_{n \to \infty} \sin(x/n)p^{n-1} \not\to 0 \]

thus is diverges.

Problem 3

We have

\[ \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_0^1 f(x)e^{-(\sin^2 x)/\epsilon}dx \]

let $y = \sin x$, $dy = (1/\sqrt{\epsilon}) \cos x \, dx$, $x = \arcsin(\sqrt{\epsilon} y)$, and $\cos x = \sqrt{1 - y^2}$. Then

\[ = \lim_{\epsilon \to 0} \int_0^{\sin^{-1} \sqrt{\epsilon}} \frac{f(\arcsin(\sqrt{\epsilon} y))}{\sqrt{1 - y^2}} e^{-y^2} dy = \lim_{\epsilon \to 0} \int_0^{\infty} \frac{f(\arcsin(\sqrt{\epsilon} y))}{\sqrt{1 - y^2}} e^{-y^2} \chi_{[0, \sin^{-1} \sqrt{\epsilon}]} \, dy \]

Now clearly

\[ \left| \frac{f(\arcsin(\sqrt{\epsilon} y))}{\sqrt{1 - y^2}} e^{-y^2} \chi_{[0, \sin^{-1} \sqrt{\epsilon}]} \right| \leq 1000e^{-y^2} \]

which is integrable on $(0, \infty)$. Hence by the dominated convergence theorem, we have

\[ = \int_0^{\infty} f(0)e^{-y^2} \, dy = f(0) \frac{\sqrt{\pi}}{2} \]

Now if the integral was from 0 to $\pi$, we have

\[ \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_0^{\pi} f(x)e^{-(\sin^2 x)/\epsilon} \, dx = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_{\pi/2}^{\pi} f(x)e^{-(\sin^2 x)/\epsilon} \, dx + \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \int_0^{\pi/2} f(x)e^{-(\sin^2 x)/\epsilon} \, dx \]

after performing the same substitution, we have

\[ = \lim_{\epsilon \to 0} \int_0^{\pi/2} \frac{f(\arcsin(\sqrt{\epsilon} y))}{\sqrt{1 - y^2}} e^{-y^2} \, dy + \lim_{\epsilon \to 0} \int_0^{1/\sqrt{\epsilon}} \frac{f(\arcsin(\sqrt{\epsilon} y))}{\sqrt{1 - y^2}} e^{-y^2} \, dy = 0 \]
Problem 4
We have \( f(x, y, z) = \sqrt{s(s - x)(s - y)(s - z)} \) where \( s = p/2 \) and \( g(x, y, z) = x + y + z = p \). Thus since \( \nabla f = \lambda \nabla g \), we have
\[
\frac{1}{2}(s(s - x)(s - y)(s - z))^{-1/2}(-s(s - y)(s - z)) = \frac{1}{2}(s(s - x)(s - y)(s - z))^{-1/2}(-s(s - x)(s - y))
\]
Thus \( x = y = z \).

Problem 5.a
Let \( F = (u, v) \). Then notice
\[
\nabla \times F = \frac{dv}{dx} - \frac{du}{dy} = 0
\]
since \( du/dy - dv/dx = 0 \). Now let \( \gamma_1 \) be the contour from \((0,0)\) to \((x,y)\) and let \( \gamma_2 \) be another contour from \((0,0)\) to \((x,y)\) such that \( \gamma_1 \cap \gamma_2 = \emptyset \). Then
\[
\int_{\gamma_1} F \cdot dS - \int_{\gamma_2} F \cdot dS = \int_{\delta E} F \cdot dS
\]
and by Green’s Theorem,
\[
= \int \int_{\Omega} \nabla \times F \cdot dV = 0
\]
Hence
\[
\int_{\gamma_1} F \cdot dS = \int_{\gamma_2} F \cdot dS
\]
Hence the line integral is path independent. So let
\[
\phi(x, y) = \int_{(0,0)}^{(x,y)} F \cdot dS
\]
Since the integration is path independent, let \( \gamma \) be the path from the origin to \((x,0)\) on the \(x\)-axis, and let \( \gamma_2 \) be the line from \((x,0)\) to \((x,y)\). Then notice
\[
\phi(x, y) = \int_{\gamma_1 + \gamma_2} F \cdot dS = \int_{\gamma_1 + \gamma_2} u(r,y)dr + v(x,t)dt
\]
\[
= \int_0^x u(r,y)dr + \int_0^0 v(x,t)dt + \int_x^x u(r,y)dr + \int_0^y v(x,t)dt
\]
Thus
\[
\frac{d\phi}{dx} = u(x,y) \quad \text{and} \quad \frac{d\phi}{dy} = v(x,y)
\]

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Problem 5.b

Notice we have

\[ u = \frac{-y}{x^2 + y^2} \Rightarrow \frac{du}{dy} = \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \]

and

\[ v = \frac{x}{x^2 + y^2} \Rightarrow \frac{dv}{dx} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \]

Thus \( \frac{du}{dy} = \frac{dv}{dx} \) in \( D \) except at \((0,0)\). Now let \( C \) be a circle centered at the origin with radius \( a \) such that it lies in \( D \). Then in the region between \( C \) and \( dD \), we have \( \frac{du}{dy} = \frac{dv}{dx} \). Thus

\[ \oint_{dD} udx + vdy = \oint_{C} udx + vdy \]

Now for \( C \), we can parameterize it by \( r(\theta) = (a \cos(\theta), a \sin(\theta)) \) for \( 0 \leq \theta < 2\pi \). Thus

\[ u = \frac{-y}{x^2 + y^2} = \frac{-\sin(\theta)}{a} \]

\[ dx = -a \sin(\theta)d\theta \]

\[ v = \frac{x}{x^2 + y^2} = \frac{\cos(\theta)}{a} \]

\[ dy = a \cos(\theta)d\theta \]

Therefore

\[ \oint_{C} udx + vdy = \int_{0}^{2\pi} \sin^2 \theta + \cos^2 \theta d\theta = 2\pi \]

This does not contradict part (a) since \( \frac{du}{dy} \neq \frac{dv}{dx} \) at \((0,0)\) in \( D \).

Jan 2002 Complex Analysis

Problem 1

The largest \( R \) is 1. Notice that

\[ w = \sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \]

which implies that

\[ e^{2iz} - 2iwe^{iz} - 1 = 0 \]

so using the quadratic formula, we have
\[ z = -i \log(iw + \sqrt{1 - w^2}) \]

where we define \( \sqrt{1 - w^2} = e^{(1/2)\log(1-w^2)} \) and \( \log z = \ln|z| + i \arg z \) where \( -\pi < \arg z < \pi \).

Hence we are using the principle branch, and deleting the negative real axis. So

\[
\text{arcsin}(z) = -i \log(iz + \sqrt{1 - z^2})
\]

So notice the domain can not include \((-\infty, -1] \cup [1, \infty)\) of the real axis. Hence we have singularities at 1 and -1 which are non-isolated. For \( f(0) = 0 \) we have the analytic continuation

\[
F(z) = \begin{cases} 
- i \log(iz + \sqrt{1 - z^2}) & |z| < 1 \\
- i \log_2(iz + \sqrt{1 - z^2}) & |z| \geq 1, z \neq 1, -1
\end{cases}
\]

Where \( \log_1 \) uses the principle branch, and \( \log_2 \) is defined as

\[
\log_2 z = \ln|z| + i \arg_2(z)
\]

where \( 0 < \arg_2(z) < 2\pi \). So for \( \log_2 \), we are deleting the positive real axis. Then \( F \) is meromorphic in \( \mathbb{C} \).

**Problem 2.a**

By performing \( u \) substitution, we can see that

\[
\int_0^\infty \frac{\sin x}{\sqrt{x}} dx = 2 \int_0^\infty \sin x^2 dx
\]

Thus let \( f(z) = e^{iz^2} \), and let \( \gamma \) be the curve along \([0, R], Re^{it} \) for \( 0 \leq t \leq \pi/4 \), and \( te^{i\pi/4} \) for \( 0 \leq t \leq R \). Thus by the residue theorem we have

\[
\int_\gamma f(z) dz = \int_0^R e^{iz^2} dz + \int_0^{\pi/4} iRe^{i\theta} e^{i(Re^{i\theta})^2} d\theta + \int_0^R e^{i(te^{i\pi/4})^2} e^{i\pi/4} dt
\]

Now notice

\[
\left| \int_0^{\pi/4} iRe^{i\theta} e^{i(Re^{i\theta})^2} d\theta \right| \leq \int_0^{\pi/4} |Re^{iR^2e^{2\theta}}| d\theta = \int_0^{\pi/4} Re^{-R^2 \sin(2\theta)} d\theta
\]

Now recall that for \( 0 < \theta < \pi/2 \) we have \( 2\theta/\pi \leq \sin \theta \). Thus our situation we have \( 0 < 2\theta < \pi/2 \) \( \Rightarrow 4\theta/\pi \leq \sin 2\theta \). Hence

\[
\int_0^{\pi/4} Re^{-R^24\theta/\pi} d\theta = \frac{\pi}{4R} \left(e^{-R^2} - 1\right) \to 0
\]

as \( R \to \infty \). Hence we have

\[
\int_0^R e^{iz^2} dz + \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \int_0^R e^{-z^2} dz = 0
\]

\(-22\)
now recall that \( \int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2 \). Thus as \( R \to \infty \) we have

\[
\int_0^\infty \cos z^2 + i \sin z^2\,dz = \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) \frac{\sqrt{\pi}}{2}
\]

Which implies that

\[
\int_0^\infty \sin z^2\,dz = \frac{\sqrt{2\pi}}{4}
\]

Hence

\[
\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \frac{\sqrt{2\pi}}{2}
\]

**Problem 2.b**

Let \( \gamma \) be the semi-circle with radius \( R \), and let \( f(z) = e^{iz}/(1 + z^2) \). Then notice

\[
\int_\gamma f(z)\,dz = 2\pi i \frac{e^{-1/2}}{2i} = \frac{\pi}{e}
\]

Notice on \( C_R \), \( z = Re^{i\theta} \) and \( dz = Rie^{i\theta} \frac{\,d\theta}{R^2 - 1} \)

\[
\left| \int_{C_R} f(z)\,dz \right| \leq \int_0^\pi \left| \frac{e^{-R\sin(\theta)}}{R^2 - 1} \right| \leq R\pi \frac{1}{R^2 - 1} \to 0
\]

as \( R \to \infty \). Thus

\[
\int_{-R}^R f(z)\,dz = \int_{-R}^R \frac{\cos(z) + i \sin(z)}{1 + z^2}\,dz = \frac{\pi}{e}
\]

Hence

\[
\int_{-R}^R \frac{\cos(z)}{1 + z^2}\,dz = \frac{\pi}{e}
\]

and since \( \frac{\cos(z)}{1 + z^2} \) is an even function, as \( R \to \infty \)

\[
\int_0^\infty \frac{\cos(z)}{1 + z^2}\,dz = \frac{\pi}{2e}
\]

**Problem 3**

We first map \( D = \{ z = (r, \theta) : r > 0, 0 < \theta < \pi/4 \} \) to the Upper half plane \( U \) by \( z^4 \). Then we must map \( U \) onto itself such that the three points \((0, -16, 1) \to (0, -1, 16) \). Thus for \( T(z) = \frac{z}{c + z^4} \), we have

\[
-1 = \frac{-16}{-16c + d} \quad \text{and} \quad 16 = \frac{1}{c + d}
\]
which implies $d = 1$ and $c = -15/16$. Hence $T(z) = \frac{16z}{-15z^4 + 16}$. Then we finally map $U$ to $D$ by $z^{1/4}$. Hence our mapping is

$$w = \left(\frac{16z^4}{-15z^4 + 16}\right)^{1/4}$$

The mapping is unique since there are 3 distinct complex numbers that maps to 3 distinct complex numbers, i.e. $(0, 1, 2e^{i\pi/4}) \mapsto (0, 2, e^{i\pi/4})$.

**Problem 4**

No, $f(z) = 0$ has an infinite number of zeros. Now if $f(0) = 1$, we know that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{-\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \cdots - \frac{1}{n!}\left(\frac{z}{a_n}\right)^n}$$

where $h$ is then genus and $\lambda$ is the order of the polynomial $g$. Since

$$|f(z)| \leq Ce^{|z|}$$

implies that $\lambda = 1$. Thus we know that $h \leq \lambda \leq h + 1$ implies $h = 0$ or $h = 1$. So for $f$ to have infinite number of zeros inside the unit circle, implies that the infinite product will not converge. So there must be a finite number of zeros inside the unit circle. We cannot give an explicit bound.

**Problem 5**

Recall the ration test theorem in Levinson: For $\sum_{j=1}^{\infty} a_j(z)$, In a region $G$, let $|a_N(z)|$ be bounded for some fixed $N \geq 1$, and for $n > N$

$$\frac{|a_{n+1}(z)|}{|a_n(z)|} \leq R < 1$$

where $R$ is constant. Then the series converges in $G$. Thus for $P(z) = \sum_{k=1}^{\infty} z^k$, we have

$$\left|\frac{z^{(k+1)!}}{z^{k!}}\right|$$

Thus for arbitrary large $n$, clearly the ratio above is less than one when $|z| < 1$. Hence $P(z)$ converges for $|z| < 1$. Now I claim that every point on $|z| = 1$ is a singular point and thus there is not an analytic continuation beyond the unit circle. Assume that $\exists$ an $\alpha$ on the unit circle such that $f$ is not singular. Then $f$ can be continued analytically in $\Delta(\alpha, \epsilon)$ for some small $\epsilon > 0$. Thus $\exists$ an arc of length $\delta$ on the unit circle inside $\Delta(\alpha, \epsilon)$, such that $f$ is analytic on. Now let $q > 2\pi/\delta$, and let $\beta = e^{2\pi ip/q}$, such that $\beta$ is on the arc. Thus $\beta^n = 1$ and for $n \geq q$ we have $\beta^n! = 1$. Thus

$$f(r\beta) = \sum_{n=1}^{q-1} r^n!\beta^n + \sum_{q}^{\infty} r^n!$$

which implies
\[ |f(r\beta)| \geq \sum_{q}^{\infty} r^{q!} - (q - 1) \]

and thus it approaches \( \infty \) as \( r \to 1^- \). Hence \( f \) cannot be analytic on \( \beta \), and thus there is no analytic continuation beyond the unit circle. (Levinson p.410)

Jan 2002 Linear Algebra

Problem 1.a

Here we have the basis \( \eta_1(x) = x^2 + x + 1, \eta_2(x) = x^2 + 1 \), and \( \eta_3(x) = x + 1 \). They correspond to \((1,1,1),(1,0,1),(0,1,1)\). Hence we change the basis from \((1,1,1),(1,0,1),(0,1,1)\) to \((1,0,0),(0,1,0),(0,0,1)\) by the matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix}
1 & 1 & -1 \\
0 & -1 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]

Now the we have the derivative operator for the standard basis as

\[
D_s = \begin{pmatrix}
0 & 0 & 0 \\
2 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

So our derivative operator \( D \) changes a vector \( v \) from the basis \((1,1,1),(1,0,1),(0,1,1)\) to the standard basis \((1,0,0),(0,1,0),(0,0,1)\), computes the derivative, and then converts it back to the original basis. Hence

\[
D = A^{-1}D_sA = \begin{pmatrix}
1 & 2 & -1 \\
-1 & -2 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]

Problem 1.b

By above we have

\[
D^2 = \begin{pmatrix}
-2 & -2 & 0 \\
2 & 2 & 0 \\
2 & 2 & 0
\end{pmatrix} \quad \text{and} \quad D^3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Problem 1.c

By above

\[
\det(I + D) = \det \begin{pmatrix}
2 & 2 & -1 \\
-1 & -1 & 1 \\
1 & 0 & 2
\end{pmatrix} = 1
\]

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Problem 2

So $[V_1, V_2]$ is an $n \times n$ matrix, $[V_1, V_3]$ is an $n \times n$ matrix. We want to show the $V_3^* V_2$ is invertible.

