Reconstructing volatility

Options on stock baskets have become a mainstay of the equity derivatives business, but pricing and hedging of such products is highly sensitive to implied volatility and correlation assumptions. Here, Marco Avellaneda, Dash Boyer-Olson, Jérôme Busca and Peter Friz address this vital problem, presenting a new approach to basket option valuation from underlying volatilities using the method of steepest descents and most-likely price configurations

Quantitative modelling in finance and option pricing theory focus mostly on models with few risk factors. This shortcoming is apparent when considering the subject of this article: the pricing of options on baskets of 20–100 stocks in relation to the values of the options on the individual stocks. Despite the theoretical and practical importance of the question, few techniques are available in the current literature (for a review, see Roelfsema, 2000). Due to the large number of stocks involved in a typical exchange-traded fund (ETF) or index, this problem lies beyond the scope of partial differential equation techniques and usually requires Monte Carlo simulation or approximate formulas. In this article, we analyse the valuation of equity index options in relation to the volatilities of the individual stocks using the method of steepest descent for diffusion kernels (see Varadhan, 1967, Azencott, 1984, and Bismut, 1984). In this approximation, the calculation of certain conditional expectations – a key step needed to characterise the local volatility function of the index – is replaced by the evaluation of a function at the most likely configuration of individual stock prices associated with a given change in index level. This procedure is based on mathematically rigorous asymptotics. It provides, in our opinion, powerful new insight on valuation of index products. Furthermore, it gives excellent results in terms of matching market quotes by 'reconstructing' the implied volatility of an option on a basket using information on the underlying stocks and their options.

We consider a basket of n stocks described by their price processes $S_i = S_i(t)$, $i = 1, \dots, n$ and an index or ETF on these stocks that consists of w_i shares of the i^{th} stock. The price of the index is:

$$B = \sum_{i=1}^{n} w_i S_i$$

with the w_i's constant.

Adopting a standard one-factor model for pricing options on stocks in the presence of a volatility skew (see Rubinstein, 1994, Dupire, 1994, and Derman & Kani, 1994), we assume a risk-neutral diffusion measure for each component:

$$\frac{dS_i}{S_i} = \sigma_i\left(S_i, t\right) dZ_i + \mu_i dt$$

where $\sigma_i(S_i, t)$ is a local volatility function associated with the i^{th} stock and μ_i is the drift associated with the cost of carry. We assume $Z_i = Z_i(t)$ are standard Brownian motions that satisfy:

$$\mathbb{E}\left(dZ_i dZ_j\right) = \rho_{ij} dt$$

where ρ_{ij} is given. For simplicity, we assume that ρ_{ij} is constant, since this is the most likely situation when correlations are estimated using historical data.[1]

An important element of our analysis is the stochastic volatility function associated with the index, that is, $\sigma_B = \sigma_B(\mathbf{S}, t)$, $\mathbf{S} = (S_1, \dots, S_n)$, which is given by:

$$\sigma_B^2 = \frac{1}{B^2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j w_i w_j S_i S_j$$

Notice that this expression depends on calendar time and the individual stock prices, and not just on the index level. A local volatility function for the index (that is, one that depends only on the price of the index and time) can be obtained by calculating the expectation of σ_B^2 conditional on the value of the index. More precisely, the function $\sigma_{B,\,loc} = \sigma_{B,\,loc}(B, t)$, defined as:

$$\sigma_{B,loc}^2 = \mathbb{E}\left\{\sigma_B^2 \middle| B(t) = B\right\}$$
$$= \mathbb{E}\left\{\frac{1}{B^2}\sum_{ij=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j S_i(t) S_j(t) \middle| \sum_{i=1}^{n} w_i S_i(t) = B\right\}$$

is such that the one-dimensional diffusion process:

$$\frac{dB}{B} = \sigma_{B,loc}\left(B,t\right) dW + \mu_B dt$$

(with μ_B representing the cost-of-carry of the ETF), returns the same prices for European-style index options as the n-dimensional model based on the dynamics for the entire basket.

To see this, we observe that $\sigma_B(\mathbf{S}, t)$ can be viewed as a stochastic volatility process that drives the index price $B(t)$, with the vector of individual stock prices $\mathbf{S}$ playing the role of ancillary risk factors. The above formula for $\sigma_{B,\,loc}^2$ expresses a well-known correspondence between the stochastic volatility of a pricing model and its corresponding (Dupire-type) local volatility (see Derman, Kani & Kamal, 1997, Britten-Jones & Neuberger, 2000, Gatheral, 2001, and Lim, 2002).

