From SABR to Geodesics

A systematic approach for modeling volatility curves with applications to option market-making and pricing multi-asset equity derivatives

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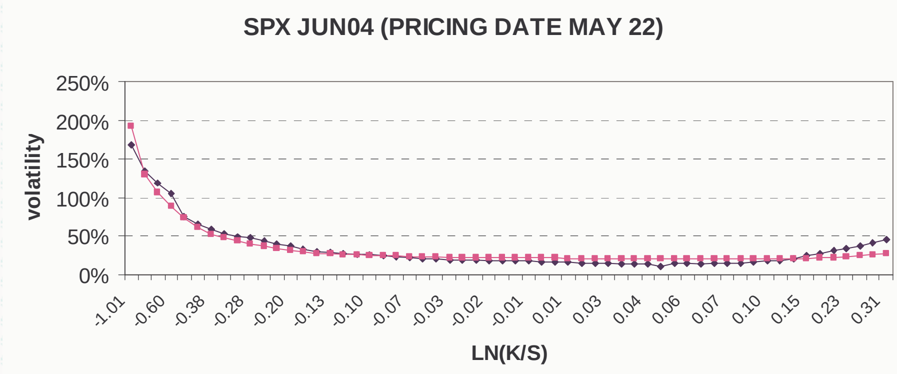
The Importance of Having a Listed Derivatives Historical Database

- Equity derivatives analysts can now access historical databases on listed options at relatively low cost. They can:
- Back-test models, especially calibration aspects
- Debunk myths about option models (there are plenty of them!)
- Back-test option strategies systematically, as is done for cash trading
- Test the stability of a skew and vol surface model with real data
- Learn more, by *observation*, get ideas…

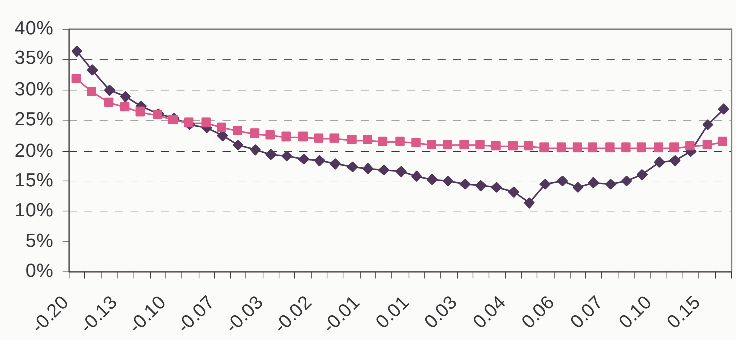
Recommendation: *IVY OptionMetrics*

Fitting Volatility Skews



Blue line= average implied vol (puts/calls)
Pink line= fitted parabola

Zoom into the region -0.2<ln(K/F)<0.2



If you zoom into the region of interest, the parabolic fit is seen as clearly inadequate

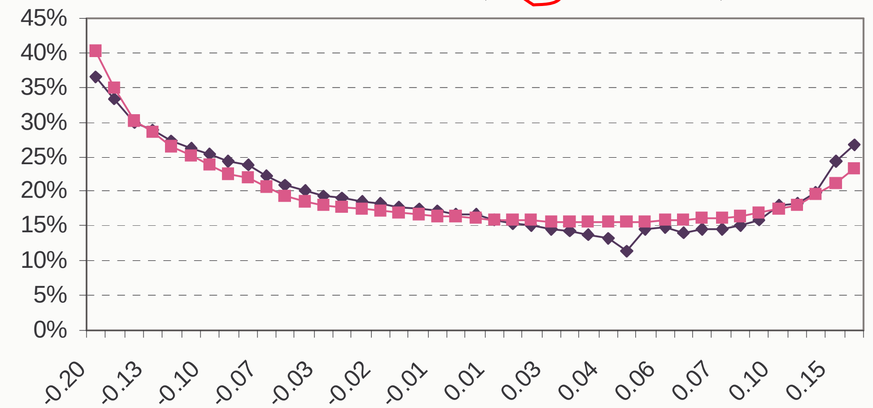
Reason: the out-of-the money options ``lift'' the curve

Parabolic fits are not consistent with arbitrage-free pricing

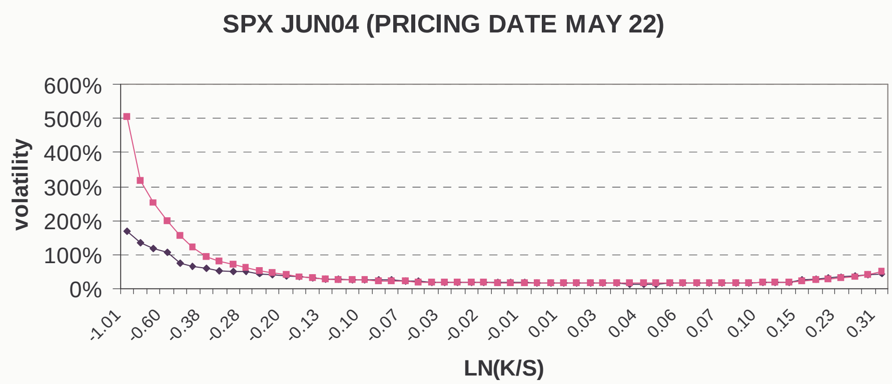
Parabolic fitting requires Delta Truncation!