Notice 

\[
\left( \begin{array}{c} V_1^* \\ V_3^* \end{array} \right) \left( \begin{array}{cc} V_1 & V_2 \end{array} \right) = \left( \begin{array}{cc} V_1^* V_1 & V_1^* V_2 \\ 0 & V_3^* V_2 \end{array} \right)
\]

which is an invertible matrix. This implies 

\[
\det(V_3^* V_2) = \det(V_1^* V_1) \det(V_3^* V_2) \neq 0.
\]

Therefore 

\[
\det(V_3^* V_2) \neq 0,
\]

which implies that $V_3^* V_2$ is invertible.

Problem 3

We have unit basis $u, v$ of $\mathbb{R}^2$. The projector onto $u$ along $v$ projects a vector onto $v$, and then $u$.

Let $P_1$ project onto $v$ and $P_2$ project onto $u$. Then

\[
P_1 = vv^T, \quad P_2 = uu^T
\]

then

\[
P = P_2 P_1 uu^T vv^T = (u^T v) uu^T
\]

so $\mu = (u^T v)$ and $x = u$ and $y = v$.

Problem 4

($\Leftarrow$) Assume that $\mu$ is a diagonal of the matrix $Q^* AQ$ for some unitary matrix $Q$. Assume that $\mu$ is in the $i$th row and $i$th column. Then we have

\[
Q^* AQ = \left( \begin{array}{cccc} \cdots & q_1^H & \cdots \\ \cdots & q_2^H & \cdots \\ \vdots & \vdots \\ \cdots & q_n^H & \cdots \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \\ \cdots \\ a_n \end{array} \right) \left( \begin{array}{c} q_1 \\ q_2 \\ \cdots \\ q_n \end{array} \right)
\]

then

\[
\mu = \frac{q_i^H A q_i}{q_i^H q_i},
\]

where $q_i \neq 0$ since $Q$ is unitary. ($\Rightarrow$) Now assume

\[
\mu = \frac{y^* A y}{y^* y}
\]

for $y \neq 0$. Now let $(b_1, \ldots, b_n)$ be the basis for $C^n$. Then perform Gram-Schmidt on $(y, b_1, \ldots, b_n)$ such that we have the orthonormal basis $(y/\sqrt{y^H y}, u_2, \ldots, u_n)$. Now define $Q$ such that
\[
Q = \begin{pmatrix}
\frac{y}{\sqrt{y^2 y}} & u_2 & \cdots & u_n
\end{pmatrix}
\]

Then we have \(Q\) unitary, and

\[
Q^H AQ = \begin{pmatrix}
\mu & x \\
x & X
\end{pmatrix}
\]

Sept 2002 Advanced Calculus

Problem 1

Since \(f\) is continuous on \([0,1]\), for all \(\epsilon > 0\) exists a polynomial \(P(x)\) such that \(|P(x) - f(x)| < \epsilon\) for all \(x \in [0,1]\). Hence

\[
\lim_{n \to \infty} n \int_0^1 (P(x) - \epsilon)x^n dx \leq \lim_{n \to \infty} n \int_0^1 f(x)x^n dx \leq \lim_{n \to \infty} n \int_0^1 (P(x) + \epsilon)x^n dx
\]

Notice

\[
\lim_{n \to \infty} n \int_0^1 \epsilon x^n dx = \lim_{n \to \infty} \frac{n}{n+1} \epsilon = \epsilon
\]

Now let \(P(x) = P_0 + P_1 x + P_2 x^2 + \cdots + P_k x^k\). Then we have

\[
\begin{align*}
\lim_{n \to \infty} n \int_0^1 P(x)x^n dx & = \lim_{n \to \infty} n \int_0^1 (P_0 + P_1 x + P_2 x^2 + \cdots + P_k x^k)x^n dx \\
& = \lim_{n \to \infty} n \left[ \frac{P_0}{n+1} + \frac{P_1}{n+2} + \cdots + \frac{P_k}{n+k} \right] = P_0 + P_1 + \cdots + P_k = P(1)
\end{align*}
\]

Hence since

\[
\lim_{n \to \infty} n \int_0^1 (P(x) - \epsilon)x^n dx \leq \lim_{n \to \infty} n \int_0^1 f(x)x^n dx \leq \lim_{n \to \infty} n \int_0^1 (P(x) + \epsilon)x^n dx
\]

we have

\[
P(1) - \epsilon \leq \lim_{n \to \infty} n \int_0^1 f(x)x^n dx \leq P(1) + \epsilon
\]

and since \(\epsilon\) is arbitrary, we have

\[
P(1) = f(1) = \lim_{n \to \infty} n \int_0^1 f(x)x^n dx
\]
Problem 2.2.a

The summation
\[ \sum_{n=0}^{\infty} \sin(e\pi n) \]
converges if and only if \( \sin(e\pi n) \to 0 \). That happens if and only if \( e\pi n \to K\pi \) for any integers \( K \). However, since \( e \) is irrational, \( \not\exists \) an integer \( n \) such that \( en \in \mathbb{Z} \). Since otherwise that would imply that \( e \) is rational. Thus the summation above does not converge.

Problem 2.2.b

the summation
\[ \sum_{n=0}^{\infty} \sin^2(e\pi n) \]
does not converge by the same argument as above.

Problem 3.1

Let \( a_n = 1/\sqrt{n} \) and \( b_n = \cos n \). Then \( a_n \searrow 0 \) and
\[ \sum_{n=1}^{N} b_n \]
is uniformly bounded since
\[ \sum_{n=1}^{N} \cos n = \frac{\sin((2N + 1)/2)}{2\sin(1/2)} \]
By Abel's Test, The summation converges.

Problem 3.2

Notice
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n} = \sum_{n=1}^{\infty} \frac{-\sqrt{n} - 1}{n - 1} \to \infty \]

Problem 4

Using polar coordinates, the first integrand breaks-up into
\[ \int_{0}^{a} dx \int_{\sqrt{a^2-x^2}}^{b} dy \ln(x^2 + y^2) = \int_{0}^{\arccos(a/b)} \int_{a}^{a/\cos \theta} r \ln r^2 dr d\theta + \int_{\arccos(a/b)}^{\pi/2} \int_{a}^{b} r \ln r^2 dr d\theta \]
and the second integrand using polar coordinates is

\[ \int_a^b dx \int_0^{\sqrt{b^2-x^2}} dy \ln(x^2+y^2) = \int_0^{\arccos(a/b)} \int_{a/\cos \theta}^b r \ln r^2 dr d\theta \]

Hence we have

\[ \int_a^b dx \int_0^{\sqrt{b^2-x^2}} dy \ln(x^2+y^2) + \int_a^b dx \int_0^{\sqrt{b^2-x^2}} dy \ln(x^2+y^2) \]

\[ = \left( \int_0^{\arccos(a/b)} \int_a^{\arccos(a/b)} r \ln r^2 dr d\theta + \int_0^{\arccos(a/b)} \int_a^{\arccos(a/b)} r \ln r^2 dr d\theta \right) \]

\[ = \left( \int_0^{\pi/2} \int_a^b r \ln r^2 dr d\theta \right) + \left( \int_0^{\arccos(a/b)} \int_a^b r \ln r^2 dr d\theta \right) \]

\[ = \int_0^{\pi/2} \int_a^b r \ln r^2 dr d\theta = \frac{\pi}{4} (b^2 \ln b^2 - b^2 - a^2 \ln a^2 + a^2) \]

Problem 5.1

By the Divergence theorem, we have

\[ \oint_{d\Omega} \mathbf{g} dS = \oint_{d\Omega} \mathbf{\nabla} \cdot \mathbf{u} ndS = \int \int \int_{\Omega} \mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{u} dV \]

\[ = \int \int \int_{\Omega} \Delta \mathbf{u} dV = \int \int \int_{\Omega} f dV \]

Problem 5.2

Again by the divergence theorem, we have

\[ \int \int \int_{\Omega} \mathbf{g} \mathbf{u} dS = \int \int \int_{\Omega} \mathbf{u} \nabla \cdot \mathbf{u} ndS = \int \int \int_{V} \nabla \cdot (\mathbf{u} \nabla \mathbf{u}) dV = \int \int \int_{V} \nabla \cdot \mathbf{u} \nabla \mathbf{u} + \mathbf{u} \Delta \mathbf{u} dV \]

\[ = \int \int \int_{\Omega} |\nabla \mathbf{u}|^2 dV + \int \int \int_{\Omega} \mathbf{u} \Delta \mathbf{u} dV \geq \int \int \int_{\Omega} \mathbf{u} \Delta \mathbf{u} dV = \int \int \int_{\Omega} f dV \]

Sept 2002 Complex Analysis

Problem 1

\[ (1 + i)^i = e^{i \log(1+i)} = e^{i (\ln(\sqrt{2}) + i(\pi/4 + 2\pi n))} = e^{-(\pi/4 + 2\pi n)} \cdot e^{i \ln(\sqrt{2})} \]

\[ = e^{-(\pi/4 + 2\pi n)} (\cos(\ln(\sqrt{2})) + i \sin(\ln(\sqrt{2}))) \]

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Problem 2
Notice on the unit circle
\[ |z^6 + 3z^4 + 1 - 3z^4| = |z^6 + 1| \leq |z^6| + 1 \leq 2 < 3|z^4| \]
thus \( z^6 + 3z^4 + 1 \) has 4 zeros inside the unit circle. Now for the semi-circle with radius \( R \) we have
\[ f(Re^{i\theta}) = R^6 e^{6i\theta} \left( 1 + \frac{3}{R^2} e^{2i\theta} + \frac{1}{R^6} e^{i\theta} \right) \]
By the argument principle we have \( 6\pi/2\pi = 3 \) zero’s in the upper half plane since \( R \to \infty \). Thus there are also 3 zero’s in the lower half of the plane. Hence there are no real zeros, and therefore we have two zeros in the upper half of the unit disk.

Problem 3
\[ w = \sin\left(\frac{\pi z}{2}\right) \]

Problem 4
notice
\[ 1 + z + z^2 + \cdots = \frac{1}{1 - z} \]
thus
\[ \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1 - z) \]
so for \( z = 3i \), we have
\[ -\log(1 - 3i) = -(\ln(\sqrt{10}) + i(\theta + 2\pi n)) \]
where \( \theta = \arctan(-3/1) \) in correct form.

Problem 5
First I claim that
\[ \int_{|z|<1} \int |f'(z)|^2 \, dx \, dy = \pi \sum_{n=1}^{\infty} n |a_n|^2 \]
Proof: Notice
\[ \int_{|z|<1} \int |f'(z)|^2 \, dx \, dy = \int_{r=0}^{1} \left( \int_{\theta=0}^{2\pi} |f'(z)|^2 \, d\theta \right) \, r \, dr \]
and by Parseval’s Theorem
\[
= \int_{r=0}^{1} 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)} dr = \frac{2\pi}{2n} \sum_{n=1}^{\infty} n^2 |a_n|^{2n} \left[ r \right]_{r=0}^{1} = \pi \sum_{n=1}^{\infty} n |a_n|^2
\]

Thus for \( f'(z) = 1 + 2z^4 + z^8 \), we have
\[
\int \int_{|z|<1} |f'(z)|^2 dxdy = \pi (1 + 10 + 9) > \pi
\]

**Sept 2002 Linear Algebra**

**Problem 1.1**

Now \( A \) is diagonalizable if and only if it has \( n \) eigenvectors that are linearly independent. Now I claim that for \( A = A_n \),

\[
\det(A_n - \lambda I) = (-1)^n \lambda^{n-2} (\lambda^2 - \lambda - (n - 1))
\]

First we will show that for \( M_k \), where

\[
M_2 = \begin{pmatrix} 0 & -\lambda \\ 1 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \\ 1 & 1 & 1 \end{pmatrix}, \ldots,
\]

\[
M_k = \begin{pmatrix} 0 & -\lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}
\]

Then clearly by induction \( \det M_k = \lambda^{k-1} \). **BASE CASE:** \( n = 2 \), we have

\[
\det \begin{pmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{pmatrix} = -\lambda (1 - \lambda) - 1 = \lambda^2 - \lambda - 1
\]

**INDUCTIVE STEP:** Notice we have

\[
\det(A_n - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & \cdots & 1 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 0 & A_{n-1} \end{pmatrix}
\]

by the inductive hypothesis, we have

\[
\det(A_n - \lambda I) = -\lambda \det(A_{n-1}) + (-1)^{n-1} \lambda^{n-2} = -\lambda((-1)^{n-1} \lambda^{n-3} (\lambda^2 - \lambda - (n - 2))) + (-1)^{n-1} \lambda^{n-2}
\]
Thus we have 2 nonzero eigenvalues at 
\(1 ± \sqrt{1 + 4(n - 1)}\), which are distinct. Now notice we have an eigenvalue \(\lambda = 0\), and it’s Algebraic Multiplicity = Geometric Multiplicity since \(AM = n - 2\) by above and \(GM = \ker A_n = n - \text{rank}(A_n) = n - 2\). Thus \(A_n\) has \(n\) linearly independent eigenvectors. Thus \(A \in \mathbb{R}^n\).

**Problem 1.2**

The relationship between the two nonzero eigenvalues are conjugates as shown above,

\[
\lambda_1 = \frac{1}{2} + \frac{\sqrt{1 + 4(n - 1)}}{2}, \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{1 + 4(n - 1)}}{2}
\]

**Problem 1.3**

Clearly by inspection we have

\[
A^2 = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & n
\end{pmatrix}
\]

Since \(A\) is diagonalizable, we know that the eigenvalues of \(A^2\) is \(\Lambda^2\). Hence the eigenvalues of \(A^2\) is \(0, \left(\frac{1}{2} + \frac{\sqrt{1 + 4(n - 1)}}{2}\right)^2, \left(\frac{1}{2} - \frac{\sqrt{1 + 4(n - 1)}}{2}\right)^2\).

**Problem 1.4**

We know that the minimal polynomial \(p\) is of the form

\[p(x) = x^k(x^2 - x - (n - 1))\]

where \(1 \leq k \leq n - 2\). Notice that \(k = 1\) suffices since

\[
A(A^2 - A - (n - 1)I)
\]
\[
\begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \cdots & 1 \\
0 & \cdots & 0 & 1 \\
1 & \cdots & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 - n & 1 & \cdots & 1 & 0 \\
1 & 2 - n & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & 0 & \vdots \\
1 & 1 & 1 & 2 - n & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

Thus the minimal polynomial is \( p(x) = x(x^2 - x - (n - 1)) \).

**Problem 2**

Let \( \Lambda \) be the eigenvalues of \( M \). Then \( \Lambda^2 \) is the eigenvalues for \( M^2 \) since

\[
M^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}
\]

where, \( S \) is the \( 3 \times 3 \) eigenvector matrix. Thus

\[
\begin{pmatrix}
3 & 0 & 0 \\
8 & 4 & 0 \\
5 & 0 & 1
\end{pmatrix} - \lambda I = 0 \Rightarrow \Lambda^2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Now notice that the eigenvector matrix \( S \) is the same for \( M \) and \( M^2 \). Thus the eigenvector for \( \lambda = 3 \) is

\[
\begin{pmatrix}
3 & 0 & 0 \\
8 & 4 & 0 \\
5 & 0 & 1
\end{pmatrix} - 3I \quad x = 0 \Rightarrow x = \left( \frac{2}{5}t, \frac{-16}{5}t, t \right)
\]

Thus for simplicity let \( t = 5 \), and we have the eigenvalue \( x = (2, -16, 5) \). Now for the eigenvector for \( \lambda = 4 \) is

\[
\begin{pmatrix}
3 & 0 & 0 \\
8 & 4 & 0 \\
5 & 0 & 1
\end{pmatrix} - 4I \quad x = 0 \Rightarrow x = (0, t, 0)
\]

Thus for simplicity let \( t = 1 \), and we have the eigenvector \( x = (0, 1, 0) \). Now the eigenvector for \( \lambda = 1 \), we have

\[
\begin{pmatrix}
3 & 0 & 0 \\
8 & 4 & 0 \\
5 & 0 & 1
\end{pmatrix} - I \quad x = 0 \Rightarrow x = (0, 0, t)
\]

Thus for simplicity let \( t = 1 \), and we have the eigenvector \( x = (0, 0, 1) \). Hence \( S = \begin{pmatrix} 2 & 0 & 0 \\ -16 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix} \).