The problem, of course, is that the conditional expectation is difficult to calculate. To provide a tractable formula for $\sigma_{B,\,loc}$, we rewrite the latter equation formally as:

$$\sigma_{B,loc}^2 = \frac{\mathbb{E}\left\{\sigma_B^2 \delta\left(B(t) - B\right)\right\}}{\mathbb{E}\left\{\delta\left(B(t) - B\right)\right\}} \qquad (1)$$

where $\delta(x)$ is the Dirac delta function, and use asymptotic analysis to determine an approximate value for $\sigma_{B,\,loc}^2$ in the limit $\bar{\sigma}^2 t << 1$.[2]

The main mathematical tool for carrying out this calculation is Varadhan's formula and the so-called method of steepest descent (see Varadhan, 1967, Azencott, 1984, and Bismut, 1984). We introduce the change of variables:

$$x^i = \ln \frac{S_i}{S_i(0) e^{\mu_i t}}$$

and introduce the diffusion matrix of the process $\mathbf{x} = (x^1, \dots, x^n)$, $a^{ij} = \sigma_i \sigma_j \rho_{ij}$. Notice that $F_i = S_i(0) e^{\mu_i t}$ is the forward price of the i^{th} stock, for delivery

[1] The results presented here apply to more general correlation/volatility structures, including, for instance, the case of multivariate stochastic volatility/stochastic correlation models

[2] Here $\bar{\sigma}$ denotes a characteristic volatility level associated with the equity basket. For example, in the case of the Dow Jones Industrial Average (DJX), the index volatility in 2001 ranged between about 20% and 25%. Taking $\bar{\sigma} = 0.20$ as the typical level, we find that a six-month option has $(\bar{\sigma})^2 t \simeq 0.02$. This regime produces distributions that have very low variance in dimensionless units

at time t. We consider the inverse of a^{ij}, which we denote by g^{ij}, and the associated Riemmanian metric:

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j \tag{2}$$

Let $\pi(\mathbf{x}_0, t_0; \mathbf{x}, t)$ denote the probability density function associated with the process $(x^1, \ldots, x^n)$, that is:

$$\pi(\mathbf{x}_0, t_0; \mathbf{x}, t) dx = \mathbb{P}\left\{\mathbf{x}(t) \in \mathbf{B}(x; |dx|) \middle| \mathbf{x}(t_0) = \mathbf{x}_0\right\}$$

where $\mathbf{B}(\mathbf{x}; r)$ is the Euclidean ball with centre x and radius r.

Varadhan's formula states that:

$$\pi(\mathbf{0}, 0; \mathbf{x}, t) \sim e^{-\frac{d^2(0,x)}{2t}} = e^{-\frac{(\bar{\sigma})^2 d^2(0,x)}{2(\bar{\sigma})^2 t}} \tag{3}$$

where:

$$d^2(\mathbf{0}, \mathbf{x}) = \inf_{\mathbf{x}(0)=0, \mathbf{x}(1)=\mathbf{x}} \int_0^1 \sum_{i,j=1}^{n} g_{ij}(\mathbf{x}(s), 0) \dot{x}^i \dot{x}^j ds \tag{4}$$

Here $\dot{\mathbf{x}}$ is the time-derivative of $\mathbf{x}$. The asymptotics in Varadhan's formula are understood in the sense that the ratio of the logarithms of the two terms tends to one as $\bar{\sigma}^2 t << 1$. The expression $d^2(\mathbf{0}, \mathbf{x})$ can be interpreted as the distance between the points $\mathbf{0}$ and $\mathbf{x}$ in the metric (2).

According to the method of steepest descent, the probability density function $\pi(\mathbf{0}, 0; \mathbf{x}, t)$ is strongly peaked near the points $\mathbf{x}$ where $d^2(\mathbf{0}, \mathbf{x})$ is minimal, for $\bar{\sigma}^2 t << 1$. Therefore, we can obtain an approximate expression for the ratio of expectations in (1) by setting:

$$\sigma_{B,loc}^2 \simeq \sum_{\mathbf{x}^* \in M} q(\mathbf{x}^*) \sigma_B^2(\mathbf{x}^*, t) \tag{5}$$

where M is the set of points on the hypersurface

$$\Gamma_B = \left\{\mathbf{x} : \sum_{i=1}^{n} w_i F_i e^{x_i^*} = B\right\}$$

that have the shortest distance to the origin in the metric d.[3] The numbers $q(\mathbf{x}^*)$ are positive weights that satisfy:

$$\sum_{\mathbf{x}^* \in M} q(\mathbf{x}^*) = 1$$

They are proportional to the curvature of the metric at the minimiser. Generically, that is, barring symmetries and isolated points, there is a unique minimiser $\mathbf{x}^*$ with $q(\mathbf{x}^*) = 1$.