Fit only volatilities such that -0.2<x<+0.2

$$\sigma_{\text{parabolic}}(x) = 0.16 - 0.34x + 4.45x^2$$
$$= 0.16 \times \left(1 - \underline{2.1}\, x + 27.33\, x^2\right)$$



Truncated Parabolic Fit: a look at the full curve



Out of the money options are not guaranteed to be well-fitted

Using a better spline to fit the data (from SABR)

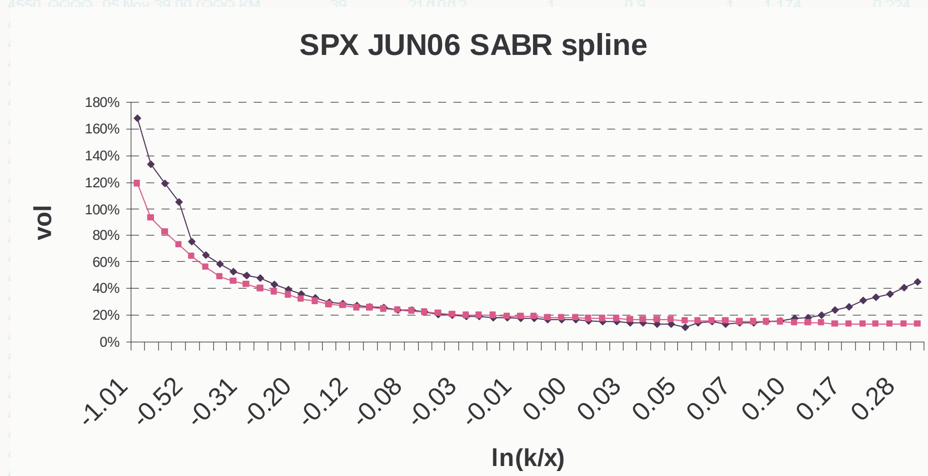
$$\sigma_{\text{imp}}(x) \approx \frac{\kappa |x|}{\ln\left(\kappa \left| \frac{1-e^{-\beta x}}{\sigma_0 \beta} \right| + \sqrt{1+\kappa^2 \left(\frac{1-e^{-\beta x}}{\sigma_0 \beta} \right)^2} \right)}$$

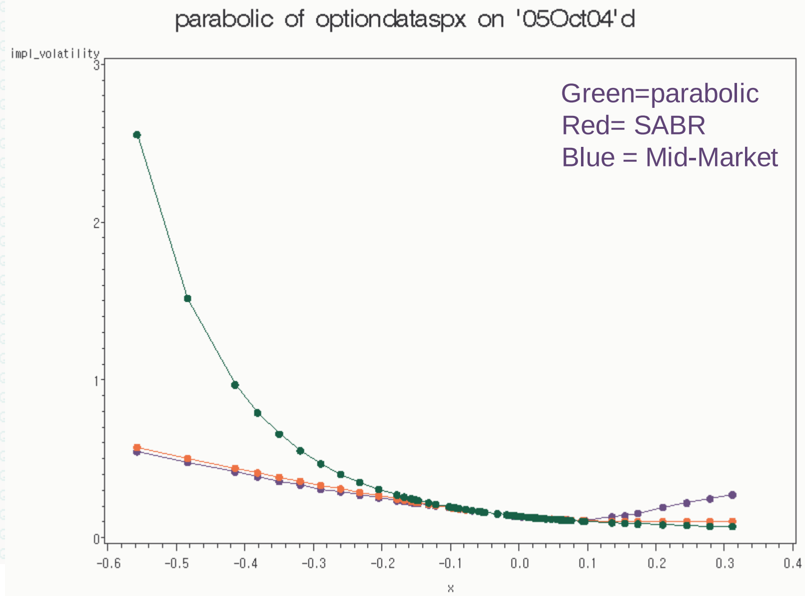
$$x = \ln\left(\frac{K}{F_0}\right) \qquad \gamma \equiv \text{slope}(x=0) = \frac{\beta}{2}$$

Sigma, beta and kappa are adjustable parameters

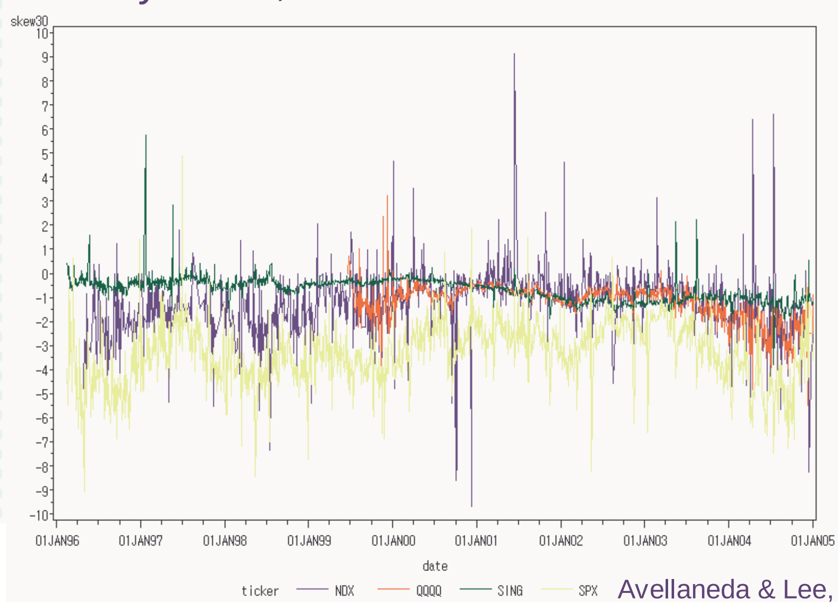
Formula is derived from a **stochastic volatility model** so it does not violate arbitrage conditions

Fitting a SABR-like spline to the SPX front-month curve



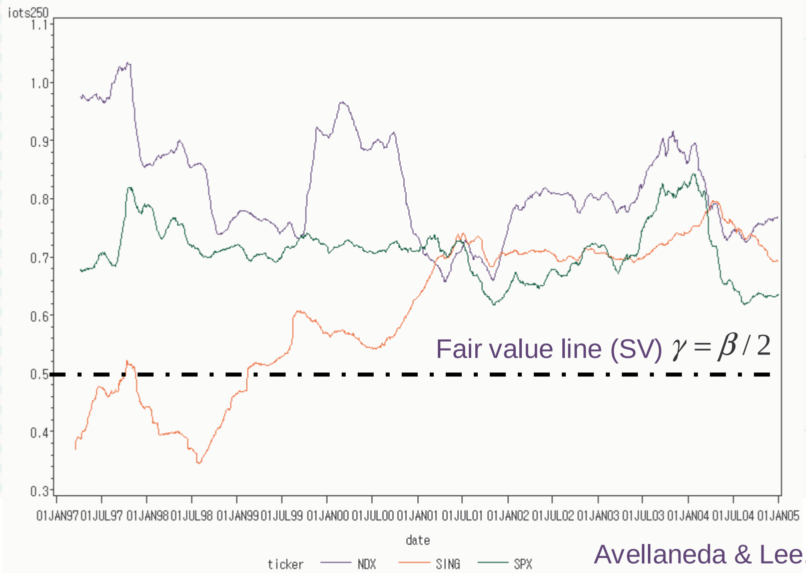


Evolution of the slope of the 30-day implied volatility curve, 1996-2004



Avellaneda & Lee, 2005

Evolution of ratio [slope/leverage coefficient] The ``roaring 90’s’’!