Now recall that

\[
S^{-1} = \frac{1}{\det(S)} C^T = \frac{1}{\det(S)} \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}^T
\]

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Hence $S^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 8 & 1 & 0 \\ -\frac{5}{2} & 0 & 1 \end{pmatrix}$. Now there are 8 possible diagonal matrices for $M$, all of which has 3 different eigenvalues which implies that $M$ is indeed diagonalizable:

$$
\Lambda_1 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\Lambda_4 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_5 = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 0 & -\sqrt{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_6 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

$$
\Lambda_7 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_8 = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 0 & -\sqrt{4} & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

Thus there are 8 possible $M$'s that satisfy the relation, and they are

$$
M_1 = S\Lambda_1 S^{-1} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ -8(-2 + \sqrt{3}) & 2 & 0 \\ \frac{5}{2}(-1 + \sqrt{3}) & 0 & 1 \end{pmatrix}, \quad M_2 = S\Lambda_2 S^{-1} = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 8(2 + \sqrt{3}) & 2 & 0 \\ -\frac{5}{2}(1 + \sqrt{3}) & 0 & 1 \end{pmatrix}
$$

$$
M_3 = S\Lambda_3 S^{-1} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ -8(-2 + \sqrt{3}) & -2 & 0 \\ \frac{5}{2}(-1 + \sqrt{3}) & 0 & 1 \end{pmatrix}, \quad M_4 = S\Lambda_4 S^{-1} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ -8(-2 + \sqrt{3}) & 2 & 0 \\ \frac{5}{2}(1 + \sqrt{3}) & 0 & -1 \end{pmatrix}
$$

$$
M_5 = S\Lambda_5 S^{-1} = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 8(-2 + \sqrt{3}) & -2 & 0 \\ -\frac{5}{2}(1 + \sqrt{3}) & 0 & 1 \end{pmatrix}, \quad M_6 = S\Lambda_6 S^{-1} = \begin{pmatrix} -\sqrt{3} & 0 & 0 \\ 8(2 + \sqrt{3}) & 2 & 0 \\ -\frac{5}{2}(-1 + \sqrt{3}) & 0 & -1 \end{pmatrix}
$$

$$
M_7 = S\Lambda_7 S^{-1} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ -8(-2 + \sqrt{3}) & -2 & 0 \\ \frac{5}{2}(1 + \sqrt{3}) & 0 & -1 \end{pmatrix}, \quad M_8 = S\Lambda_8 S^{-1} = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 8(-2 + \sqrt{3}) & -2 & 0 \\ -\frac{5}{2}(-1 + \sqrt{3}) & 0 & -1 \end{pmatrix}
$$

**Problem 3.1**

Let $A$ be monotone. Assume that $A$ is not invertible. Then one of the columns of $A$ is a linear combination of the other columns. For simplicity, let $c_1 = a_2 c_2 + \cdots + a_n c_n$. Then notice

$$
A \begin{pmatrix} \frac{-1}{2} \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \geq 0
$$
However clearly \[
\begin{pmatrix}
-1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
\] is not \(\geq 0\). Hence we have a contradiction and thus \(A\) is invertible.

**Problem 3.2**

(\(\Rightarrow\)) Assume that \(A\) is monotone. Then by above we know that it is invertible. Now assume that \(\exists\) a negative entry in \(A^{-1}\), say \(a_{i,j}\) in the \(j\)th column \(c_j\). Then notice for

\[
Ax = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

where the 1 is in the \(j\)th position. Since \(A\) is monotone, this implies that \(x \geq 0\). However notice

\[
A^{-1}Ax = x = c_j \not\geq 0
\]

Thus we have a contradiction.

(\(\Leftarrow\)). Now assume \(A\) is invertible and all of the entries of \(A^{-1}\) are nonnegative. Clearly if \(Ax \geq 0\), then by multiplying \(A^{-1}\) on both sides we have

\[
A^{-1}Ax \geq 0 \Rightarrow x \geq 0
\]

Thus \(A\) is monotone. Since all of the entries of \(A^{-1}\) is greater than or equal to zero, the inequality sign does not change.

**Problem 4**

Notice that \(\det A = 28\) and \(\text{Tr}(A) = -3\). Thus the eigenvalues must satisfy \(\lambda_1 < 0\), \(\lambda_2 < 0\), and \(\lambda_3 > 0\). Also \(\det B = -25\) and \(\text{Tr}(B) = 11\). Hence \(\mu_1 < 0\), \(\mu_2 > 0\), and \(\mu_3 > 0\). Hence it suffices to show that \(\mu_3^4 > \lambda_3^4\). Thus for \(A^4\), we have

\[
A^4 = \begin{pmatrix}
140 & -14 & 348 \\
-14 & 329 & -262 \\
348 & -262 & 1036
\end{pmatrix}
\]

with eigenvalues \(\lambda_1^4, \lambda_2^4, \lambda_3^4\), which are all positive. Since \(\text{Tr}(A^4) = 1505\), this implies that \(\lambda_3^4 \leq 1505\). Now for \(B^4\) we have

\[
B^4 = \begin{pmatrix}
485 & 1992 & 348 \\
1992 & 8577 & 1544 \\
348 & 1544 & 285
\end{pmatrix}
\]

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with eigenvalues $\mu_1, \mu_2, \mu_3, \mu_4$, which are all positive. Since $\text{Tr}(B^4) = 9347$, we have $\mu_3^4 \geq 9347/3 > 1505$. Hence $\mu_3 > \lambda_3$. Note that $|\mu_1| < |\mu_3|$, since if not, then $\mu_2 > 11$. However $|\mu_1\mu_2\mu_3| = 25$ implies that $\mu_3 < 1$. Hence $\mu_3 < \mu_2$ and we have a contradiction.

**Problem 5.1**

Clearly $x^T(B+C)x = x^T B x + x^T C x > 0$ since $B$ is positive definite and $C$ is positive semi-definite.

**Problem 5.2**

First since $B$ is hermitian, this implies that $x^H B x$ is real for all $x \in \mathbb{C}^n$. Same for $B + C$. Now also notice that $\forall x \in \mathbb{C}^n$, we have

$$0 < x^H B x \leq x^H (B + C) x$$

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $B$ and $0 < \mu_1 \leq \mu_2 \leq \cdots \mu_n$ be the eigenvalues of $B + C$. Now I claim that $\lambda_i \leq \mu_i$. Recall the min-max theorem. WLOG we can assume that $x^H x = 1$ where

$$\min_{\dim(S) = i} \max_{x \in S} x^H B x = \lambda_i$$

and

$$\min_{\dim(S) = i} \max_{x \in S} x^H (B + C) x = \mu_i$$

So let $T$ be the subspace with dimension $i$ be the subspace that minimizes $\max_{x \in T} x^H (B + C) x = \mu_i$. Also let $y \in T$ such that it maximizes $\max_{x \in T} x^H B x = y^H B y$. Then we have

$$\lambda_i = \min_{\dim(S) = i} \max_{x \in S} x^H B x \leq \max_{x \in T} x^H B x = y^H B y \leq y^H (B + C) y \leq \max_{x \in T} x^H (B + C) x = \mu_i$$

Hence $0 < \lambda_i \leq \mu_i \forall i$. Therefore

$$\det(B) = \prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} \mu_i = \det(B + C)$$

and clearly we have equality when $\lambda_i = \mu_i$ for all $i$.

**Problem 5.3**

We have 2 cases. **CASE 1:** Assume that $B + C = I$. Then we know that all of it’s eigenvalues are 1 and $\lambda_i \leq 1$ for all $i$. This implies $1/\lambda_i \geq 1$ for all $i$. Now we want to show that for all $x \in \mathbb{C}^n$,

$$x^H B^{-1} x \geq x^H (B + C)^{-1} x$$

notice that
\[
\frac{x^H (B + C)^{-1} x}{x^H x} \leq 1 \leq \frac{1}{\lambda_n} \leq \frac{x^H B^{-1} x}{x^H x}
\]

which implies that \( x^H (B + C)^{-1} x \leq x^H B^{-1} x \) for all \( x \). **CASE 2:** Since \( \mu_i > 0 \) for all \( i \) we can define

\[
B + C = U^H \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} U
\]

\[
(B + C)^{1/2} = U^H \begin{pmatrix} \sqrt{\mu_1} & & \\ & \ddots & \\ & & \sqrt{\mu_n} \end{pmatrix} U
\]

\[
(B + C)^{-1/2} = U^H \begin{pmatrix} 1/\sqrt{\mu_1} & & \\ & \ddots & \\ & & 1/\sqrt{\mu_n} \end{pmatrix} U
\]

Then \((B + C)^{1/2}(B + C)^{-1/2} = (B + C)^{-1/2}(B + C)^{1/2} = I\). So \((B + C)^{1/2}\) and \((B + C)^{-1/2}\) are both invertible and their range is \(\mathbb{C}^n\). So we know that for all \( x \in \mathbb{C}^n \), we have

\[
x^H Bx \leq x^H (B + C)x = x^H (B + C)^{1/2}(B + C)^{1/2}x
\]

Now I claim that for all \( x \), we have

\[
x^H (B + C)^{-1/2} B(B + C)^{-1/2} x \leq x^H Ix
\]

indeed we know \( \exists y \) such that \( (B + C)^{-1/2} x = y \). So we have

\[
x^H (B + C)^{-1/2} B(B + C)^{-1/2} x = y^H By \leq y^H (B + C)^{1/2}(B + C)^{1/2}y
\]

\[
= x^H (B + C)^{-1/2}(B + C)^{1/2}(B + C)^{1/2}(B + C)^{-1/2}x = x^H Ix
\]

Thus by case 1, we have for all \( x \)

\[
x^H (B + C)^{1/2} B^{-1}(B + C)^{1/2} x \geq x^H x
\]

Now I claim that for all \( y \) we have

\[
y^H B^{-1} y \geq y^H (B + C)^{-1} y
\]

we know \( \exists x \) such that \((B + C)^{1/2} x = y\). This implies \((B + C)^{-1/2} y = x\) and

\[
y^H B^{-1} y = x^H (B + C)^{1/2} B^{-1}(B + C)^{1/2} x \geq x^H x = y^H (B + C)^{-1/2}(B + C)^{-1/2}y = y^H (B + C)^{-1} y
\]

this completes the proof.
Jan 2003 Advanced Calculus

Problem 2

Here we have \( x = \cos u \sin v \), \( y = \sin u \sin v \), and \( z = \cos v \) for \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq \pi \) that parameterizes \( S \). Hence we have

\[
\begin{align*}
    dx &= -\sin u \sin v du + \cos v \cos u dv \\
    dy &= \cos u \sin v du + \cos v \sin u dv \\
    dz &= -\sin v dv
\end{align*}
\]

Thus we have \( dxdy = \cos v \sin u dv du \), \( dzdx = -\sin u \sin^2 v dv du \), and \( dxdy = -\cos v \sin u dv du \). Hence we have

\[
\int \int_S (yz - xy) dxdy + y^2 dzdx = \int \int \cos u \cos v \sin^3 u - \cos^2 v \sin^2 v \sin u - \sin^3 u \sin^4 v dv du
\]

Notice each integrand over \( u \) is a sine function over its period. Hence the integral is zero when integrating respect to \( u \) for each three integrals. Hence

\[
\int \int_S (yz - xy) dxdy + y^2 dzdx = 0
\]

Problem 3.a

So we have

\[
V(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}
\]

which implies

\[
F(x, y, z) = \nabla \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \left( \frac{-x}{\sqrt{x^2 + y^2 + z^2}}, \frac{-y}{\sqrt{x^2 + y^2 + z^2}}, \frac{-z}{\sqrt{x^2 + y^2 + z^2}} \right)
\]

so we have \( x = t \), \( dx = dt \), \( y = t^2 \), \( dy = 2tdt \), \( z = \cos(\pi t/2) \) and \( dz = -(\pi/2) \sin(\pi t/2) \). So

\[
\int_C F \cdot dl = \int_0^1 \frac{-t - 2t^3 + (\pi/2) \cos(\pi t/2) \sin(\pi t/2)}{\sqrt{t^2 + t^4 + \cos^2(\pi t/2)}} dt
\]

then by \( u \)-substitution, we have \( u = t^2 + t^4 + \cos^2(\pi t/2) \), and \( du = 2t + 4t^3 + \pi \cos(\pi t/2) \sin(\pi t/2) \). Then we have

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$$\int \frac{-1}{\sqrt{u}} du = -u^{1/2} = -\sqrt{t^2 + t^4 + \cos^2(\pi t/2)} \bigg|_0^1 = 1 - \sqrt{2}$$

**Problem 3.b**

For $F = (-x/r, -y/r, -z/r)$, we have by the divergence theorem,

$$\int \int \int_S F \cdot ndS = \int \int \int_V \nabla \cdot F dV$$

then on the unit sphere, we use spherical coordinates and have

$$= \int \int \int \frac{-x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{3/2}} dx dy dz$$

$$= -\int \int \int \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV = -\int \int \int \frac{1}{\rho^2} \rho^2 \sin \phi d\phi d\theta d\phi = \frac{1}{2} 2\pi \pi = \pi^2$$

**Problem 4**

Let’s look at $f(x) = -\pi/4$ for $x \in (-\pi, 0)$ and $f(x) = \pi/4$ for $x \in (0, \pi)$. Then let it repeat. This is what $f(x)$ is. We are given the Fourier expansion of it.

$$f(x) = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots$$

Recall that

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^\pi f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos(nx) dx = 0$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin(nx) dx = \frac{1}{2n} - \frac{\cos(n\pi)}{2n}$$
**Problem 5.1**

Notice

\[ D_n = \frac{1}{2\pi} \left( e^{-inx} + e^{-i(n-1)x} + \cdots + e^{i(n-1)x} + e^{inx} \right) \]

\[ = \frac{1}{\pi} \left( \frac{1}{2} + \cos(x) + \cos(2x) + \cdots + \cos((n-1)x) + \cos(x) \right) \]

So notice

\[ \sin(x) \left( \frac{1}{2} + \cos(x) + \cos(2x) + \cdots + \cos((n-1)x) + \cos(x) \right) \]

\[ = \frac{1}{2} \sin(x/2) + \frac{1}{2} \left( \sin(-x/2) + \sin(3x/2) \right) + \cdots + (1/2) \left( \sin((-2n-1)x/2) + \sin((2n+1)x/2) \right) \]

\[ = (1/2) \sin(x/2) + (1/2) \sin((n + 1/2)x) \]

So

\[ D_n = \frac{\sin((n + 1/2)x)}{2\pi \sin(x/2)} \]

and

\[ \int_{-\pi}^{\pi} \sum_{k=-n}^{n} \frac{e^{ikx}}{2\pi} = \sum_{k \neq 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} dx + \int_{-\pi}^{\pi} \frac{1}{2\pi} dx \]

\[ = \sum_{k \neq 0} \frac{1}{2\pi} \left( \frac{e^{ikx}}{ik} \right) \bigg|_{-\pi}^{\pi} + 1 = \sum_{k \neq 0} \frac{1}{\pi k} \sin(k\pi) + 1 = 1 \]

**Problem 5.2**

Notice since \( f \) is periodic

\[ \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(g)e^{-ikt} dt e^{ikx} = S_n(x) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{k=-n}^{n} f(t)e^{ik(x-t)} dt \]

\[ = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{k=-n}^{n} f(x-t) e^{ikt} dt = \int_{-\pi}^{\pi} f(x-t)D_n(t) dt \]
Jan 2003 Complex Variables

Problem 1

Let’s define the function $h(z) = f(z) - z$. Then notice
$$|h(z) + z| = |f(z)| < 1 = |z|$$
for all $z$ on the unit circle. Thus by Rouche’s theorem, $h(z)$ has exactly one zero inside the unit circle. Thus $\exists$ a unique $z$ with $|z| < 1$ such that $f(z) = z$. If $f(z) \leq 1$ for $|z| = 1$, then we cannot apply Rouche’s Theorem. Thus we could have $f(z) = 1$ for $|z| \leq 1$, and so there does not exist any $z$ in the unit disk such that $f(z) = z$.