Formula (5) expresses a relation between the local volatility of the index, the correlation matrix and the local volatilities of the component stocks. It admits a simple interpretation. Notice that the surfaces:

$$\Theta_\delta = \{\mathbf{x} : d(\mathbf{0}, \mathbf{x}) = \delta\}$$

correspond approximately to the level sets of the probability density function of the multivariate price process. The vector(s) $\mathbf{x}^*$ that produces the minimum value of the distance to the 'price manifold' Γ_B correspond therefore to the most probable vector(s) $\mathbf{S} = (S_1, \ldots, S_n)$ such that:

$$\sum_{i=1}^{n} w_i S_i = B$$

Thus, the method of steepest descent equates the local volatility of the index at a given level to its stochastic volatility evaluated at the most probable price configuration $\mathbf{S}$ conditional on reaching that level.

Calculating the most likely configuration

Let us concentrate on the case where the minimiser $\mathbf{x}^*$ is unique, since this is the generic case. Abusing somewhat the notations, we denote by $\sigma_i(x^i) = \sigma_i(F_i e^{x^i}, 0)$ and $\sigma_{B,loc}(\bar{x}) = \sigma_{B,loc}(B(0)e^{\bar{x}}, 0)$, $\bar{x} = \ln(B/(B(0)e^{\mu_B t}))$ the local volatilities of the underlying assets (respectively, the basket), at time to maturity 0, as a function of log-moneyness.

With these notations, (5) reads:

$$\sigma_{B,loc}^2 = \sum_{i,j=1}^{n} \rho_{ij} \sigma_i(x_i^*) \sigma_j(x_j^*) p_i(\mathbf{x}^*) p_j(\mathbf{x}^*) \tag{6}$$

where:

$$p_i(\mathbf{x}) = \frac{F_i e^{x_i} w_i}{\sum_{k=1}^{n} F_k e^{x_k} w_k} \tag{7}$$

represents the percentage of stock i represented in the index when $F_i = S_i e^{x_i}$.

To obtain useful formulas, we need to characterise $\mathbf{x}^*$. Notice that the metric satisfies:

$$\sum_{i,j} g_{ij}(\mathbf{x}, 0) \dot{x}^i \dot{x}^j = \sum_{i,j} (\rho^{-1})_{ij} \frac{\dot{x}^i}{\sigma_i(x^i)} \frac{\dot{x}^j}{\sigma_j(x^j)}$$

Introducing the change of variables:

$$y^i = \int_0^{x^i} \frac{du}{\sigma_i(u)}$$

we obtain from (4) the simple problem of calculus of variations:

$$\int_0^1 \sum_{i,j=1}^{n} (\rho^{-1})_{ij} \dot{y}^i \dot{y}^j ds = \text{minimum}$$

subject to the non-linear constraint

$$\sum_{i=1}^{n} w_i F_i e^{x^i(y^i)} = B \tag{8}$$

Given the simple structure of this problem, the solution is such that $\dot{\mathbf{y}}$ is constant. The Euler-Lagrange first-order conditions for a minimum can be expressed in the form:

$$\sum_{j=1}^{n} (\rho^{-1})_{ij} \dot{y}^j = \lambda w_i F_i e^{x^i(y^i)} \frac{\partial x^i(y^i)}{\partial y^i}$$
$$= \lambda w_i F_i e^{x^i(\mathbf{y}^i)} \sigma_i(x^i)$$

where λ is a Lagrange multiplier associated with the price constraint (8). At this point, it is convenient to recast the Lagrange multiplier λ as λ/B since it then becomes dimensionless. Using this redefinition and multiplying both sides of the equation by the correlation matrix ρ, we obtain:

$$\dot{y}^i = \lambda \frac{\sum_{j=1}^{n} w_j F_j e^{x^j(\mathbf{y}^j)} \rho_{ij} \sigma_j(x^j)}{B}$$
$$= \lambda \sum_{j=1}^{n} p_j(\mathbf{x}) \rho_{ij} \sigma_j(x^j)$$

Now, since $\dot{y}^i$ is constant, it is equal to its average over the interval (0, 1), so that $\dot{y}^i = y^i$.