Avellaneda & Lee, 2005

Differential Geometry and Implied Volatility Modeling

Factor Models and Diffusion Kernels

CIR-type setting,
X= state variables
W= m-dim Brownian motion

$$\mathrm{x}(t) = (X_1(t),\ldots,X_n(t))$$
$$\mathrm{w}(t) = (W_1(t),\ldots,W_m(t))$$

$$dX_i = \sum_{k=1}^{m} \sigma_j^k dW_k + b_i dt, \qquad i = 1,2,3\ldots,n$$

$$\pi(\mathrm{x},t;\mathrm{y},T) = \mathrm{Prob.}\{\mathrm{x}(T) = \mathrm{y} \mid \mathrm{x}(t) = \mathrm{x}\}$$

Diffusion kernel

$$E\{F(\mathrm{x}(T)) \mid \mathrm{x}(t) = \mathrm{x}\} = \int_{\mathrm{y}\in R^n} F(\mathrm{y})\pi(\mathrm{x},t;\mathrm{y},T)d^n\mathrm{y}$$

Fokker-Planck Equation and Dimensionless Time

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sum_{ij=1}^{n} a_{ij}\frac{\partial^2 \pi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \pi}{\partial x_i} = 0$$

$$\pi(\mathrm{x},T;\mathrm{y},T) = \delta(\mathrm{x}-\mathrm{y})$$

$$a_{ij} = \sum_{k=1}^{m} \sigma_i^k \sigma_j^k$$

Covariance matrix of state variables

$$(\bar{\sigma})^2 \equiv E\left\{\frac{1}{n}\sum_{i=1}^{n} a_{ii}\right\}$$

$$\tau \equiv (\bar{\sigma})^2 t$$

volatility of S&P=0.15
t=1 yr. corresponds to tau=0.0225<<1

``typical variance'' of x

Dimensionless time

Varadhan Asymptotics for the Diffusion Kernel

$$\lim_{\tau \to 0} \tau \ln \pi(\mathrm{x},0;\mathrm{y},T) = -\frac{L^2(\mathrm{x},\mathrm{y})}{2}; \qquad \tau = (\bar{\sigma})^2 T,$$

$$L(\mathrm{x},\mathrm{y}) = \text{geodesic distance between x and y}$$

$$L(\mathrm{x},\mathrm{y}) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} \int_0^1 \left\| \frac{\mathrm{d}\gamma}{\mathrm{dt}} \right\|_\gamma dt,$$

$$\|v\|_{\mathrm{x}}^2 = \sum_{ij=1}^{n} g_{ij}(\mathrm{x}) v_i v_j$$

$$g_{ij} = (\bar{\sigma})^2 \left(\mathrm{a}^{-1}\right)_{ij}$$

Dimensionless Riemann tensor

Heuristically: Diffusion Kernels ``resemble'' Gaussian Kernels with |x-y| replaced by L(x,y)

$$\pi(x,0;y,T) \approx c(\tau) \cdot e^{-\frac{(L(x,y))^2}{2\tau}} \qquad \tau << 1$$

$$(dL)^2 = \sum_{ij=1}^{n} g_{ij}(\mathrm{x}) dx_i dx_j$$

We shall use this approximation to compute option prices and implied volatilities assuming tau is small

Example 1: Local volatility model

$$\frac{dF_t}{F_t} = \sigma(F_t,t)dW_t \qquad x = \ln\left(\frac{F}{F_0}\right)$$

$$dx_t = \sigma(x_t,t)dW_t + (\ldots)dt$$

$$(dL)^2 = \frac{\left(\bar{\sigma}\right)^2 dx^2}{(\sigma(x,0))^2} = \frac{dx^2}{\left(\frac{\sigma(x,0)}{\bar{\sigma}}\right)^2} = \left(\frac{dx}{\tilde{\sigma}(x)}\right)^2$$

$$L(x,y) = \left|\int_x^y \frac{du}{\tilde{\sigma}(u)}\right| = \left|G(y) - G(x)\right|$$

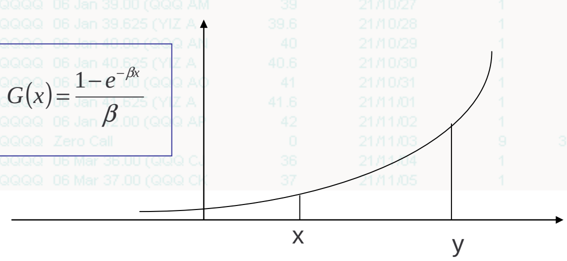
1-dimensional distances are always `trivial'

Special solvable 2-D case: the CEV Model

$$\sigma(F,t) = \bar{\sigma}\left(\frac{F}{F_0}\right)^{\beta} \quad \therefore \quad \sigma(x,t) = \bar{\sigma}e^{\beta x}$$

$$\tilde{\sigma}(x) = e^{\beta x} \qquad L(x,y) = \left|\frac{e^{-\beta x} - e^{-\beta y}}{\beta}\right|$$

$$G(x) = \frac{1 - e^{-\beta x}}{\beta}$$



Negative beta for Equities (leverage)

Distance= area under the curve

Stochastic Volatility Models

$$\frac{dF_t}{F_t} = \sigma_t dW_t$$ **Forward price**

$$\frac{d\sigma_t}{\sigma_t} = \kappa dZ_t$$ **Stochastic vol.**

$$E\{dW_t dZ_t\} = \rho dt \qquad \beta \equiv \frac{\kappa\rho}{\sigma}$$ **Leverage**

$$\frac{d\sigma_t}{\sigma_t} = \beta \frac{dF_t}{F_t} + \varepsilon$$

Beta= regression coefficient of vol on stock returns

Equivalent Model with Independent Brownian Motions (SABR)

$$\sigma_t = \sigma_t^{(0)} \exp(\beta x_t) \qquad x_t = \ln\left(\frac{F_t}{F_0}\right)$$