Problem 2a

Let $\gamma$ be the unit circle in the complex plane and let $z = e^{i\theta}$. Then we have $dz = izd\theta$. Hence
$$\int_0^{2\pi} \frac{d\theta}{a \cos(\theta)} = \int_{\gamma} \frac{dz}{iz(a + (1/2)z + (1/2)z^{-1})}$$
$$= \int_{\gamma} \frac{2/i}{z^2 + 2az + 1} dz = \int_{\gamma} \frac{2/i}{z - (-a + \sqrt{a^2 - 1})}(z - (-a - \sqrt{a^2 - 1})) dz$$
Thus by the residue theorem, we have
$$\int_0^{2\pi} \frac{d\theta}{a \cos(\theta)} = \frac{2\pi}{\sqrt{a^2 - 1}}$$

Problem 2b

Let $\gamma$ be the semi-circle with radius $R$, and $f(z) = e^{iaz}/(1 + z^2)^2$. Then by the Residue theorem
$$\int_{\gamma} f(z) dz = 2\pi i \left( \frac{iae^{-a}2i - 2e^{-a}}{-8i} \right) = \frac{\pi e^{-a}(a + 1)}{2}$$
notice for $z = Re^{i\theta}$, we have
$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)|dz \leq R\pi \frac{\max(e^{aR \sin(\theta)})}{(R^2 - 1)^4} \to 0$$
as $R \to \infty$. Thus
$$\int_{-R}^{R} f(z) dz = \int_{-R}^{R} \frac{\cos(az) + i \sin(az)}{(1 + z^2)^2} dz$$
which implies as $R \to \infty$
$$\int_{-\infty}^{\infty} \frac{\cos(az)}{(1 + z^2)^2} dz = \frac{\pi e^{-a}(a + 1)}{2}$$
and since the function above is even, we have
\[
\int_0^\infty \frac{\cos(az)}{(1+z^2)^2}dz = \frac{\pi e^{-a}(a + 1)}{4}
\]

**Problem 3**

Let \( f(z) = 2z^5 - 6z^2 + z + 1 \). For \(|z| = 1\), notice
\[
|2z^5 - 6z^2 + z + 1 + 6z^2| = |2z^5 + z + 1| \leq |2z^5| + |z| + 1 = 4 < 6 = |6z^2|
\]
thus there are two zero’s inside the unit circle for \( f(z) \). Now for \(|z| = 2\), notice
\[
|f(z) - 2z^5| = |6z^2 + z + 1| \leq |6z^2| + |z| + 1 = 27 < 64 = |2z^5|
\]
Thus there are 5 zero’s inside the circle of radius 2. Hence there are 3 zeros inside the annulus \( 1 \leq |z| \leq 2 \).

**Problem 4**

(⇒) Assume \( \exists g \) such that \( g^n = f \) and \( g \) is entire. Since \( g \) is entire, all of it’s zeros are of integer order. Now Assume that \( f \) has a zero at \( z_0 \) of order \( m_0 \) such that \( m_0 \) does not divide \( n \). Then \( g(z_0) = 0 \) since otherwise \( f(z_0) = g^n(z_0) \neq 0 \). So \( g \) has a zero at \( z_0 \) of an integer order \( k \). Since \( g^n = f \), \( f \) has a zero of \( kn \) and so we have a contradiction.

(⇐) Now assume that all zeros of \( f \) are divisible by \( n \). Since \( f \) is entire, it has the form
\[
f(z) = z^{m_0} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{m_n} e^{-\left(\frac{z}{a_n}\right)^2 - \cdots - \frac{1}{k} \left(\frac{z}{a_n}\right)^h}
\]
let \( g \) be defined as
\[
g(z) = z^{m_0/n} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{m_n/n} e^{-\left(\frac{z}{a_n}\right)^2 - \cdots - \frac{1}{k} \left(\frac{z}{a_n}\right)^h/n}
\]
Then the \( g \) is entire and it converges and \( g^n = f \).

**Problem 5**

Let \( w \) be some region in the \( u - v \) plane. Then the area not in \( w \) is some closed region \( R \). So
\[
Area(R) = \int \int_R dudv = \frac{1}{2} \oint_C udv - vdu = \frac{1}{2} \int_0^{2\pi} u \frac{dv}{d\theta} - v \frac{du}{d\theta} d\theta
\]
you can think of the last equality by cancelling out the \( d\theta \). Now recall that
\[
f'(z) = \frac{1}{iz} \left( \frac{dv}{d\theta} + i \frac{dv}{d\theta} \right)
\]
Since $f$ is analytic $|z| > 1$ (does not contain the origin), we have the C-R equations in polar coordinates

$$\frac{du}{dr} = \frac{1}{r} \frac{dv}{d\theta} \quad -\frac{1}{r} \frac{du}{d\theta} = \frac{dv}{dr}$$

Now since $f'(z) = 1/e^{i\theta} \left( \frac{du}{dr} + iv\frac{dv}{dr} \right)$, we have

$$f'(z) = \frac{1}{iz} \left( \frac{du}{d\theta} + i \frac{dv}{d\theta} \right)$$

This is obvious if we just work it out. So is we look at

$$\frac{1}{2} \oint_{|z|=r>1} f'(z)f(z)dz = \frac{1}{2} \int_0^{2\pi} \frac{1}{iz} \left( \frac{du}{d\theta} + i \frac{dv}{d\theta} \right) (u - iv)izd\theta$$

So the imaginary part of the integral is equal to the area of the region not mapped. So

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^{-k}$$

$$f'(z) = 1 + \sum_{k=1}^{\infty} -k a_k z^{-k-1}$$

$$\bar{f}(z) = \bar{z} + \sum_{k=1}^{\infty} \bar{a}_k \bar{z}^{-k}$$

So we also have

$$\frac{1}{2} \oint_{|z|=r>1} f'(z)f(z)dz = \frac{1}{2} \int_0^{2\pi} \left( 1 + \sum_{k=1}^{\infty} -k a_k z^{-k-1} \right) \left( \bar{z} + \sum_{k=1}^{\infty} \bar{a}_k \bar{z}^{-k} \right) ire^{i\theta} d\theta$$

$$= \frac{i}{2} \int_0^{2\pi} \left( r^2 - \sum_{k=1}^{\infty} n(a_n)^2 r^{-2n} \right) d\theta = i\pi \left( r^2 - \sum_{n=1}^{\infty} n|a_n|^2 r^{-2n} \right)$$

Now notice for $k \in \mathbb{Z}$ and $k \neq 0$ we have

$$\int_0^{2\pi} re^{i\theta} d\theta = 0$$

So

$$= \frac{i}{2} \int_0^{2\pi} \left( r^2 - \sum_{n=1}^{\infty} n(a_n)^2 r^{-2n} \right) d\theta = i\pi \left( r^2 - \sum_{n=1}^{\infty} n|a_n|^2 r^{-2n} \right)$$

Since the area is $> 0$, we have

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\[ r^2 - \sum_{n=1}^{\infty} n|a_n|^2r^{-2n} > 0 \]

which implies
\[ 1 \geq \sum_{n=1}^{\infty} n|a_n|^2 \]

**Jan 2003 Linear Algebra**

**Problem 2.a**

Let 

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } b \neq c. \]

Thus we have

\[ AB + BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix} + \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ = \begin{pmatrix} ax + b & -a + bx \\ cx + d & -c + dx \end{pmatrix} + \begin{pmatrix} ax - c & -d + bx \\ cx + a & b + dx \end{pmatrix} = \begin{pmatrix} 2ax + b - c & 2bx - d - a \\ 2cx + d + a & 2dx + b - c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Thus by combining equations we have \( 2x(a - d) = 0 \) and \( 2x(b + c) = 0 \). Clearly if \( x = 0 \), we have infinite number of solutions for \( A \). Thus for \( x \neq 0 \), we have \( a = d \) and \( b = -c \), and \( 2ax + 2b = 1 \) and \(-2a + 2bx = 0 \). Thus

\[ b = \frac{1}{2x^2 + 2}, c = \frac{-1}{2x^2 + 2}, a = d = \frac{x}{2x^2 + 2} \]

Hence \( x \neq 0 \), is when we have a unique solution for \( A \).

**Problem 2.b**

\[ M(k) = \begin{pmatrix} 7 & -9 \\ 11 & -13 \end{pmatrix} - k^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

so for \( k \in R \), \( \lambda_1\text{and}\lambda_2 \) are complex conjugates are purely real. Thus

\[ Tr(M(k)) = 7 - k^2 - 13 - k^2 = -6 - k^2 = \lambda_1 + \lambda_2 \]

This implies \( \lambda_1 \) and \( \lambda_2 \) have the same real part and it negative. Now

\[ \det(M(k)) = k^4 + 6k^2 + 8 > 0 \]

so both \( \lambda_1 \) and \( \lambda_2 \) must be negative.
Problem 2.c

Notice

\[ \det(N(k)) = 4k^4 - 15k^2 + 8 = \lambda_1\lambda_2 \]

and

\[ Tr(N(k)) = -6 - k^2 \]

So if \( \lambda_1 \) and \( \lambda_2 \) are real, then \( k = 1 \) will satisfy \( \lambda_1 \) or \( \lambda_2 > 0 \). Since \( \lambda_1\lambda_2 < 0 \), If \( k \) is real, \( \lambda_1\lambda_2 = |\lambda_1|^2 \) and \( \lambda_1 + \lambda_2 \in \mathbb{R} \). Let \( k = 1 \), then \( |\lambda| < 0 \) and we have a contradiction. So \( \lambda_1 \) and \( \lambda_2 \) must be real.

Problem 2.d

No. If \( k \) is real, then either \( \lambda_1 \) and \( \lambda_2 \) are real, or \( \lambda_1 \) and \( \lambda_2 \) are complex conjugates. If \( \lambda_1 \) and \( \lambda_2 \) are real, then \( Tr(N(k)) < 0 \). \( \forall k \in \mathbb{R} \), implies that \( \lambda_1 \) or \( \lambda_2 \) must be negative. If \( \lambda_1 \) and \( \lambda_2 \) are complex conjugates, we have

\[ \lambda_1 + \lambda_2 = 2Re(\lambda_1) < 0 \]

and hence we have a contradiction.

Problem 3.a

We start with a quick claim: matrix \( A \) is nilpotent if and only if all of it’s eigenvalues are zero.

\( \Rightarrow \), assume \( \exists n \) such that \( A^n = 0 \). Let \( \lambda \) be some eigenvalue and \( x \) a nonzero eigenvector. Thus \( Ax = \lambda x \). By induction \( A^n x = \lambda^n x = 0 \). Hence \( \lambda = 0 \). \( \Leftarrow \), now if all of the eigenvalues of \( A \) are zero, then the characteristic polynomial for \( A \) is \( p(x) = x^n \). By the Cayley-Hamilton Theorem, we have \( A^n = 0 \). This completes the proof. Since the eigenvalues of a triangular matrix are on the diagonal, and all the diagonal entries are zero in \( A \), \( A \) is nilpotent.

Problem 4

So we have the system \( Ax = b \) below

\[
\begin{pmatrix}
1 & 5 & 7 & 9 \\
1 & 5 & 7 & 9 \\
1 & 1 & 1 & 1 \\
1 & 3 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix}
=
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4
\end{pmatrix}
\]

We know that we have a unique solution when \( \det A \neq 0 \). Thus we need to choose \( a \) such that \( \det A = 0 \). Hence

\[ \det A = -28a = 0 \Rightarrow a = 0 \]

Thus we have
\[
\begin{pmatrix}
1 & 5 & 7 & 9 \\
0 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -3 & -1 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix} =
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 \\
r_4
\end{pmatrix}
\]

By performing Gaussian elimination, we have
\[
\begin{pmatrix}
1 & 5 & 7 & 9 \\
0 & 1 & 1 & 1 \\
0 & -6 & -6 & -10 \\
0 & -8 & -8 & -11
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix} =
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 - r_1 \\
r_4 - r_1
\end{pmatrix}
\]

Hence
\[
\begin{pmatrix}
1 & 5 & 7 & 9 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & -4 \\
0 & 0 & 0 & -3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix} =
\begin{pmatrix}
r_1 \\
r_2 \\
r_3 - r_1 + 6r_2 \\
r_4 - r_1 + 8r_2
\end{pmatrix}
\]

Hence the last two equations must make sense, i.e. we need to avoid no solutions. Thus
\[
\frac{r_3 - r_1 + 6r_2}{-4} = \frac{r_4 - r_1 + 8r_2}{-3} \Rightarrow r_1 - 14r_2 + 3r_3 - 4r_4 = 0
\]

When \( a = 0 \) and \( r_1 \)'s satisfy above, we have an infinite number of solutions.

**Sept 2003 Advanced Calculus**

**Problem 1.1**

I claim that it converges to \( 1/2 \). Notice
\[
\lim_{p \to \infty} \frac{1}{p} \int_0^p e^{x^2-p^2}(x^2 + 1)dx = \lim_{p \to \infty} \frac{1}{p} \int_0^p x^2e^{x^2-p^2}dx + \lim_{p \to \infty} \frac{1}{p} \int_0^p e^{x^2-p^2}dx
\]

Now notice by L’Hopitals Rule
\[
\lim_{p \to \infty} \frac{1}{p} \int_0^p e^{x^2-p^2}dx = \lim_{p \to \infty} \int_0^p e^{x^2}dx = \lim_{p \to \infty} \frac{e^{p^2}}{pe^{p^2}} = 0
\]

and also
\[
\lim_{p \to \infty} \frac{1}{p} \int_0^p x^2e^{x^2-p^2}dx = \lim_{p \to \infty} \frac{p^2e^{p^2}}{e^{p^2} + 2p^2e^{p^2}} = \lim_{p \to \infty} \frac{p^2}{1 + 2p^2} = \frac{1}{2}
\]

Thus
\[
\lim_{p \to \infty} \frac{1}{p} \int_0^p e^{x^2-p^2}(x^2 + 1)dx = \frac{1}{2}
\]
Problem 1.2

Clearly this summation converges since
\[\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2 + n}} \leq \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1} + n\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty\]

Hence the series converges. In fact it converges to one by telescoping
\[\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2 + n}} = \frac{\sqrt{2} - 1}{\sqrt{2}} + \frac{\sqrt{3} - \sqrt{2}}{\sqrt{6}} + \frac{\sqrt{4} - \sqrt{3}}{\sqrt{12}} + \cdots = 1 - \sqrt{1/2} + \sqrt{1/2} - \sqrt{1/3} + \sqrt{1/3} - \sqrt{1/4} + \sqrt{1/4} \pm \cdots = 1\]

Problem 1.3

We know that \(\lim_{x \to 0} x \ln x = 0\), which implies that \(\lim_{x \to 0} x^x = 1\). Thus
\[\lim_{x \to 0} x^x = 0\]

Problem 1.4

For \(x > 1\), notice that \(p = |(1 - x)/(1 + x)| < 1\), and
\[\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left( \frac{1-x}{1+x} \right)^n = \sum_{n=1}^{\infty} \frac{1}{2n-2} p^n < \infty\]

For \(x = 0\), the series clearly converges since we are summing zero’s. Now for \(0 \leq x < 1\), notice \(0 < p = (1 - x)/(1 + x) < 1\) and we have
\[\sum_{n=1}^{\infty} \frac{(-1)^n p^n}{2n-1} < \infty\]

since \(\frac{p^n}{2n-1} \searrow 0\), and by the alternating series theorem, the summation converges. Now for \(x < 0\) and \(x \neq -1\), \(|p| = |(1 - x)/(1 + x)| > 1\). Thus
\[\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left( \frac{1-x}{1+x} \right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n p^n}{2n-1} \not\to \infty\]

Since
\[\lim_{n \to \infty} \frac{(-1)^n p^n}{2n-1} \not\to 0\]

by L’Hopital’s rule.

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Problem 2

We have

\[ r(u, v) = ((a + b \cos u) \sin v, (a + b \cos u) \cos v, b \sin u) \]

for \(0 < u, v \leq 2\pi\). Thus

\[ r'_u = (-b \sin v \sin u, -b \sin u \cos v, b \cos u) \]

and

\[ r'_v = ((a + b \cos u) \cos v, -(a + b \cos u) \sin v, 0) \]

Therefore

\[ N(u, v) = r'_u \times r'_v = (b(a + b \cos u) \sin v \cos u, b(a + b \cos u) \cos v \cos u, b(a + b \cos u) \sin u) \]

Hence

\[ ||N(u, v)|| = \sqrt{b^2(a + b \cos u)^2 \cos^2 u + b^2(a + b \cos u)^2 \sin^2 u} = b(a + b \cos u) \]

Hence the area of our surface \(S\) is

\[ \int \int_{\Omega} ||N(u, v)|| \, du \, dv = \int_0^{2\pi} \int_0^{2\pi} b(a + b \cos u) \, du \, dv = 4\pi^2 ab \]

Problem 3

Easy, this a precalc question, graph the rational function and \(y = c\). Then there are \(n\) intersections.