We have established the following result: in the limit $\bar{\sigma}^2 t << 1$, the local volatility of the index is given by:

$$\sigma_{B,loc}^2(\bar{x}) = \sum_{i,j=1}^{n} \rho_{ij} \sigma_i(x_i^*) \sigma_j(x_j^*) p_i(\mathbf{x}^*) p_j(\mathbf{x}^*) \tag{9}$$

where p_i is defined in (7) and $\mathbf{x}^*$ is the solution of the non-linear system:

$$\begin{cases} \int_0^{x_i^*} \frac{du}{\sigma_i(u)} = \lambda \sum_{j=1}^{n} \rho_{ij} p_j(x^*) \sigma_j(x_j^*), & \forall i = 1, \ldots, n \\ \sum_{i=1}^{n} w_i F_i e^{x_i^*} = B \end{cases} \tag{10}$$

[3] *We committed a slight abuse of notation in (5), writing* $\sigma_B(\mathbf{x}^*, t)$ *instead of* $\sigma_B(\mathbf{S}^*, t)$

From local volatilities to Black-Scholes implied volatilities

We translate formulas (9) and (10) into a relation between the (Black-Scholes) implied volatilities of index options and those of options on individual stocks. For this, we take advantage of a recent result by Berestycki, Busca & Florent (2000, 2002), which states that the implied volatility is the 'harmonic mean' of the local volatility function in the steepest-descent approximation. More precisely, we have:

$$\sigma_B^I(\bar{x}) \simeq \left(\frac{1}{\bar{x}}\int_0^{\bar{x}} \frac{du}{\sigma_{B,loc}(u)}\right)^{-1} \quad (11)$$

and, conversely:

$$\sigma_i(x^i) \simeq \left(\frac{d}{dy}\left(\frac{y}{\sigma_i^I(y)}\right)\Bigg|_{y=x^i}\right)^{-1}, \quad \forall i = 1,\ldots,n \quad (12)$$

Equations (9) and (10), with (11) and (12), provide a direct link between the Black-Scholes implied volatilities of the index and the underlying assets.

To derive tractable formulas, we consider a linear approximation to the price constraint (8), namely:

$$\sum_{i=1}^n p_i(0)\left(1 + \lambda\sigma_i(0)\sum_{j=1}^n \rho_{ij} p_j(\mathbf{0})\sigma_j(0) - x^i\right) \simeq 1 \quad (13)$$

which is exact to first order in $|x_i|$. Noting that:

$$\bar{x} \simeq \sum_{i=1}^n p_k(0) x_k$$

this yields:

$$\lambda \simeq \frac{\bar{x}}{\sigma_B(0)^2} \quad (14)$$

and, from (10):

$$x_i^* \simeq \frac{\bar{x}}{\sigma_B(0)^2}\sum_{j=1}^n \rho_{ij} p_j(\mathbf{0})\sigma_i(0)\sigma_j(0) \quad (15)$$

for all $i = 1, \ldots, n$.

To simplify the relations (9), (10), (11) and (12), we use the fact that the harmonic-mean relation between implied and local volatilities (11) and (12) gives rise to the approximations:

$$\sigma_B^I(\bar{x}) \simeq \frac{\sigma_{B,loc}(\bar{x}) + \sigma_B(0)}{2} \quad (16)$$

and:

$$\sigma_i(x^i) \simeq 2\sigma_i^I(x^i) - \sigma_i^I(0), \quad \forall i = 1,\ldots,n \quad (17)$$

which, again, are valid to first order for $|x_i| << 1$. Relation (16) is referred sometimes as the '1/2-slope rule' (see Gatheral, 2001). Thus (9) reduces to:

$$\sigma_{B,loc} = \sqrt{\sum_{i,j=1}^n \rho_{ij} p_i(\mathbf{x}^*) p_j(\mathbf{x}^*)\left(2\sigma_i^I(x_i^*) - \sigma_i^I(0)\right)\left(2\sigma_j^I(x_j^*) - \sigma_j^I(0)\right)} \quad (18)$$

which, with (16) and (15), provides a convenient approximate link between implied volatilities. This approximation is exploited further in the next section.