``Parametric leverage'' SV for tails

$$\frac{d\sigma_t}{\sigma_t} = \frac{d\sigma_t^{(0)}}{\sigma_t^{(0)}} + \beta dx_t$$

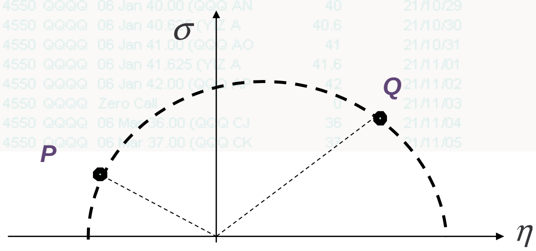
``CEV'' with stochastic <u>independent</u> volatility is equivalent to SV model with correlated volatility, from the Riemann viewpoint

$$\begin{cases} dx_t = e^{\beta x_t} \sigma_t^{(0)} dW_t \\ \dfrac{d\sigma_t^{(0)}}{\sigma_t^{(0)}} = \kappa dZ_t \\ E(dW_t dZ_t) = 0 \end{cases}$$

Riemann Metric for SV / SABR: The Poincare Upper Half-Space Model

$$\eta \equiv \kappa\left(\frac{1-e^{-\beta x}}{\beta}\right), \qquad \sigma \equiv \sigma^{(0)}$$

$$dL^2 = \frac{\bar{\sigma}^2}{\kappa^2} \cdot \frac{d\eta^2 + d\sigma^2}{\sigma^2}$$



Geodesics are half-circles with center on the horizontal axis

$$L(P,Q) = \frac{\bar{\sigma}}{\kappa}\left|\int_{\theta_P}^{\theta_Q} \frac{d\theta}{\sin\theta}\right|$$

Using the asymptotics to compute option prices

$$F_T = F(x_T) \qquad P_0 = F(0) < K$$

$$CALL = \int_{R^n} \max(F(y)-K,0)\,\pi(0,0;y,T)\,d^n y$$

$$\approx c\int_{R^n} \max(F(y)-K,0)\,e^{-\frac{L^2(0,y)}{2\tau}}\,d^n y$$

$$\approx c\int_{\{y:F(y)>K\}} (F(y)-K)\,e^{-\frac{L^2(0,y)}{2\tau}}\,d^n y$$

$$\approx c\int_{\{y:F(y)>K\}} e^{-\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right]}\,d^n y$$

Steepest-descent approximation for computing implied volatilities

$$\int_{\{y:F(y)>K\}} e^{-\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right]} d^n y \approx e^{-\min_{y:F(y)>K}\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right]}$$

$$\min_{y:F(y)>K}\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right] = \frac{1}{\tau}\min_{y:F(y)>K}\left[\tau\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2}\right]$$

$$\approx \frac{1}{2\tau}\min\left\{L^2(0,y)\,\middle|\,y:F(y)>K\right\}, \quad \tau << 1$$

Equate formulas for OTM calls with Black-Scholes …

$$L^*(K) = \min\left\{L(0,y)\,\middle|\,y:F(y)>K\right\}$$

Minimum distance from 0 to the region {F(y)>0}

$$\ln CALL \approx -\frac{\left(L^*(K)\right)^2}{2\tau}$$

Small-tau asymptotics **(model)**

$$\ln CALL \approx -\frac{\left(\ln(K/F_0)\right)^2}{2\sigma_{imp}^2(K)T} = -\frac{\left(\ln(K/F_0)\right)^2}{2\left(\frac{\sigma_{imp}^2(K)}{(\bar{\sigma})^2}\right)\tau}$$

Small-tau asymptotics **(Black-Scholes)**

Approximation for Implied Volatility for general diffusion model

$$\sigma_{\text{imp}}(K) = \bar{\sigma}\,\frac{|\ln(K/F_0)|}{\min\{L(0,\mathrm{y}) \mid \mathrm{y} : F(\mathrm{y}) > K\}}$$

$$= \frac{|\ln(K/F_0)|}{\min\{L_1(0,\mathrm{y}) \mid \mathrm{y} : F(\mathrm{y}) > K\}}$$

$$L_1(x,y) \equiv \min_{\substack{\gamma(0)=\mathrm{x}\\ \gamma(1)=\mathrm{y}}} \int_0^1 \sqrt{\sum_{ij=1}^{n} \left(\mathrm{a}^{-1}\right)_{ij} \dot{\gamma}_i(t)\dot{\gamma}_j(t)}\,dt$$

Example 1: Local Volatility Model

$$\sigma_{\text{imp}}(K) \approx \frac{\ln(K/F_0)}{\displaystyle\int_0^{\ln(K/F_0)} \frac{du}{\sigma(u,0)}} = \left(\frac{1}{x}\int_0^x \frac{du}{\sigma(u,0)}\right)^{-1}$$

$$x = \ln\left(\frac{K}{F_0}\right)$$

Implied Volatility= Harmonic Mean of Local Volatility

Berestycki, Busca and Florent, 2001

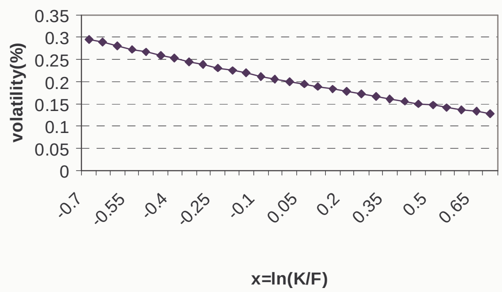
Example 2: Constant Elasticity of Variance

$$\sigma(x,t) = \sigma_0 e^{\beta x}$$

CEV

$$\sigma_{\text{imp}}(x) \approx \sigma_0 \left| \frac{\beta x}{1 - e^{-\beta x}} \right|$$