Problem 4

Here we want to min/max the function \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\), on the curve of of intersection of the plane \(z = x + y\) and \(x^2 + y^2 + 2z^2 = 1\). Thus, WLOG, we need to min/max the function \(f(x, y) = 2x^2 + 2y^2 + 2xy\), and \(g(x, y) = 3x^2 + 3^2 + 4xy = 1\). By Lagrange we have \(\nabla f = \lambda \nabla g\), hence

\[ \frac{4x + 2y}{6x + 4y} = \frac{4y + 2x}{6y + 4x} \Rightarrow x^2 = y^2 \]

Thus one solution is \(x = y = \frac{1}{\sqrt{10}}\) and \(z = \frac{2}{\sqrt{10}}\) as well as \(x = y = \frac{-1}{\sqrt{10}}\) and \(z = \frac{-2}{\sqrt{10}}\). Thus at both of these coordinate we are \(f = \sqrt{3/5}\) distance away from the origin. Now if \(x = -y\), we have \(x = 1/\sqrt{2}\), \(y = -1/\sqrt{2}\), and \(z = 0\) as well as \(x = -1/\sqrt{2}\), \(y = 1/\sqrt{2}\), and \(z = 0\). At these coordinates, we are 1 distance away from the origin. Hence the minimum distance is
\[ f \left( \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}} \right) = f \left( -\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}} \right) = \sqrt{3/5} \]

and the maximum distance is
\[ f \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) = f \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) = 1 \]

**Problem 5**

\( F(u, v) : \mathbb{R}^2 \rightarrow \mathbb{R} \) with \( F(1, 2) = 0 \). Now let \( g(x, y, z) = F(xy, \sqrt{x^2 + z^2}) \). Then \( g(1, 1, \sqrt{3}) = 0 \) and
\[
\frac{dg}{dz}(1, 1, \sqrt{3}) = F_v(1, 2) \frac{1}{2} 2\sqrt{3} \neq 0
\]

Thus by the implicit function theorem, we can define a smooth function \( z(x, y) \) in a neighborhood of \((1, 1)\). Thus we have a smooth surface \((x, y, z(x, y))\) in a neighborhood of \((1, 1, \sqrt{3})\). Now for the second part we have \( \nabla F = (dF/dx, dF/dy, dF/dz) \) normal vector. Hence at \((1, 1, \sqrt{3})\) we have
\[
\begin{align*}
\frac{dF}{dx} &= \frac{dF}{dv} \frac{dv}{dx} + \frac{dF}{du} \frac{du}{dx} = 2 \\
\frac{dF}{dy} &= \frac{dF}{dv} \frac{dv}{dy} + \frac{dF}{du} \frac{du}{dy} = 1 \\
\frac{dF}{dz} &= \frac{dF}{du} \frac{du}{dz} + \frac{dF}{dv} \frac{dv}{dz} = \sqrt{3}
\end{align*}
\]

Thus \((2, 1, \sqrt{3})\) is our normal vector at \((1, 1, \sqrt{3})\).

**Sept 2003 Complex Variables**

**Problem 1.a**

for \( z = x + iy \) we have
\[
\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n} = \sum_{n=1}^{\infty} \frac{e^{-ny}(\cos(nx) + i \sin(nx))}{i2^{n+1}} + \sum_{n=1}^{\infty} \frac{e^{ny}(\cos(nx) - i \sin(nx))}{i2^{n+1}}
\]

notice for convergence we must have \( e^y < 2 \) and \( e^{-y} < 2 \) in order to have both summations to converge. Thus implies \(-\ln 2 < y < \ln 2\). Otherwise one of the summations will diverge and the other will converge. Which implies the summation as a whole diverges. So for \( z = x + iy \), the summation converges for \( x \in \mathbb{R} \) and \(-\ln 2 < y < \ln 2\).
Problem 1.b

notice for

\[ \sum_{n=1}^{\infty} \left( \frac{e^{iz}}{2} \right)^n = s \]

we have

\[ s \left( 1 - \frac{e^{iz}}{2} \right) = \frac{e^{iz}}{2} \]

which implies

\[ s = \frac{\cos(z) + i \sin(z)}{(2 - \cos(z)) - i \sin(z)} \]

thus

\[ \text{Im}(s) = \sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n} = \frac{2 \sin(z)}{5 - 4 \cos(z)} \]

Problem 2

Notice

\[ f(z) = \sum_{n=0}^{\infty} (n+1)(n+2)(z+1)^n = \sum_{n=0}^{\infty} (n^2 + 3n + 2)(z+1)^n \]

\[ = \sum_{n=0}^{\infty} n^2(z+1)^n + 3 \sum_{n=0}^{\infty} n(z+1)^n + 2 \sum_{n=0}^{\infty} (z+1)^n \]

Therefore for \(|z+1| < 1\) we have

\[ \sum_{n=0}^{\infty} (z+1)^n = \frac{-1}{z} \]

so

\[ \sum_{n=0}^{\infty} n(z+1)^{n-1} = \frac{1}{z^2} \]

which implies

\[ \sum_{n=0}^{\infty} n(z+1)^n = \frac{z+1}{z} \]

and finally

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\[
\sum_{n=0}^{\infty} n^2(z + 1)^{n-1} = \frac{-z - 2}{z^3}
\]

thus

\[
\sum_{n=0}^{\infty} n^2(z + 1)^n = \frac{-(z + 2)(z + 1)}{z^3}
\]

thus for analytic continuation, we have

\[
f(z) = \frac{-(z + 2)(z + 1)}{z^3} + \frac{3(z + 1)}{z^2} - \frac{2}{z}
\]

hence

\[
\lim_{z \to 1} f(z) = 2
\]

**Problem 3**

\[
T(z) = \frac{iz - 1}{iz + 1}
\]

works. Now the inverse mapping is

\[
z = \frac{w + 1}{w - 1}
\]

which maps the upper half of the unit circle to the real axis, and the lower half of the unit circle to the negative real axis. Also it maps the interior to the UHP. Now by Poisson’s Integral formula for a harmonic function in the UHP, we have

\[
\phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{-y}{y^2 + (x - w)^2} dw + \frac{1}{\pi} \int_{0}^{\infty} \frac{y}{y^2 + (x - w)^2} dw = \frac{2}{\pi} \arctan(x/y)
\]

Now notice the mapping from the disk to the UHP with \(w = x + iy\), we have

\[
z = i \frac{(x + iy + 1)}{x - 1 + iy} = \frac{2y}{(x - 1)^2 + y^2} + i \frac{(x^2 - 1)}{(x - 1)^2 + y^2} = \frac{y}{1 - x} + i \frac{x^2 - 1}{2x + 2}
\]

So the harmonic function we want for the unit disk is

\[
\phi(x, y) = \frac{\pi}{2} - \frac{2}{\pi} \arctan \left( \frac{(x^2 - 1)(1-x)}{y(2x + 2)} \right)
\]

and with \(x = r \cos \theta\) and \(y = r \sin \theta\), we have

\[
\phi(r, \theta) = \frac{\pi}{2} - \frac{2}{\pi} \arctan \left( \frac{(r^2 \cos^2 \theta - 1)(1 - r \cos \theta)}{r \sin \theta(2r \cos \theta + 2)} \right)
\]
Problem 4.1

Let \( f(z) = \frac{\pi iz}{(z-(-2+i))(z-(-2-2i))} \). Also let \( \gamma \) be the upper semi-circle with radius \( R \). Then by the Residue Theorem, we have

\[
\int_\gamma f(z) \, dz = 2\pi i \left( \frac{e^{-2-2i}}{2} \right) = \pi e^{-1-2i} = \pi e^{-1} (\cos(2) - i \sin(2))
\]

So notice for \( z = Re^{i\theta} \), we have

\[
\left| \int_{C_R} f(z) \, dz \right| \leq \int_{C_R} |f(z)| \, dz \leq R\pi \max(e^{-r \sin(\theta)}) \to 0
\]
as \( R \to \infty \). Thus we have

\[
\int_{-R}^{R} f(z) \, dz = \int_{-R}^{R} \frac{\cos(z) + i \sin(z)}{z^2 + 4z + 5} \, dz = \pi e^{-1} (\cos(2) - i \sin(2))
\]

Thus as \( R \to \infty \) we have

\[
\int_{-\infty}^{\infty} \frac{\sin(z)}{z^2 + 4z + 5} \, dz = -\pi e^{-1} \sin(2)
\]

Problem 4.2

For \( m = 5 \), notice that the integral over \( C_R \) would still approach zero. Thus

\[
\int_{-\infty}^{\infty} \frac{\sin(x)}{(x^2 + 4x + 5)^5} \, dx = \text{Im} \left( \frac{1}{4!} \lim_{z \to -2+i, \text{dz}^4} f(z) (z - (-2 + i))^5 \right)
\]

Problem 5

Let \( g(z) = f(1/z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \). Now clearly \( z_0 = re^{i\theta} \) a zero of \( f \) if and only if \( (1/r)e^{-i\theta} \) is a zero of \( g \). So we will show that all the zeros of \( g \) lie outside the circle with radius \( \sqrt{n} \) for all \( n \). Notice by Rouche we have

\[
|g(z) - e^z| = \left| \frac{z^{n+1}}{(n+1)!} + \frac{z^{n+2}}{(n+2)!} + \cdots \right| \leq \frac{\sqrt{n}^{n+1}}{(n+1)!} + \frac{\sqrt{n}^{n+2}}{(n+2)!} + \cdots < \frac{\sqrt{n}^n}{n!}
\]

We can see the last inequality since

\[
\frac{n!}{(\sqrt{n})^n} \left( \frac{\sqrt{n}^{n+1}}{(n+1)!} + \frac{\sqrt{n}^{n+2}}{(n+2)!} + \cdots \right) = \frac{n}{n+1} + \frac{\sqrt{n}}{(n+1)(n+2)} + \cdots < \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1
\]

Now since

\[
\lim_{n \to \infty} \frac{\sqrt{n}}{n! (1/n)} \to 0
\]

We have

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\[
\frac{\sqrt{n}}{(n!)(1/n)} + \frac{1}{\sqrt{n}} \leq 1
\]
for large enough \( n \) Hence
\[
\frac{\sqrt{n}}{(n!)(1/n)} \leq 1 - \frac{1}{\sqrt{n}} = 1 - \frac{\sqrt{n}}{n}
\]
which implies
\[
\left(\frac{\sqrt{n}}{n!}\right)^n \leq \left(1 - \frac{\sqrt{n}}{n}\right)^n \leq e^{-\sqrt{n}}
\]
Thus
\[
|g(z) - e^z| < |e^{-\sqrt{n}}| \leq |e^z|
\]
for \( |z| = \sqrt{n} \). Hence there are no zeros inside the circle \( |z| = \sqrt{n} \) which implies all zeros of \( f \) are inside the circle \( |z| = 1/\sqrt{n} \). SO for any \( r > 0 \), \( \exists N \) such that for \( n > N \) \( f \) has all zeros inside \( |z| = r \).

**Sept 2003 Linear Algebra**

**Problem 1**

Let
\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
Then we have \( Tr(M) = a + d \), \( Det(M) = ad - bc \), and
\[
M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}
\]
Thus
\[
\frac{1}{2}(Tr(M)^2 - Tr(M^2)) = \frac{(a + d)^2 - a^2 - 2bc - d^2}{2} = ad - bc = Det(M)
\]
We know for any \( 2 \times 2 \) matrix, the polynomial \( p(x) = x^2 - Tr(M)x + Det(M) \), we have \( p(M) = 0 \). Thus
\[
M^2 - Tr(M)M + \frac{1}{2}(Tr(M)^2 - Tr(M^2))I_2 = 0
\]
and for the \( 2 \times 2 \) matrix \( M + N \), we have
\[
(M + N)^2 - Tr(M + N)(M + N) + \frac{1}{2}((Tr(M + N))^2 - Tr((M + N)^2))I_2
\]
\[
= MN + NM - Tr(M)N - Tr(N)M + (Tr(M)Tr(N) - Tr(MN))I_2 = 0
\]
Problem 3a
Assume that \( S_1 \cap S_2 = \{0\} \). Also let \( \{b_1, \ldots, b_{n_1}\} \) be the basis for \( S_1 \) and let \( \{v_1, \ldots, v_{n_2}\} \) be the basis for \( S_2 \). Since \( S_1 \cap S_2 = \{0\} \), \( \{b_1, \ldots, b_{n_1}, v_1, \ldots, v_{n_2}\} \) is a linearly independent set of vectors in an \( n \) dimension vector space. However since \( n_1 + n_2 > n \), we have a contradiction.

Jan 2004 Advanced Calculus

Problem 1.a

Notice that

\[
\sum_{k=1}^{\infty} \frac{1}{k \log^p(k+1)} \leq \frac{1}{\log^p 2} + \sum_{k=2}^{\infty} \frac{1}{k \log^p k}
\]

Hence in order to show that \( \sum_{k=1}^{\infty} \frac{1}{k \log^p(k+1)} \) converges, it suffices to show that \( \sum_{k=2}^{\infty} \frac{1}{k \log^p k} \) converges. By the integral theorem, it converges if and only if

\[
\int_{2}^{\infty} \frac{1}{x \log^p x} \, dx
\]

converges. Notice using \( u \) substitution, we have

\[
\int_{2}^{\infty} \frac{1}{x \log^p x} \, dx = \left. \frac{(\log x)^{1-p}}{1-p} \right|_{2}^{\infty}
\]

Hence the integral converges (and thus the summation) when \( p > 1 \). Keeping that in mind, recall that

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \log k + O(1)
\]

Hence notice

\[
\sum_{k=1}^{\infty} \frac{1}{k \log^p(k+1)} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) = \sum_{k=1}^{\infty} \frac{\log k + O(1)}{k \log^p(k+1)} \leq \sum_{k=2}^{\infty} \frac{\log k}{k \log^p k} + \sum_{k=2}^{\infty} \frac{M}{k \log^p k} + O(1)
\]

for some constant \( M \). Thus the summation converges when \( p > 2 \).

Problem 1.b

Let’s consider the case when \( p < 0 \), and let \( q = |p| \). Then we have

\[
\sum_{k=1}^{\infty} \frac{k^p}{p^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^q q^k}
\]

clearly \( k^q q^k \downarrow 0 \), thus by the alternating series theorem, the summation converges. Now for \( p > 1 \), notice
\[
\lim_{k \to \infty} \frac{(k+1)^p p^k}{k^{p+1} k^p} = 1/p < 1
\]

Hence the summation converges by the ratio theorem. Also notice by the ration test, the summation diverges for \(0 < p < 1\). Clearly the sequence diverges for \(p = \pm 1\).