The formula in terms of Black-Scholes deltas

In view of applications to arbitrage and market-making in index options, we formulate the correspondence between index volatilities and component volatilities more transparently. To do this, we express the most probable configuration for stock prices at time t in terms of log-moneyness normalised by volatility. The latter quantity is closely related to the Black-Scholes delta.

Notice that (with all functions evaluated at $s = 1$), the components of the vector $\mathbf{y}$* satisfy:

$$y^i = \frac{x^i}{\sigma_i^I(x)} = \frac{1}{\sigma_i^I(S^*, t)}\ln\frac{S^*}{F_i}$$

Recall also that the Black-Scholes formula for the delta of a call option with strike K is $\Delta = N(d_1)$, with $N(x) = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-y^2/2)dy$ where:

$$d_1 = \frac{1}{\sigma\sqrt{t}}\ln\frac{F}{K} + O\left(\sigma\sqrt{t}\right)$$

We conclude that the delta of a call option on the ith underlying stock with strike $S^* = F_i e^{x^*_i}$ is given, to leading order as $\sigma\sqrt{t}$, by:

$$\Delta_i \simeq N\left(-\frac{y^i}{\sqrt{t}}\right)$$

which provides an alternative interpretation for the vector $\mathbf{y}$*. A similar approximation holds for the delta of a call option with strike B. Using equation (14), we find that

$$\Delta_B \simeq N\left(-\frac{1}{\sigma_B^I\sqrt{t}}\ln\frac{B}{F_B}\right) = N\left(-\frac{\lambda\sigma_B^I}{\sqrt{t}}\right)$$

Applying the inverse-normal distribution function to both sides and using the first-order optimality conditions (15) for the most probable configuration, we conclude that:

$$N^{-1}(\Delta_i) \simeq N^{-1}(\Delta_B) \times \sum_{j=1}^n \rho_{ij}\left(\frac{\sigma_j^I}{\sigma_B^I}\right) p_j$$

or, finally:

$$\Delta_i \simeq N\left[N^{-1}(\Delta_B) \times \sum_{j=1}^n \rho_{ij}\left(\frac{\sigma_j^I}{\sigma_B^I}\right) p_j\right] \quad (19)$$

which gives a simple relation between the delta of an index call option and the deltas of the call options on the components that are used in the volatility reconstruction formula.

To shed light on equations (16), (17) and (18) using Black-Scholes deltas, we assume that the expression:

$$\sum_{j=1}^n \rho_{ij}\left(\frac{\sigma_j^I}{\sigma_B^I}\right) p_j$$

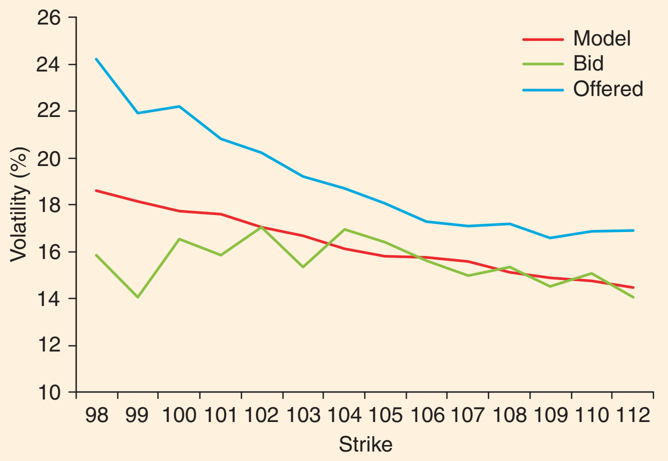
is evaluated using at-the-money implied volatilities. Equation (19) then gives, for a given value of Δ_B, the corresponding vector $(\Delta_1, \ldots, \Delta_n)$ of call-option deltas corresponding to different strikes $(S_*^1, \ldots, S_n^*)$ used in connection with the volatility skews of the components. In fact, the implied volatilities $(\sigma_1^I(S_1^*, t), \ldots, \sigma_n^I(S_n^*, t))$ can then be substituted directly into equations (16), (17) and (18) to generate a curve of implied volatilities $\sigma_B^I(B, t)$ for index options. This represents a useful shortcut for reconstructing the implied volatility skew of an ETF option, without having to solve problem (10).