Implied volatility

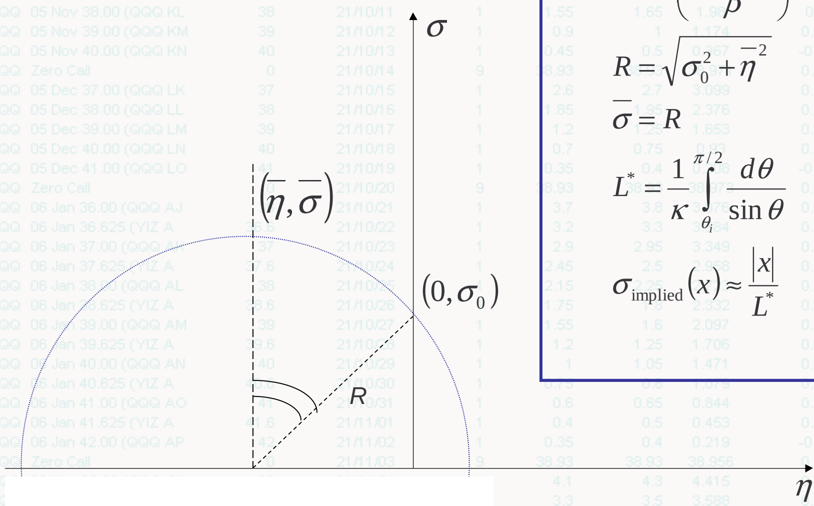


Example 3: Stochastic Volatility / SABR

$$\sigma_{\text{imp}}(x) \approx \frac{\kappa |x|}{\ln\left(\kappa \left| \frac{1 - e^{-\beta x}}{\sigma_0 \beta} \right| + \sqrt{1 + \kappa^2 \left(\frac{1 - e^{-\beta x}}{\sigma_0 \beta} \right)^2} \right)}$$

$$x = \ln\left(\frac{K}{F_0} \right)$$

Minimizing the distance to the line eta =const. in the Poincare plane



$$\bar{\eta} = \kappa\left(\frac{1-e^{-\beta x}}{\beta}\right)$$

$$R = \sqrt{\sigma_0^2 + \bar{\eta}^2}$$

$$\bar{\sigma} = R$$

$$L^* = \frac{1}{\kappa}\int_{\theta_i}^{\pi/2}\frac{d\theta}{\sin\theta}$$

$$\sigma_{\text{implied}}(x) \approx \frac{|x|}{L^*}$$

Example 4: the Heston Model
A variant of the Poincare Half-Space

$$\frac{dS_t}{S_t} = \sqrt{V_t}\,dW_t$$

$$dV_t = \kappa\sqrt{V_t}\,dZ_t \qquad E(dW_t dZ_t) = \rho dt$$

$$dL_1^2 = \frac{1}{\kappa^2}\frac{d\xi^2 + dV^2}{V} \qquad \xi = \frac{\kappa\left(1-e^{-\beta x}\right)}{\beta}$$

Note: V, <u>not V squared</u>

Closed-form solution for geodesics

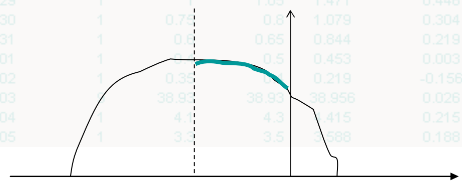
$$\xi = \kappa\left(\frac{1-e^{-\beta x}}{\beta}\right) \qquad dL^2 = \frac{d\xi^2 + dV^2}{\kappa^2 V}$$

$$\xi(\theta) = \frac{R^2}{2}\left(\theta - \sin\theta\cos\theta\right) + \xi(0)$$

$$V(\theta) = R^2 \sin^2\theta \qquad 0 \le \theta \le \pi$$

$$dL = \frac{2R^2}{\kappa}\sin\theta\, d\theta$$

Geodesics are cycloids



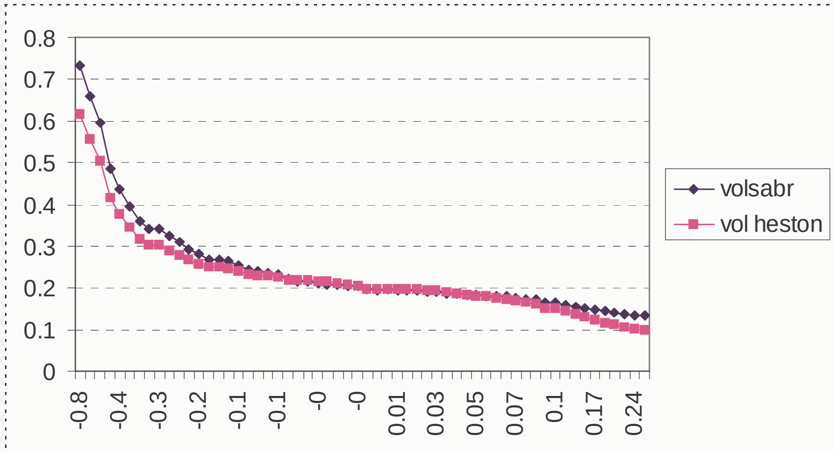
Implied volatility curve for Heston model is obtained as an algebraic system

$$\xi = \frac{\sigma_0^2}{\sin^2\theta_{\text{init}}}\left(\frac{\pi}{2} - \theta_{\text{init}} + \sin\theta_{\text{init}}\cos\theta_{\text{init}}\right)$$

$$\sigma(\xi) = \frac{\kappa|\xi|\sin^2\theta_{\text{init}}}{2\sigma_0^2\left|\cos\theta_{\text{init}}\right|}$$

Given xi, solve for theta_init, and substitute in the second equation

Auto-calibration of SABR and Heston



$$\sigma_0 = 20\% \qquad \kappa_{\text{sabr}} = 0.5$$

$$\beta = -4 \qquad \kappa_{\text{Heston}} = 2\sigma_0\kappa_{\text{sabr}} = 0.2$$

Multi-Asset Derivatives

Multi-Asset Derivatives: Index Options, Rainbows

Derive index volatility skew from **single-stock skews** and **correlation matrix**

$$dx_i = \sigma(x_i, t)dW_i, \qquad i = 1,2,\ldots,n$$

$$E(dW_i dW_j) = \rho_{ij} dt$$

N equations for the index components

$$I = \sum_{i=1}^{n} w_i S_i = \sum_{i=1}^{n} w_i S_i(0) e^{x_i} \qquad \bar{x} = \ln\left(\frac{F_I}{I(0)}\right)$$