**Problem 2**

First assume that \(0 < x < \pi/2\). Then let \(y = \sqrt{n} \sin z\). Then

\[
\lim_{n \to \infty} \sqrt{n} \int_0^x z^2 e^{n(\cos^2 z - 1)} \, dz = \lim_{n \to \infty} \frac{z^2}{\cos z} e^{-y^2} \, dy
\]

\(z = \arcsin(y/\sqrt{n})\) and so \(\cos z = \sqrt{1 - y^2/n}\). Therefore

\[
= \lim_{n \to \infty} \int_0^{\sqrt{n} \sin x} \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \, dy = \lim_{n \to \infty} \int_0^\infty \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \chi_{[0,\sqrt{n} \sin x]} \, dy
\]

Now let

\[
f_n = \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \chi_{[0,\sqrt{n} \sin x]}
\]

Then \(|f_n| \leq 1000 e^{-y^2}\), which is integrable. Therefore by the dominated convergence theorem, we have

\[
= \lim_{n \to \infty} \int_0^\infty \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \chi_{[0,\sqrt{n} \sin x]} \, dy = \int_0^\infty 0 \, dy = 0
\]

Now assume that \(\pi/2 < x < \pi\). Then by the same substitution, we have

\[
\lim_{n \to \infty} \sqrt{n} \int_0^x z^2 e^{n(\cos^2 z - 1)} \, dz = \int_0^\infty \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \chi_{[0,\sqrt{n} \sin x]} \, dy = \pi^2 \int_0^\infty e^{-u^2} \, du = \frac{\pi^2 \sqrt{\pi}}{2}
\]

Likewise if \(\pi < x < 3\pi/2\), we have

\[
\lim_{n \to \infty} \sqrt{n} \int_0^x z^2 e^{n(\cos^2 z - 1)} \, dz = \int_0^\infty \frac{\arcsin^2(y/\sqrt{n})}{\sqrt{1 - y^2/n}} e^{-y^2} \chi_{[0,\sqrt{n} \sin x]} \, dy = \pi^2 \int_0^- e^{-u^2} \, du = -\frac{\pi^2 \sqrt{\pi}}{2}
\]

So in closed form, we have

\[
g(x) = \begin{cases} 
(k\pi)^2 \frac{\sqrt{\pi}}{2} & k\pi/2 < x < (k + 1)\pi/2, k \text{ odd} \\
-(k\pi)^2 \frac{\sqrt{\pi}}{2} & k\pi/2 < x < (k + 1)\pi/2, k \text{ even} \\
\text{undefined} & \text{else}
\end{cases}
\]

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Problem 3

We have \( f(x, y) = 2x^2 + \frac{\sqrt{3}}{4}y^2 \) and \( g(x, y) = 6x + 3y = 1 \). Thus \( y = (1 - 6x)/3 \), and

\[
f(x) = 2x^2 + \frac{\sqrt{3}}{36}(1 - 6x)^2
\]

therefore

\[
f'(x)4x - \frac{\sqrt{3}}{3}(1 - 6x) = 0 \Rightarrow x = \frac{\sqrt{3}}{12 + 6\sqrt{3}}
\]

Notice this is the minimum since we are dealing with a parabola opening upwards. Thus the maximum total area is \( 1/18 \) when we use the wire to make a triangle and no rectangles.

Problem 4a

First notice that \( f(x) = f(x+0) = f(x) + f(0) \) which implies that \( f(0) = 0 \). Thus \( f(0) = f(x-x) = f(x) + f(-x) \), which implies \( f(-x) = -f(x) \). Also for some positive integer \( m \), we have

\[
f(mx) = f(x) + f((m-1)x) = \cdots = mf(x)
\]

Also notice for any positive integer \( n \) we have \( f(x) = f(nx/n) = nf(x/n) \), which implies that \( f(x/n) = (1/n)f(x) \). Now let \( p \) be some real number. Then we know \( \exists \) a sequence of rational numbers \( \{r_n\} \) such that \( r_n \to p \). By continuity, we have

\[
f(px) = f(\lim r_n x) = \lim_n r_n f(x) = pf(x)
\]

Thus \( f(x) = xf(1) \), for all \( x \in [0, \infty) \).

Problem 4b

First notice that \( f(x) = f(x+0) = f(x)f(0) \), which implies that \( f(0) = 1 \). Also \( f(0) = f(x-x) = f(x)f(-x) \), which implies \( f(-x) = f(x)^{-1} \). Let \( m \) be some positive integer. Then

\[
f(mx) = f(x)f((m-1)x) = \cdots = f(x)^m
\]

Also for any positive integer \( n \) we have

\[
f(x) = f(nx/n) = f(x/n)^n \Rightarrow f(x/n) = f(x)^{1/n}
\]

Thus for any rational number \( r \) we have \( f(rx) = f(x)^r \). Now let \( p \) be some real number. Then we know \( \exists \) a sequence of rational number \( \{r_n\} \) such that \( r_n \to p \). By continuity we have

\[
f(px) = f(\lim r_n x) = \lim_n f(r_n x) = \lim_n f(x)^{r_n} = f(x)^p
\]

Thus \( f(x) = f(1)^x \) for all \( x \).
Problem 4c
Notice for all \(x\), \(f(x) = f(x+0) = f(x)f(0) + f(0) + f(x)\), which implies that \(f(0) = 0\) or \(f(x) = -1\). However we know that \(f(x) \geq 0\), hence \(f(0) = 0\). Now I claim that \(f(x) = 0\) for all \(x\). Assume \(\exists x_0\) such that \(f(x_0) \neq 0\). Then notice
\[
f(0) = f(x_0 - x_0) = f(x_0)f(-x_0) + f(x_0) + f(-x_0) = 0 \Rightarrow f(-x_0) = \frac{-f(x_0)}{1 + f(-x_0)}
\]
notice \(f(-x_0) = 0\), which implies \(f(x_0) = 0\). Thus we have a contradiction and so \(f(x) = 0\) for all \(x\).

Problem 5
We know that the area of \(S\) is
\[
\int \int N(u, v) \, dudv
\]
where \(N(u, v)\) is the fundamental vector product \(r_u \times r_v\) where \(r\) is the vector that parameterizes the surface \(S\). Thus let \(r = (u, f(u) \cos(v), f(u) \sin(v))\) where \(a \leq u \leq b\) and \(0 \leq v \leq 2\pi\). Then we have
\[
N(u, v) = r_u \times r_v = \begin{vmatrix} 1 & f'(u) \cos(v) & f'(u) \sin(v) \\ 0 & -f(u) \sin(v) & f(u) \cos(v) \end{vmatrix}
\]
\[
= (f'(u)f(u) \cos^2(v) + f'(u)f(u) \sin^2(v), -f(u) \cos(v) - f(u) \sin(v)) = (f'(u)f(u), -f(u) \cos(v), -f(u) \sin(v))
\]
Thus \(\|N(u, v)\| = \sqrt{(f'(u)f(u))^2 + f^2(u)} = f(u) \sqrt{(f'(u))^2 + 1}\). Hence the area of \(S\) is
\[
\int_a^b \int_0^{2\pi} \sqrt{(f'(u)f(u))^2 + f^2(u)} \, dudv = \int_a^b \int_0^{2\pi} f(u) \sqrt{(f'(u))^2 + 1} \, dudv = \int_a^b 2\pi f(u) \sqrt{(f'(u))^2 + 1} \, du
\]

Jan 2004 Complex Variables

Problem 1
Notice for \(z = e^{i\theta}\)
\[
|z^{16} + 3z^7 + z^3 + 1 - 3z^7 - 1| = |z^{16} + z^3| = |z^3| |z^{13} + 1| = |z^{13} + 1| \leq 2 \leq |3z^7 + 1|
\]
However notice that \(|z^{13} + 1| = 2\) only when \(z = 1\) and \(z = e^{i2\pi n/13}\). Thus \(2 < 4 = |3z^7 + 1|\) when \(z = 1\) and when \(z = e^{i2\pi n/13}\), we have
\[
|3e^{i14\pi n/13} + 1| > 2
\]
Since \(e^{i14\pi n/13} \neq -1\) for all integers \(n\). Thus
\[ |z^{16} + 3z^7 + z^3 + 1 - z^7 - 1| = |z^{16} + z^3| < |3z^7 + 1| \]
and by Rouche’s theorem, the number of zeros inside the unit circle of \( z^{16} + 3z^7 + z^3 + 1 \) is the same as \( 3z^7 + 1 \). Hence we have 7 roots.

**Problem 2**

we want to compute \( \sum_{n=1}^{\infty} 1/n^4 \). Let \( \gamma_N \) be the square in the complex plane with vertices at \( \pm(N + 1/2) \pm i(N + 1/2) \), where \( N \) is a positive integer. Let \( f(z) = \frac{\pi \cot(\pi z)}{z^4} \), then we know as \( N \to \infty \), we have

\[ \int_{\gamma_N} f(z)\,dz = 0 \]

and so by the Residue Theorem, we have

\[ \sum \text{Res}(f(z), z) = \sum_{n=-\infty}^{\infty} \frac{1}{n^4} + \text{Res}(f(z), z = 0) \]

Hence

\[ \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2} \text{Res}(f(z), z = 0) \]

So we need to compute the coefficient \( a_1 \) of the Laurent series for \( f(z) \). Hence

\[ \frac{\pi \cot(\pi z)}{z^4} = \frac{\pi \cos(\pi z)}{z^4 \sin(\pi z)} = \left( a_{-5} + a_{-4} \frac{1}{z^4} + a_{-3} \frac{1}{z^3} + a_{-2} \frac{1}{z^2} + a_{-1} \frac{1}{z} + \cdots \right) \]

which implies

\[ \left( \pi - \frac{\pi^3 z^2}{2!} + \frac{\pi^5 z^4}{4!} + \cdots \right) = \left( \pi z^5 - \frac{\pi^3 z^7}{3!} + \frac{\pi^5 z^9}{5!} + \cdots \right) \left( a_{-5} + a_{-4} \frac{1}{z^4} + a_{-3} \frac{1}{z^3} + a_{-2} \frac{1}{z^2} + a_{-1} \frac{1}{z} + \cdots \right) \]

Which implies \( a_{-5} = 1, a_{-4} = 0, a_{-3} = -\pi^2/3, a_{-2} = 0, \) and \( a_{-1} = -\pi^4/45 \). Hence

\[ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \]

**Problem 3**

By the Schwartz-Christoffel equation, this is an equilateral triangle with \( \infty \) mapping to 0. Also

\[ f(1) = \int_1^{\infty} t^{-2/3}(t - 1)^{-2/3} \,dt > 0 \]

So let
\[ B = \int_{1}^{\infty} t^{-2/3}(t-1)^{-2/3}dt \]

Now notice that
\[ f'(z) = -x^{-2/3}(x-1)^{-2/3} \]

So for \( x \in (0, 1) \), we have
\[ \arg(f'(x)) = \pi + 0 - 2\pi/3 = \pi/3 \]

and for \( x < 0 \), we have
\[ \arg(f'(x)) = \pi - 2\pi/3 - 2\pi/3 = \pi - 4\pi/3 = -\pi/3 \]

we have an equilateral triangle with vertices 0, B, B/2 − i√3B/2.

**Problem 4**

Notice that \( w = z^2 \) will map \( D = D_1 \) to the region \( D_2 = \{ x + iy : x < 1, y > 0 \} \). Indeed since for \( z = x + iy \) such that \( x^2 - y^2 = 1 \), we have
\[ z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi = 1 + 2xyi \]

Thus \( w = -z^2 \) will give us \( D_2 = \{ x + iy : x > -1, y < 0 \} \). Thus \( w = (1 - z^2)^2 \) give us \( D_3 = \{ z : \text{Im}(z) < 0 \} \). Finally \( w = -(1 - z^2)^2 \) gives us \( U \). Now in order for \( (0, 1, \infty) \) to be mapped to \( (0, 1, \infty) \), we need \( w = 1 - (1 - z^2)^2 \).

**Problem 5**

Let \( \gamma_1 \) be the semi-circle with radius \( R \) in the LHP with a very small loop below the \([0, 1]\) real line. Likewise make \( \gamma_2 \) be the same contour in the UHP. Then let
\[ f(z) = \frac{1}{1 + z^2} \frac{dz}{\sqrt{z^2 - 1}} \]

Then by the residue theorem, we have
\[ \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 2\pi i \left( \frac{1}{2i\sqrt{i+1}} + \frac{1}{-2i\sqrt{-2(1+i)}} \right) = \frac{\pi}{\sqrt{i(1-i)}} - \frac{\pi}{\sqrt{-i(1+i)}} \]

Note we have the branch cut for \( 0 < \arg(z) < 2\pi \) So
\[ = \frac{\pi}{e^{1/2(\ln(\sqrt{2} + i\pi/4))}} - \frac{\pi}{e^{1/2(\ln(\sqrt{2} + i\pi/4))}} = \frac{\pi}{\sqrt{2}} 2\cos(\pi/8) \]

Now as we make that loop approach the real axis, we have
\[ \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 2 \int_{0}^{1} f(z)dz = \frac{\pi}{\sqrt{2}} 2\cos(\pi/8) \]

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Jan 2004 Linear Algebra

Problem 1
We have the system $Ax = b$ below

$$
\begin{pmatrix}
2 + 2\alpha & 0 & 1 + \alpha \\
1 + \alpha & 1 + \alpha & 1 + \alpha \\
0 & 1 + \alpha & -1 + 2\alpha
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
1 + \beta \\
2 - \beta
\end{pmatrix}
$$

We know that the system of equations has a unique solution when $A$ is invertible, i.e. $\det A \neq 0$. Thus

$$
\det A = (2 + 2\alpha)((1 + \alpha)(-1 + 2\alpha) - (1 + \alpha)^2) + (1 + \alpha)^3 = (1 + \alpha)^2(-3 + 3\alpha)
$$

Thus when $\alpha \neq 1$ or $\alpha \neq -1$, the system of equations has a unique solution. When $\alpha = -1$, we have no solutions since this would imply $0x + 0y + 0z = 0 = 2$. Now when $\alpha = 1$, we have

$$
\begin{pmatrix}
4 & 0 & 2 \\
2 & 2 & 2 \\
0 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
1 + \beta \\
2 - \beta
\end{pmatrix}
$$

Thus by performing Gaussian elimination, we have

$$
\begin{pmatrix}
4 & 0 & 2 \\
0 & 2 & 1 \\
0 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
\beta \\
2 - \beta
\end{pmatrix}
$$

Thus if $\beta \neq 1$, we have no solutions. If $\beta = 1$, we have infinite number of solutions.

Problem 2
The adjoint $L^*$ is defined such that

$$
\langle L(f), g \rangle = \langle f, L^*(g) \rangle
$$

thus

$$
\int_0^{2\pi} f(x + 1)g(x)h(x)dx = \int_0^{2\pi} f(x)L^*(x)h(x)dx
$$

which implies

$$
\int_0^{2\pi} f(x)g(x - 1)h(x-1)dx = \int_0^{2\pi} f(x)L^*(x)h(x)dx
$$

so

$$
L^*(g(x)) = g(x - 1)\frac{h(x - 1)}{h(x)}
$$
Problem 3

We will apply Gram Schmidt to \( e_1, e_2, e_3 \). Then notice

\[
u_1 = \sqrt{3}(1,0,0)T
\]

Then

\[
u_2 = e_2 - (e_2, u_1)u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1 \\ 0 \end{pmatrix}
\]

then after normalizing it we have

\[
u_2 = 4\sqrt{5} \begin{pmatrix} -3/4 \\ 1 \\ 0 \end{pmatrix}
\]

Then

\[
u_3 = e_3 - (e_3, u_1)u_1 - (e_3, u_2)u_2
\]

Then we have

\[
(e_3, u_1) = \int_0^1 x^2 x^2 \sqrt{3} dx = \frac{\sqrt{3}}{5}
\]

and

\[
(e_3, u_2) = \int_0^1 x^2 x^2 (-3\sqrt{5} + 4\sqrt{5}x) dx = \frac{\sqrt{5}}{15}
\]

Hence

\[
u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{\sqrt{3}}{5} \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} - \frac{\sqrt{5}}{15} \begin{pmatrix} -3\sqrt{5} \\ 4\sqrt{5} \\ 0 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -4/3 \\ 1 \end{pmatrix}
\]

After normalizing it we have

\[
\int_0^1 x^2 (2/5 - (4/3)x + x^2)^2 dx \sim 0.000635
\]

hence

\[
u_3 = \sqrt{1575} \begin{pmatrix} 2/5 \\ -4/3 \\ 1 \end{pmatrix}
\]
Problem 4

Given that $A$ is an $n \times m$ matrix and $B$ is an $m \times n$ matrix, we have $AB$ an $n \times n$ matrix and $BA$ an $m \times m$ matrix. Let $\lambda$ be a nonzero eigenvalue of $AB$. Then $\exists$ a vector $x \in \mathbb{R}^n$ such that

$$ABx = \lambda x \Rightarrow BABx = B\lambda x = \lambda Bx$$

Hence $\lambda$ is an eigenvalue for $BA$ with eigenvector $Bx$. Same argument for $BA$ to $AB$. If $x_1, \ldots, x_n$ are the eigenvectors for $AB$, then $Bx_1, \ldots, Bx_n$ are eigenvectors for $BA$.