Two limiting cases seem noteworthy. Consider first the case of perfectly correlated stocks. In this case, from (18), we have:

$$\sigma_B^I \simeq \sum_{j=1}^n p_j \sigma_j^I$$

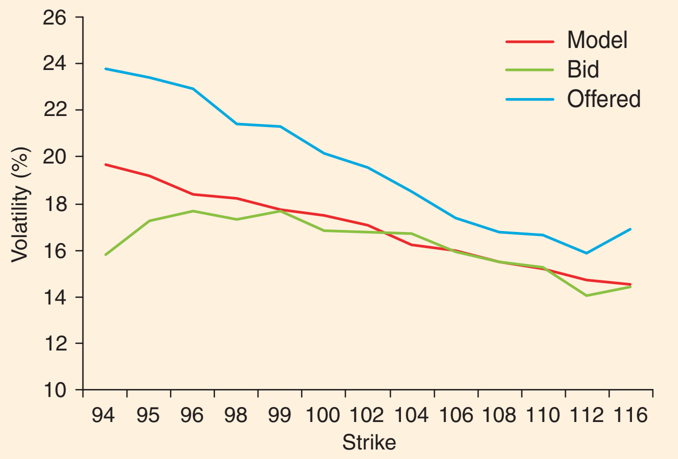
[4] *Since these calculations are exact to first order in $|x_i|$ for small values of the parameter, it follows that the above relation gives an exact formula for the 'slope' of the index volatility function at $|\bar{x}| = 0$*

1. DJX April 2002 index implied volatilities: model v. market



Market quotes are expressed in terms of implied volatilities (bid, offered) and compared with the predictions of the steepest-descent method using (7a) and (7b) and the approximation (8) for calculating the most probable configuration

2. DJX May 2002 index implied volatilities: model v. market



so our results imply that the most probable point corresponds to strikes that all have approximately the same delta as the index option. This equal-delta approximation for pricing index options in terms of the component volatility skews is apparently well known to professionals.[5] Our analysis suggests that an equal-delta approximation is indeed appropriate for ETFs that exhibit high correlation among the components, but also indicates that it may not be optimal for low-correlation indexes.

Consider next the (more unlikely) case where all stocks are uncorrelated. Equation (19) shows that the most probable price configuration corresponds to a set of deltas such that:

$$\Delta_i \simeq N\left(N^{-1}\left(\Delta_B\right)\times\left(\frac{\sigma_i^I}{\sigma_B^I}\right)p_i\right)$$
$$= N\left(N^{-1}\left(\Delta_B\right)\times\frac{p_i\sigma_i^I}{\sqrt{\sum_j p_j^2\left(\sigma_j^I\right)^2}}\right)$$

For example, assume that the index option is an out-of-the-money call with $\Delta_B < 0.5$. (In this case, $N^{-1}(\Delta_B\}) < 0$.) Since the fraction:

$$\frac{p_i\sigma_i^I}{\sqrt{\sum_j p_j^2\left(\sigma_j^I\right)^2}}$$

is less than unity, the deltas of the individual options associated with the steepest-descent approximation will be higher than Δ_B, that is, the relevant strikes are closer-to-the-money.

Finally, consider the case of an uncorrelated basket with a single, exceptionally volatile stock $\sigma_1^I >> \sigma_i^I$, $i \neq 1$. In this case, the most probable configuration will have $x_i \approx 0$, $i \geq 2$, corresponding to 50-delta implied volatilities $i \neq 1$ for the low-volatility stocks and a delta (or strike) for the high-volatility stock that coincides roughly with that of the index.

[5] *From a private communication with an options specialist at the American Stock Exchange*

Empirical results

We compared the results of the method of steepest descent with contemporaneous market quotes taken from two indexes traded on US markets on March 20, 2002.

We considered options on the Dow Jones Industrial Average (CBOE symbol: DJX) as well as options on the Merrill Lynch Biotech Holdr (Amex symbol: BBH). Two short-term expiries were considered: April (front month) and May. The experiment used historical estimates for correlations between the index components derived using one-year historical data. Implied volatilities for options on the individual stocks (bid/offered) were recorded simultaneously with the implied volatilities for the index. We used equation (19) to derive the deltas associated with different strikes and formulas (16) and (18) to calculate the theoretical implied volatilities of the index call options. In the reconstruction formula, we used mid-market implied volatilities for the components of the index. The calculations were then compared with the actual market quotes on index. The results presented in the four graphs are representative of other indexes that we analysed as well.