BBH: ETF of 20 Biotechnology Stocks

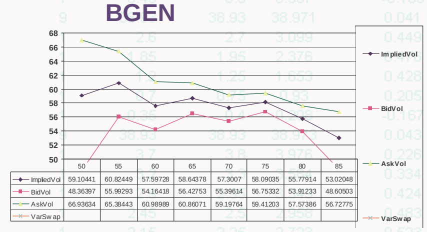
(Components of IBH)

Ticker	Shares	ATM ImVol	Ticker	Shares	ATM ImVol
ABI	18	55	GILD	8	46
AFFX	4	64	HGSI	8	84
ALKS	4	106	ICOS	4	64
AMGN	**46**	40	IDPH	12	72
BGEN	13	41	MEDI	15	82
CHIR	16	37	MLNM	12	92
CRA	4	55	QLTI	5	64
DNA	44	53.5	SEPR	6	84
ENZN	3	81	SHPGY	6.8271	47
GENZ	14	56	BBH	-	32

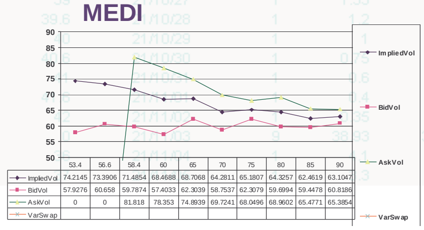
Implied Volatility Skews

Multiple Names, Multiple Expirations

	60	65	70	75	80	85	90	95
ImpliedVol	67.5042	66.93523	67.08418	64.42438	60.53124	57.80586	55.33041	55.29034
BidVol	63.23	64.55163	65.02824	62.43146	58.36119	56.02341	54.08097	51.39689
AskVol	71.59664	69.29996	69.1395	66.41618	62.68222	59.54988	56.54053	58.64633
VarSwap								



	50	55	60	65	70	75	80	85
ImpliedVol	59.10441	60.82449	57.59728	58.64378	57.3007	58.09035	55.77914	53.02048
BidVol	48.36397	55.99293	54.16418	56.42753	55.39614	56.75332	53.91233	48.60503
AskVol	66.93634	65.38443	60.98989	60.86071	59.19764	59.41203	57.57386	56.72775
VarSwap								



	53.4	56.6	58.4	60	65	70	75	80	85	90
ImpliedVol	74.2145	73.3906	71.4854	68.4688	68.7068	64.2811	65.1807	64.3257	62.4619	63.1047
BidVol	57.9276	60.658	59.7874	57.4033	62.3039	58.7537	62.3079	59.6994	59.4478	60.8186
AskVol	0	0	81.818	78.353	74.8939	69.7241	68.0496	68.9602	65.4771	65.3854
VarSwap										

BBH March 2003 Implied Vols

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What is the `fair value' of the index volatility reconstructed from the components?

Riemannian metric for the multi-D local vol model

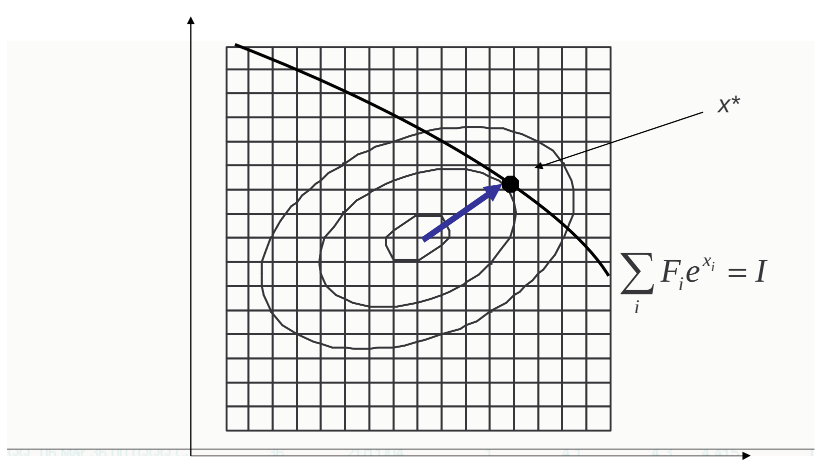
$$dL^2 = \sum_{ij=1}^{n} \left(\rho^{-1}\right)_{ij} \frac{dx_i}{\sigma(x_i,0)} \frac{dx_j}{\sigma(x_j,0)}$$

$$= \sum_{ij=1}^{n} \left(\rho^{-1}\right)_{ij} dy_i dy_j, \qquad dy_i \equiv \frac{dx_i}{\sigma(x_i,0)}$$

If correlations are constant, the metric is ``flat'': it is Euclidean metric after making the change of variables x->y.

Geodesics are straight lines in the y-coordinates

Steepest Descent=Most Likely Stock Price Configuration



Replace conditional distribution by “Dirac function” at most likely configuration

Exact solution: Euler-Lagrange Equations

$$\sigma_{\text{impl. }I}\left(\bar{x}\right)=\frac{\left|\bar{x}\right|}{\sqrt{\sum_{ij=1}^{n}\left(\rho^{-1}\right)_{ij}\int_{0}^{x_i^*}\frac{du}{\sigma_i(u,0)}\int_{0}^{x_j^*}\frac{du}{\sigma_j(u,0)}}}$$

$$=\frac{\left|\bar{x}\right|}{\sqrt{\sum_{ij=1}^{n}\left(\rho^{-1}\right)_{ij}\frac{x_i^*}{\sigma_{\text{impl.},i}\left(x_i^*\right)}\frac{x_j^*}{\sigma_{\text{impl.},j}\left(x_j^*\right)}}}$$

Euler - Lagrange equations

$$\int_{0}^{x_i^*}\frac{du}{\sigma_i(u,0)}=\Lambda\sum_{j=1}^{n}\rho_{ij}\,p_j\left(x_j^*\right)\sigma_j\left(x_j^*,0\right)\qquad i=1,2,\ldots n$$