Problem 5

We first solve $A_1x_1 = b$, then solve $A_2x_2 = x_1$, then solve $A_3x_3 = x_2$. Then we have the solution $A_1A_2A_3x_3 = b$

Sept 2004 Advanced Calculus

Problem 2

$$\frac{df}{dx} = \frac{e^{x^3}}{x^2} (2x) + \int_y^x e^t dt = 2e^x + e^x - \frac{e^x}{x} = 3e^x - \frac{e^x}{x}$$

Problem 3.a

Recall

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin(2x)| \sin(nx) dx$$

Notice that $|\sin(2x)| \sin(nx)$ is an odd function. Hence the integral is zero.

Problem 3.b

Now notice

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n^2}{\pi} \int_{-\pi}^{\pi} |\sin(2x)| \cos(nx) dx = \frac{2n^2}{\pi} \int_{0}^{\pi} \sin(2x) \cos(nx) dx$$

Since $|\sin(2x)| \cos(nx)$ is an even function. Hence let $u = \sin(2x)$, $du = 2 \cos(2x)$, $v = (1/n) \sin(nx)$, and $dv = \cos(nx) dx$. Then using integration by parts we have

$$\int_{0}^{\pi} \sin(2x) \cos(nx) dx = \frac{1}{n} \sin(2x) \sin(nx) - \frac{2}{n} \int_{0}^{\pi} \cos(2x) \sin(nx) dx$$

again let $u = \cos(2x)$, $du = -2 \sin(2x)$, $v = -(1/n) \cos(nx)$, and $dv = \sin(nx) dx$. Then using integration by parts we have

$$\int_{0}^{\pi} \sin(2x) \cos(nx) dx = \frac{1}{n} \sin(2x) \sin(nx) - \frac{2}{n} \left( -\frac{1}{n} \cos(nx) \cos(2x) - \frac{2}{n} \int_{0}^{\pi} \sin(2x) \cos(nx) dx \right)$$
Thus
\[\int_0^\pi \sin(2x) \cos(nx)dx = \frac{\frac{1}{n} \sin(2x) \sin(nx) + \frac{2}{n^2} \cos(nx) \cos(2x)}{1 - 4/n^2} =\]
\[\frac{n \sin(2x) \sin(nx) + 2 \cos(nx) \cos(2x)}{n^2 - 4}\]
\[\int_0^\pi = \frac{2 \cos(n\pi)}{n^2 - 4} = \frac{2}{n^2 - 4} = \frac{2(-1)^n - 2}{n^2 - 4}\]

hence
\[\lim_{n \to \infty} b_n n^2 = \lim_{n \to \infty} \frac{2(-1)^n - 2}{1 - 4/n^2}\]

Hence it does not converge

Problem 3.c

By above we have
\[b_3 = \frac{-4}{5\pi}\] and \[b_6 = 0\]

Problem 4

We have \[f(x, y, z) = xyz + x^2 + y^2 + z^2\] and \[g(x, y, z) = x^2 + y^2/2 + z^2/2 = 3\]. By Lagrange we have \(\nabla f = \lambda \nabla g\), which implies
\[\frac{2x + yz}{2x} = \frac{2y + xz}{y} = \frac{2z + xy}{z}\]

Thus by the last equality, we have \(z^2 = y^2\). Thus if \(z = y\), we have \(x^2 + y^2 = 3\) and
\[2xy + y^3 = 4xy + 2x^2 y \Rightarrow y^2 = 2x^2 + 2x\]

Thus \(x = \frac{-1}{3} + \frac{\sqrt{10}}{3}\). For \(x = \frac{-1}{3} + \frac{\sqrt{10}}{3}\), this implies that \(y = \pm \sqrt{3 - (-1/3 + \sqrt{10}/3)^2} = z\). Hence we have two solutions at
\[(x, y, z) = \left(\frac{-1}{3} + \frac{\sqrt{10}}{3}, \sqrt{3 - (-1/3 + \sqrt{10}/3)^2}, \sqrt{3 - (-1/3 + \sqrt{10}/3)^2}\right)\]

and
\[(x, y, z) = \left(\frac{-1}{3} + \frac{\sqrt{10}}{3}, -\sqrt{3 - (-1/3 + \sqrt{10}/3)^2}, -\sqrt{3 - (-1/3 + \sqrt{10}/3)^2}\right)\]

and both have values of
\[f(x, y, z) = \frac{1}{27}(85 + 14\sqrt{10})\]

For \(x = \frac{-1}{3} - \frac{\sqrt{10}}{3}\), we have the solution

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\[(x, y, z) = \left(\frac{-1}{3} - \frac{\sqrt{10}}{3}, \sqrt{3 - (-1/3 - \sqrt{10}/3)^2}, \sqrt{3 - (-1/3 - \sqrt{10}/3)^2}\right)\]

and

\[(x, y, z) = \left(\frac{-1}{3} - \frac{\sqrt{10}}{3}, -\sqrt{3 - (-1/3 - \sqrt{10}/3)^2}, -\sqrt{3 - (-1/3 - \sqrt{10}/3)^2}\right)\]

and both have values of

\[f(x, y, z) = \frac{1}{27}(85 - 14\sqrt{10})\]

Now if \(y = -z\), we have \(x^2 + y^2 = 3\) and

\[2xy - y^3 = 4xy - 2x^2y \Rightarrow y^2 = 2x^2 - 2x\]

Thus \(x = \frac{1}{3} \pm \frac{\sqrt{10}}{3}\). For \(x = \frac{1}{3} + \frac{\sqrt{10}}{3}\), this implies that \(y = \pm\sqrt{3 - (-1/3 + \sqrt{10}/3)^2}\) and \(z = \mp\sqrt{3 - (-1/3 + \sqrt{10}/3)^2}\). Hence we have two solutions at

\[(x, y, z) = \left(\frac{1}{3} + \frac{\sqrt{10}}{3}, \sqrt{3 - (1/3 + \sqrt{10}/3)^2}, -\sqrt{3 - (1/3 + \sqrt{10}/3)^2}\right)\]

and

\[(x, y, z) = \left(\frac{1}{3} + \frac{\sqrt{10}}{3}, -\sqrt{3 - (1/3 + \sqrt{10}/3)^2}, \sqrt{3 - (1/3 + \sqrt{10}/3)^2}\right)\]

and both have values of

\[f(x, y, z) = \frac{1}{27}(85 - 14\sqrt{10})\]

For \(x = \frac{1}{3} - \frac{\sqrt{10}}{3}\), we have the solution

\[(x, y, z) = \left(\frac{1}{3} - \frac{\sqrt{10}}{3}, -\sqrt{3 - (1/3 - \sqrt{10}/3)^2}, -\sqrt{3 - (1/3 - \sqrt{10}/3)^2}\right)\]

and

\[(x, y, z) = \left(\frac{1}{3} - \frac{\sqrt{10}}{3}, -\sqrt{3 - (1/3 - \sqrt{10}/3)^2}, \sqrt{3 - (1/3 - \sqrt{10}/3)^2}\right)\]

and both have values of

\[f(x, y, z) = \frac{1}{27}(85 + 14\sqrt{10})\]
Sept 2004 Complex Variables

Problem 1

We first map the domain $D = D_1$ to the strip $D_2 = \{ x + iy : -1/2 < x < 1/2 \}$. Then we will switch the borders by multiplying by $-1$. Then we will rotate it by $\pi/2$ by multiplying by $i$. Now we have the domain $D_3 = \{ x + iy : -\pi/2 < y < \pi/2 \}$. Then we will shift the domain up by $\pi/2$ and map it to the upper half plane by $w = e^{\pi i/2}$. Hence our mapping is

$$w = e^{i(-\pi/z + \pi/2)}$$

Problem 2.a

Let $\gamma$ be the contour of two semicircles with radius $R$ and $\epsilon$ and $f(z) = \frac{\sqrt{z} \log z}{z^2+1}$. By the Residue Theorem, we have

$$\int_{\gamma} f(z)dz = 2\pi i \frac{\sqrt{i} \log i}{2i} = \frac{\pi^2}{2} \left( \frac{-\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)$$

Now on $C_R$, we have $z = Re^{i\theta}$. Notice

$$\left| \int_{C_R} \frac{\sqrt{z} \log z}{z^2+1} \right| = \int_{C_R} \frac{\sqrt{Re^{i\theta} \log Re^{i\theta}}}{(Re^{i\theta})^2+1} \, dz \leq \pi R \frac{\sqrt{R \log^2 |R| + \pi^2}}{R^2 - 1} \to 0$$

as $R \to \infty$. Also notice on $C_\epsilon$, we have

$$\left| \int_{C_\epsilon} \frac{\sqrt{\epsilon e^{i\theta} \log \epsilon e^{i\theta}}}{(\epsilon e^{i\theta})^2+1} \, dz \right| \leq \pi \epsilon \frac{\epsilon^{1/2} \sqrt{(\log^2 |\epsilon| + \pi^2)}}{\epsilon^2 - 1} \to 0$$

as $\epsilon \to 0$. Thus

$$\int_{-\epsilon}^{\epsilon} \frac{\sqrt{z} \log z}{z^2+1} \, dz + \int_{\epsilon}^{R} \frac{\sqrt{z} \log z}{z^2+1} \, dz = \int_{\epsilon}^{R} \frac{i \sqrt{z} (\log z + i\pi)}{z^2+1} \, dz + \int_{\epsilon}^{R} \frac{\sqrt{z} \log z}{z^2+1} \, dz$$

Thus matching up the imaginary parts and letting $\epsilon \to 0$ and $R \to \infty$, we have

$$\int_{0}^{\infty} \frac{\sqrt{z} \log z}{z^2+1} \, dz = \frac{\pi^2 \sqrt{2}}{4}$$

Problem 2.b

Let $C$ be the contour of two semicircles with radius $R$ and $\epsilon$ and $f(z) = \frac{(\log z)^2}{z^2+1}$. By the Residue Theorem, we have

$$\int_{C} f(z)dz = 2\pi i \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}$$

Now notice on $C_R$ we have $z = Re^{i\theta}$. Hence
\[ \left| \int_C f(z) \, dz \right| \leq R\pi \frac{(\log R)^2}{R^2 - 1} \to 0 \]
as \( R \to \infty \). Also notice on \( C_\epsilon \), we have
\[ \left| \int_C f(z) \, dz \right| \leq \epsilon\pi \frac{(\log(\epsilon))^2}{\epsilon^2 - 1} \to 0 \]
as \( \epsilon \to 0 \). Thus we have
\[
\int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1} \, dz + \int_{-R}^{-\epsilon} \frac{(\log z)^2}{z^2 + 1} \, dz = \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1} \, dz + \int_{\epsilon}^{R} \frac{(\log z + i\pi)^2}{z^2 + 1} \, dz
\]
\[
= \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1} \, dz + \int_{\epsilon}^{R} \frac{(\log z)^2}{z^2 + 1} \, dz + 2\pi i \int_{\epsilon}^{R} \frac{\log z}{z^2 + 1} \, dz + \int_{\epsilon}^{R} \frac{-\pi^2}{z^2 + 1} \, dz
\]
Thus as \( R \to \infty \) and \( \epsilon \to 0 \), we have
\[
\int_{0}^{\infty} \frac{\log z}{z^2 + 1} \, dz = 0
\]
Now Recall that
\[
\int_{0}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}
\]
Indeed since we can take the integral around the semicircle of radius \( R \) and apply the Residue Theorem.

**Problem 2.c**

Thus by above
\[
\int_{0}^{\infty} \frac{\log z}{z^2 + 1} \, dz = \frac{-\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{8}
\]

**Sept 2004 Linear Algebra**

**Problem 2.a**

Notice that \( \dim \ker A_1 = 31 - \dim \im A_1 \leq 17 \), \( \dim \ker A_2 = 31 - \dim \im A_2 \leq 21 \), and \( \dim \ker A_3 = 31 - \dim \im A_3 \leq 27 \). Since for all \( i \), \( \ker A_i \subset \mathbb{R}^{31} \), we have
\[
\dim (\ker A_1 \cap \ker A_2) \geq 7
\]
\[
\dim (\ker A_1 \cap \ker A_3) \geq 13
\]
\[
\dim (\ker A_2 \cap \ker A_3) \geq 17
\]
Thus \( \dim(\ker A_1 \cap \ker A_2 \cap \ker A_3) \geq 6 \) which implies

\[
\ker A_1 \cap \ker A_2 \cap \ker A_3 \neq \emptyset
\]

**Problem 4**

Since \( O \) is orthogonal we know that for any eigenvalue \( \lambda, |\lambda| = 1 \). Also \( \det O = \pm 1 \), which implies that all eigenvalues are nonzero. Now I claim that if \( \lambda \) is an eigenvalue, then so is \( 1/\lambda \). Indeed since

\[
p(\lambda) = \det(A - \lambda I) = \det \left( (-\lambda A \left( A^T - \frac{1}{\lambda} I \right) \right) = \det(-\lambda A) \det \left( \left( A - \frac{1}{\lambda} I \right)^T \right)
\]

\[
= (\lambda)^n \det A \det \left( A - \frac{1}{\lambda} I \right) = \pm \lambda^n p\left( \frac{1}{\lambda} \right)
\]

Since \( O \) is a \( 3 \times 3 \), \( \exists \) an eigenvalue \( \lambda \) such that

\[
\lambda = \frac{1}{\lambda} \implies \lambda = \frac{\lambda}{|\lambda|^2} \implies \lambda = \bar{\lambda}
\]

Hence \( \lambda = \pm 1 \). If all of \( O \)'s eigenvalues are not 1's, then \( \exists \) an eigenvalue \( \lambda = -1 \). Hence \( \exists \) and eigenvector \( x \) such that

\[
Ox = -x \implies O O x = -(-x) \implies O^2 x = x
\]

Thus \( O \) has an eigenvalue of 1.

**Problem 5.a**

Recall that \( \text{Tr}(AB) = \text{Tr}(BA) \) since

\[
\text{Tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} b_{j,i} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{j,i} a_{i,j} = \text{Tr}(BA)
\]

Thus \( \text{Tr}(AB^2 A) = \text{Tr}(ABB A) = \text{Tr}(BA AB) = \text{Tr}(BA^2 B) \).

**Problem 5.b**

No. Let

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

Then we have

\[
ABC = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad CBA = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}
\]

Thus \( \text{Tr}(ABC) = 4 \neq 3 = \text{Tr}(CBA) \).
Problem 5.c
False. Let $A = (1)$ a $1 \times 1$ matrix. Then

$$\text{Tr} \ (A^2) = 1 \neq 0 = (\text{Tr} \ A)^2 - n \det A$$

Jan 2005 Advanced Calculus

Problem 1
We want to show that $(x_1 x_2^2 x_3^3 \cdot x_n^n)^{1/n^2} \to \sqrt{a}$. Hence it suffices to show that

$$\log (x_1 x_2^2 x_3^3 \cdot x_n^n)^{1/n^2} = \frac{1}{n^2} (\log x_1 + 2 \log x_2 + 3 \log x_3 + \cdots + n \log n) \to \frac{1}{2} \log a$$

Notice that

$$\frac{1}{n^2} (\log x_1 + 2 \log x_2 + 3 \log x_3 + \cdots + n \log n) = \frac{1}{n^2} \left( \frac{n}{2} + \frac{1}{2} \right) \sum_{i=1}^{n} \log x_i$$

Now I claim that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log x_i = \log a$$

Indeed since for all $\epsilon > 0$, $\exists N$ such that $\forall n > N$, we have

$$|\log x_n - \log a| < \epsilon$$

Hence

$$\left| \frac{1}{n} \sum_{i=1}^{n} \log x_i - \log n \right| = \left| \frac{\sum_{i=1}^{n} (\log x_i - \log a)}{n} \right|$$

$$= \left| \sum_{i=1}^{N} (\log x_i - \log a) + \sum_{i=N+1}^{n} (\log x_i - \log a) \right| \frac{1}{n}$$

Hence since $\sum_{i=1}^{N} (\log x_i - \log a) = K$ constant, we have

$$\leq \frac{K + \sum_{i=N+1}^{n} |\epsilon|}{n} = \frac{(n - N - 1) \epsilon}{n} \to \epsilon$$

as $n \to \infty$. Thus $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log x_i = \log a$. Therefore
\[
\frac{1}{n^2} \left( \frac{n}{2} + \frac{1}{2} \right) \sum_{i=1}^{n} \log x_i = \frac{1}{2n} \sum_{i=1}^{n} \log x_i + \frac{1}{2n^2} \sum_{i=1}^{n} \log x_i \to \log a \over 2 + 0
\]

**Problem 2.b**

Notice we have

\[
1 - \left( \frac{1}{2} + \frac{1}{3} \right) + \left( \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) - \left( \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) + \cdots
\]

\[
= \sum_{n=1}^{\infty} (-1)^n a_n \quad \text{where} \quad a_n = \sum_{k=1}^{n} \frac{1}{k + n(n-1)/2}
\]

Now I claim that for all \( n \), \( a_{n+1} < a_n \), ie

\[
\sum_{k=1}^{n+1} \frac{1}{k + n(n+1)/2} < \sum_{k=1}^{n} \frac{1}{k + n(n-1)/2}
\]

Thus it suffices to show that

\[
\sum_{k=1}^{n} \frac{2}{2k + n(n-1)} - \frac{2}{2k + n(n+1)} > \frac{2}{n^2 + 2n + 3}
\]

Notice we have

\[
\sum_{k=1}^{n} \frac{2}{2k + n(n-1)} - \frac{2}{2k + n(n+1)} = \sum_{k=1}^{n} \frac{4n}{4k^2 + 2kn^2 + n^4 - n^2} \geq \frac{4n^2}{4n^2 + 4n^3 + n^4 - n^2}
\]

\[
= \frac{4}{4 + 4n + n^2 - 1} = \frac{2}{n^2/2 + 2n + 3/2} > \frac{2}{n^2 + 2n + 3}
\]

Also notice that \( \lim_{n \to \infty} a_n = 0 \). Hence \( a_n \downarrow 0 \), and by the alternating series theorem, the series does converge.