Conclusions

We derived a simple formula, based on the method of steepest descent, that links the local volatility function of an index with the local volatility functions for the index components and a given correlation matrix. The intuition behind the steepest-descent approximation is that, if the dimensionless time scale $(\bar{\sigma})^2 t$ is sufficiently small, the local volatility of the index should be determined from the most likely configuration of stock prices conditional on arriving at a given index level.

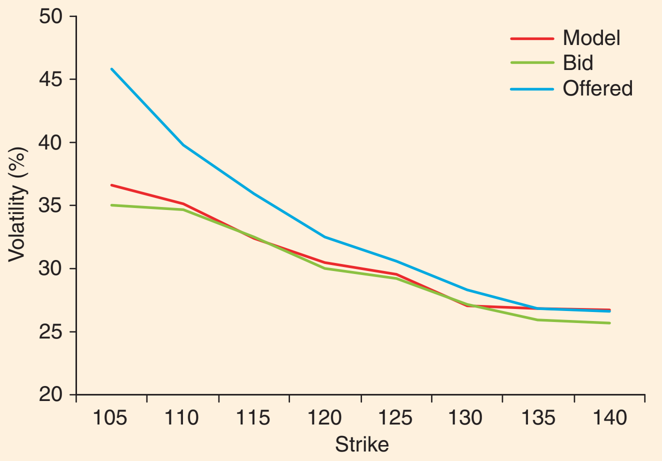
In a second step, we obtained an approximation that operates at the level of the implied volatilities, that is, of option prices. This approximation uses an estimate for the most likely configuration arising from the Euler-Lagrange equations with an asymptotic relation between local and implied volatilities derived by Berestycki, Busca & Florent (2000, 2002) (also in the limit $\bar{\sigma}^2 t << 1$).

Finally, we characterised the most likely price configuration to a vector of Black-Scholes deltas that determine which points on the volatility skews of the component stocks contribute the most to the index implied volatility. The resulting formula is useful because it can be tested directly on market data.

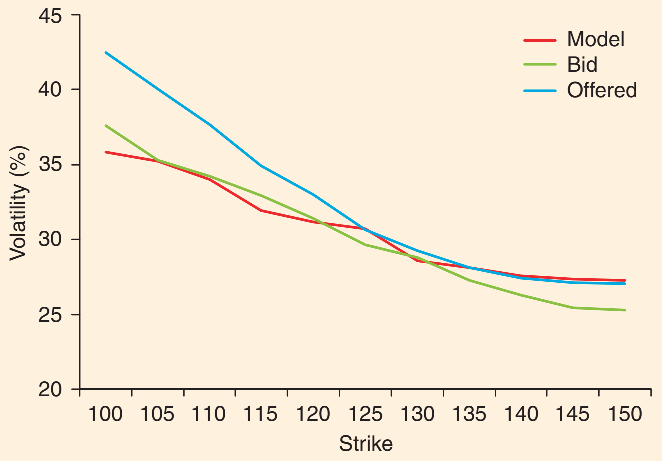
The predictions of the formula, which is based on historical correlations and the volatility skews of the components, are in good agreement with contemporaneous quotes for index options.

We note, however, that we have not undertaken at this point an extensive statistical study in this direction, leaving it for a future publication.

3. BBH April 2002 index implied volatilities: model v. market



4. BBH May 2002 index implied volatilities: model v. market



Despite observing good agreement with the data, discrepancies from market quotes may arise due to effects not contemplated in the model. First, we expect that the short-dimensionless-time asymptotics will break down, or at least be less accurate, for longer expiries. Discrepancies may also arise from our simple treatment of asset correlations.

In this regard, we believe that the steepest-descent model could be useful to analyse correlation 'risk premiums'. By this, we mean that observed differences between the shape of the actual index volatility curve and the one predicted by the steepest-descent approximation (using, say, historical correlations) can be attributed to expectations about future correlations that are dependent on the index level – that is, to a 'correlation skew' that might thus be observable through index option prices. We note also that empirical studies by Duffee (2001) indicate the presence of a larger idiosyncratic (or stock-specific) correlation of returns for upside market moves. It would be interesting to incorporate such effects in this pricing framework.

As a last remark, we point out that the method of steepest descent can be applied in principle to other asset-pricing models in which correlations and volatility skews play a significant role. Most notably, its application to Libor market models would allow one to derive local volatility models to price European-style swaptions in a model expressed in terms of forward Libor rates. ■

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