Approximate solution: introduce the stock betas

$$x_i=\beta_i\,\bar{x}+\varepsilon_i \qquad\qquad x_i^*=\beta_i\,\bar{x}$$

Regression relation between stock and index returns

Approximate formula for the optimal stock configuration

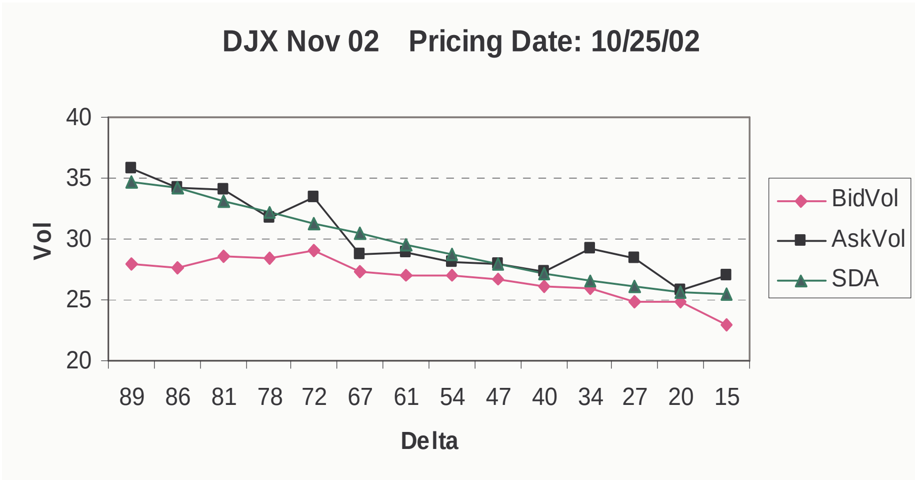
$$\frac{1}{\sigma_{\text{imp},I}\left(\bar{x}\right)}\approx\sqrt{\sum_{ij=1}^{n}\frac{\left(\rho^{-1}\right)_{ij}\beta_i\beta_j}{\sigma_{\text{imp},i}\left(\beta_i\,\bar{x}\right)\sigma_{\text{imp},i}\left(\beta_i\,\bar{x}\right)}}$$

$$\sigma_{\text{imp},I}\left(\bar{x}\right)\approx\sqrt{\sum_{ij=1}^{n}\rho_{ij}\,p_i\,p_j\,\sigma_{\text{imp},i}\left(\beta_i\,\bar{x}\right)\sigma_{\text{imp},i}\left(\beta_i\,\bar{x}\right)}$$