**Problem 3.a**

TRUE. The sequence is Cauchy, and hence it converges.

**Problem 3.b**

TRUE. Since \( f \) is continuous and unbounded on \( \mathbb{R} \), \( \exists x_0 \) such that \( \forall x \geq x_0 \), we have \( f(x) \geq f(x_0) \). Thus

\[
\int_{x_0}^{\infty} f(x)dx \geq \int_{x_0}^{\infty} f(x_0)dx = \int_{x_0}^{\infty} f(x_0)dx = \infty f(x_0)
\]

Hence it diverges since \( f(x_0) > 0 \).
Problem 3.c
FALSE. let us define
\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & \text{else} \end{cases} \]
Clearly \( f_x \) and \( f_y \) exists for all \( x \) and \( y \), and hence \( \nabla f \) exists everywhere. However the function is not continuous at \((0, 0)\). Notice as we approach \((0, 0)\) along the \( y = x \) line, we have
\[
\lim_{x \to 0} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \neq 0
\]

Problem 4.a
Recall that
\[
\int \int_S H(x, y, z) dS = \int \int_\Omega H(x(u, v), y(u, v), z(u, v)) \cdot ||N(u, v)|| dudv
\]
where
\[
N(u, v) = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \end{vmatrix}
\]
So for \( x = u \cos v, y = u \sin v \) and \( z = v \) we have
\[
||N(u, v)|| = \sqrt{1 - u^2}
\]
So by simple \( u \)- substitution
\[
\int \int_S H(x, y, z) dS = \int_0^{2\pi} \int_0^1 u \sqrt{1 + u^2} dudv = \frac{2\pi}{3}
\]

Problem 5
Using Lagrange, we have \( \nabla f = \lambda \nabla g \), and so
\[
(a_1, a_2, ..., a_n) = \lambda (2\mu_1 x_1, 2\mu_2 x_2, ..., 2\mu_n x_n)
\]
which implies
\[
\frac{a_1}{\mu_1 x_1} = \frac{a_2}{\mu_2 x_2} = \cdots = \frac{a_n}{\mu_n x_n}
\]
Hence for \( i \leftarrow 1 \) to \( n \), we have
\[
x_i = \frac{a_i \mu_1 x_1}{a_1 \mu_i}
\]
thus
Thus solving for $x_1$ we have

$$\mu_1 x_1^2 = \left( \frac{a_i \mu_1 x_1}{a_1 \mu_i} \right)^2 = \frac{a_i^2 \mu_i^2 x_1^2}{a_1^2 \mu_i}$$

$$\mu_2 x_2 + \frac{a_2^2 \mu_2^2 x_2^2}{a_1^2 \mu_2} + \cdots + \frac{a_n^2 \mu_2^2 x_n^2}{a_1^2 \mu_n} = \sum_{k=1}^{n} \frac{a_k^2 \mu_1^2 x_1^2}{a_1^2 \mu_k} = 1$$

Hence

$$x_1^2 = \frac{1}{\sqrt{\sum_{k=1}^{n} \frac{a_k^2 \mu_1^2 x_1^2}{a_1^2 \mu_k}}} \Rightarrow x_1 = \frac{\pm 1}{\sqrt{\sum_{k=1}^{n} \frac{a_k^2 \mu_1^2 x_1^2}{a_1^2 \mu_k}}}$$

Hence following the exact same method, we have for $j \leftarrow 1$ to $n$

$$x_j = \frac{\pm 1}{\sqrt{\sum_{k=1}^{n} \frac{a_k^2 \mu_j^2 x_j^2}{a_j^2 \mu_k}}}$$

Hence the maximum value of $f$ is

$$f_{\text{max}} = \sum_{j=1}^{n} a_j \left| \frac{1}{\sqrt{\sum_{k=1}^{n} \frac{a_k^2 \mu_j^2 x_j^2}{a_j^2 \mu_k}}} \right|$$

and the minimum value of

$$f_{\text{min}} = -\sum_{j=1}^{n} a_j \left| \frac{1}{\sqrt{\sum_{k=1}^{n} \frac{a_k^2 \mu_j^2 x_j^2}{a_j^2 \mu_k}}} \right|$$

Jan 2005 Complex Variables

Problem 1

Here we are going to map the triangle in the RHP to the triangle in the UHP with vertices $-1, 1, i\sqrt{3}$. Then with Schwartz-Christoffel, we’ll map that to the UHP, then to the RHP. We will do this by computing the inverse, and then taking it’s inverse. So from RHP to UHP we have $w = i z$. Then from UHP to $\Delta(-1, 1, i\sqrt{3})$, we have

$$g'(w) = A(w-1)^{-2/3}(w+1)^{2/3}$$

which implies
\[ g(w) = A \int_1^w \frac{1}{z^2 - 1} \, dz + B \]

with
\[ g(1) = 1 = 0 + B \]

and
\[ g(\infty) = i\sqrt{3} = A \int_1^\infty \frac{1}{(z^2 - 1)^{2/3}} \, dz + 1 \]

So let
\[ \beta = \int_1^\infty \frac{1}{(z^2 - 1)^{2/3}} \, dz \]

and we have
\[ A = \frac{i\sqrt{3} - 1}{\beta} \]

Hence
\[ g(w) = \frac{i\sqrt{3} - 1}{\beta} \int_1^w \frac{1}{(z^2 - 1)^{2/3}} \, dz + 1 \]

So let \( f(z) = g^{-1}(w) \). Then \( \phi(z) = -if(iz) \). Then we have our \( \phi \).

**Problem 2**

Let
\[ f_n(z) = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} \]

Then clearly \( f_n \to e^z - 1 \) pointwise. Let \( D_1 = \{ z : |z| \leq 2\pi + \epsilon \} \) and \( D_2 = \{ z : |z| \leq 1\pi - \epsilon \} \) for \( \epsilon > 0 \). Then \( f_n(z) \) converges uniformly on \( D_1 \) (and thus \( D_2 \)) since for
\[ f_n(z) = \sum_{k=1}^{n} \frac{z^k}{k!} = \sum_{k=1}^{n} u_k(z) \]

and \( |u_k(z)| \leq (2\pi + \epsilon)^k/k! \). Notice
\[ \sum_{k=1}^{\infty} \frac{(2\pi + \epsilon)^k}{k!} < \infty \]

since by the ratio test we have
\[ \lim_{k \to \infty} \frac{(2\pi + \epsilon)^{k+1}}{(k+1)!} \frac{k!}{(2\pi + \epsilon)^k} = \frac{2\pi + \epsilon}{k + 1} \to 0 \]
Thus \( f_n \to e^z - 1 \) uniformly on \( D_1 \) by the Weierstrass \( M \) test. So

\[
\lim_{n \to \infty} \int_{|z|=2\pi+\epsilon} \frac{f'_n(z)}{f_n(z)} \, dz = \# \text{Zeros} - \# \text{Poles}
\]

Since we have uniform convergence, we have \( f'_n(z) = e^z \). Therefore

\[
\lim_{n \to \infty} \int_{|z|=2\pi+\epsilon} \frac{e^z}{e^z - 1} \, dz = \text{Res}(f,0) + \text{Res}(f,2\pi i) + \text{Res}(f,-2\pi i) = 1 + 1 + 1 = 3
\]

Since \( f_n(z) \) has no poles inside \( D_1 \), this implies that there are 3 zeros. Likewise notice

\[
\lim_{n \to \infty} \int_{|z|=2\pi-\epsilon} e^z \frac{e^z - 1}{dz} = \text{Res}(f,0) = 1
\]

Therefore we know there is a zero at \( z_{1,n} = 0 \), and so \( |z_{2,n}|, |z_{3,n}| \to 2\pi \).

**Problem 3**

Let \( g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \). Then for \( |z| < 1 \) notice

\[
\sum_{n=1}^{\infty} \frac{|z|^n}{n!} < \infty
\]

which implies \( g \) converges for \( |z| < 1 \). Also for \( |z| = 1 \) we have

\[
|g(z)| \leq \sum_{n=1}^{\infty} \frac{1}{n!} < \infty
\]

Hence \( g \) converges on \( |z| = 1 \). Finally notice for \( |z| > 1 \). Then let \( z = re^{i\theta} \) and

\[
\lim_{n \to \infty} \frac{z^n}{n!} = \lim_{n \to \infty} \frac{r^n}{n!} e^{i\theta n!} \not\to 0
\]

Hence \( g \) does not converge outside the unit circle. Now I claim that \( |z| = 1 \) is the natural boundary of \( g \), and thus cannot be continued analytically beyond \( |z| = 1 \). Notice for

\[
f(z) = \sum_{n=1}^{\infty} z^n
\]

we have \( zg'(z) = f(z) \). Hence it suffices to show that \( |z| = 1 \) is the natural boundary of \( f \). Recall the ration test theorem in Levinson: For \( \sum_{j=1}^{\infty} a_j(z) \), In a region \( G \), let \( |a_N(z)| \) be bounded for some fixed \( N \geq 1 \), and for \( n > N \)

\[
\frac{|a_{n+1}(z)|}{|a_n(z)|} \leq R < 1
\]

where \( R \) is constant. Then the series converges in \( G \). Thus for \( P(z) = \sum_{k=1}^{\infty} z^{k!} \), we have

\[
\left| \frac{z^{(k+1)!}}{z^{k!}} \right| < 1
\]

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Thus for arbitrary large $n$, clearly the ratio above is less than one when $|z| < 1$. Hence $P(z)$ converges for $|z| < 1$. Now I claim that every point on $|z| = 1$ is a singular point and thus there is not an analytic continuation beyond the unit circle. Assume that $\exists\ an\ \alpha$ on the unit circle such that $f$ is not singular. Then $f$ can be continued analytically in $\Delta(\alpha, \epsilon)$ for some small $\epsilon > 0$. Thus $\exists$ an arc of length $\delta$ on the unit circle inside $\Delta(\alpha, \epsilon)$, such that $f$ is analytic on. Now let $q > 2\pi/\delta$, and let $\beta = e^{2\pi p/q}$, such that $\beta$ is on the arc. Thus $\beta^q = 1$ and for $n \geq q$ we have $\beta^{n!} = 1$. Thus

$$f(r\beta) = \sum_{n=1}^{q-1} r^{n!} \beta^n + \sum_{q}^{\infty} r^{n!}$$

which implies

$$|f(r\beta)| \geq \sum_{q}^{\infty} r^{n!} - (q - 1)$$

and thus it approaches $\infty$ as $r \to 1^-$. Hence $f$ cannot be analytic on $\beta$, and thus there is no analytic continuation beyond the unit circle. (Levinson p.410)

**Problem 4**

Let $\gamma$ be the contour in the upper half plane with radius $R$. Then let $f(z) = e^{iz}/(1 + z^4)$. Then by the residue theorem, we have

$$\int_{\gamma} f(z) dz = 2\pi i \left( \text{Res}(f, e^{i\pi/4}) + \text{Res}(f, e^{i3\pi/4}) \right)$$

Now notice on $C_R$, we have $z = Re^{i\theta}$. Hence

$$\left| \int_{C_R} f(z) dz \right| \leq \int_{0}^{\pi} \left| \frac{e^{iR(\cos\theta + i\sin\theta)} R e^{i\theta}}{R^2 - 1} \right| d\theta \leq \frac{R}{R^2 - 1} \int_{0}^{\pi} e^{-R\sin\theta} d\theta \leq \frac{R}{R^2 - 1} \int_{0}^{\pi} e^{-2R\theta/\pi} d\theta = \frac{R}{R^2 - 1} (e^{-2R-1}) \to 0$$
as $R \to \infty$. Hence

$$\int_{0}^{\infty} f(z) dz = 2\pi i \left( \text{Res}(f, e^{i\pi/4}) + \text{Res}(f, e^{i3\pi/4}) \right)$$

Now

$$\text{Res}(f, e^{i\pi/4}) = \frac{e^{i(\sqrt{2}/2 + i\sqrt{2}/2)}}{4e^{3i\pi/4}} = \frac{1}{4} e^{-\sqrt{2}/2} \left( \cos(\sqrt{2}/2 - 3\pi/4) + i \sin(\sqrt{2}/2 - 3\pi/4) \right)$$

and

$$\text{Res}(f, e^{i3\pi/4}) = \frac{e^{i(-\sqrt{2}/2 + i\sqrt{2}/2)}}{4e^{9i\pi/4}} = \frac{1}{4} e^{-\sqrt{2}/2} \left( \cos(-\sqrt{2}/2 + \pi/4) + i \sin(-\sqrt{2}/2 - \pi/4) \right)$$

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Hence
\[
\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4} \, dx = \frac{\pi e^{-\sqrt{2}/2}}{2} (-\sin(\sqrt{2}/2 - 3\pi/4)) - \frac{\pi e^{-\sqrt{2}/2}}{2} \sin(-\sqrt{2}/2 - \pi/4)
\]

Jan 2005 Linear Algebra

Problem 1

A is the orthogonal projection on the \(x + y + z = 0\) plane. Then let

\[
P = \begin{pmatrix}
1 & 1 \\
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

Then \(A = P(P^TP)^{-1}P^T\), and so

\[
A = \frac{1}{3} \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
\]

Now to find \(be\), we have the unit perpendicular vertex \(u = (1/\sqrt{3})(1,1,1)^T\). So

\[
B = I - 2uu^T = \begin{pmatrix}
1/3 & -2/3 & -2/3 \\
-2/3 & 1/3 & -2/3 \\
-2/3 & -2/3 & 1/3
\end{pmatrix}
\]

Now if we want to rotate a vector through an angle of \(\pi/2\) about the line \(x = y = z\). This line is the vector \((1,1,1)\). Now if we want to rotate a vector through an angle of \(\theta\) about the \(z\) axis, then we have

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

So if we want to rotate a vector though an angle of \(\pi/2\) about the \(z\) axis, we have

\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Now we need to change basis to \((1,-1,0),(1/2,1/2,-1),(1,1,1)\). Then we would have the rotation of \(\pi/2\) about the line \((1,1,1)\). Hence we need

\[
P = \begin{pmatrix}
1 & 1/2 & 1 \\
-1 & 1/2 & 1 \\
0 & -1 & 1
\end{pmatrix}
\]

which implies

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\[ P^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \]

So

\[ C = P \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1/4 & -1/4 & 1 \\ 11/12 & 5/12 & -1/3 \\ -1/6 & 5/6 & 1/3 \end{pmatrix} \]

Since \( A \) is projection, and \( B \) is a reflection, we have \( A^2 = A \) and \( B^2 = I \). Clearly \( C^4 = I \). Also \( AB - BA = AC - CA = BC - CB = 0 \).