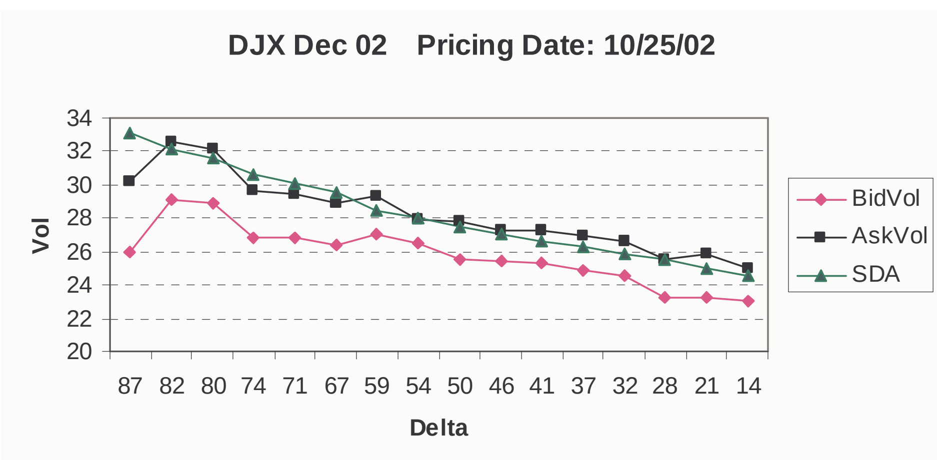
Performs well in the range -0.2<x<+0.2

DJX: Dow Jones Industrial Average

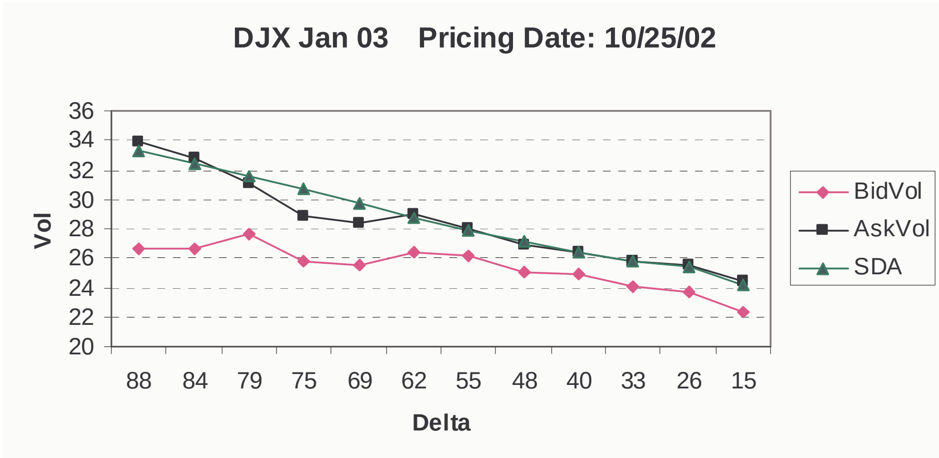
T=1 month



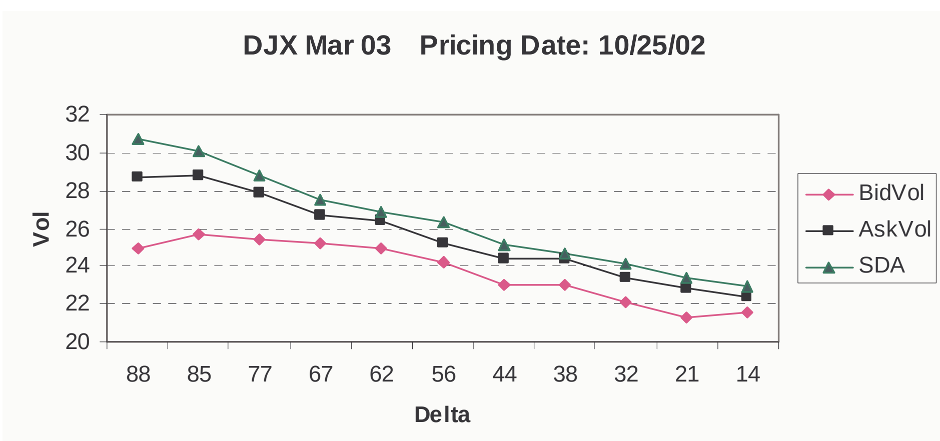
T= 2 months



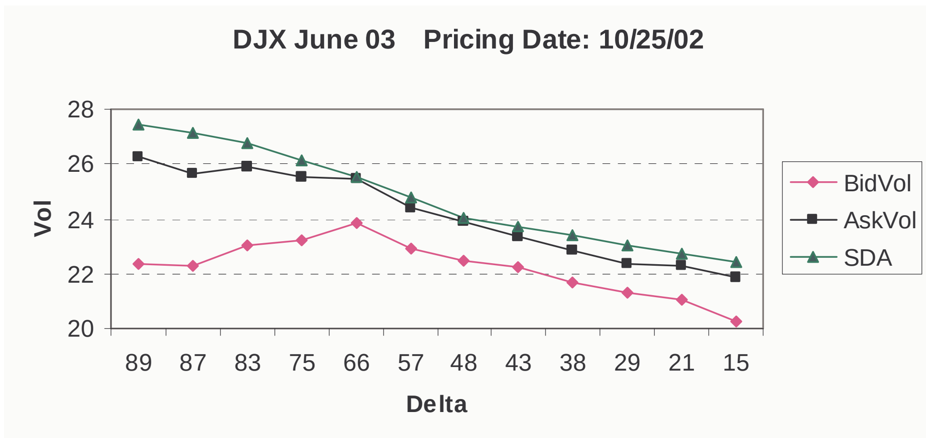
T=3 months



T= 5 months

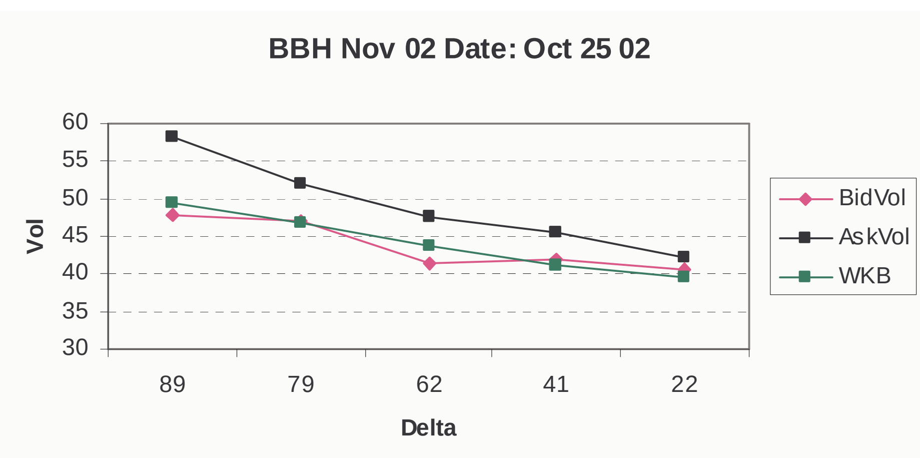


T=7 months

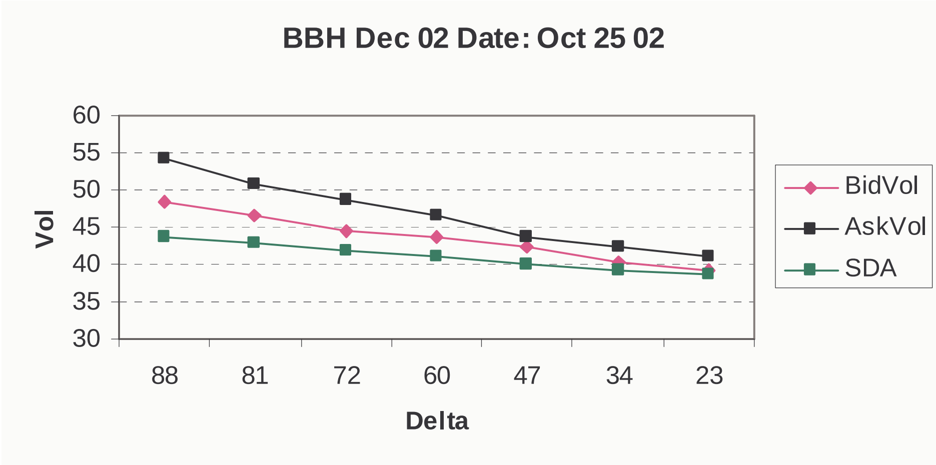


BBH: Biotechnology HLDR

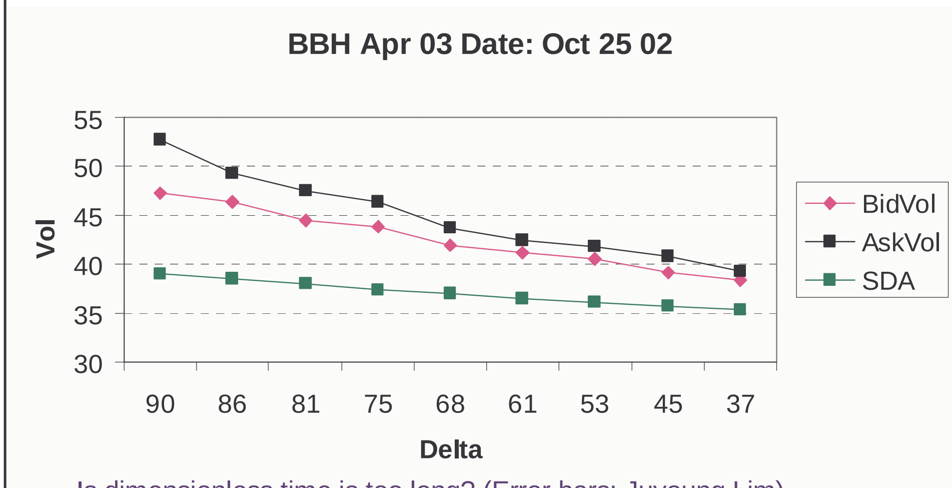
T = 1 month



T = 2 months



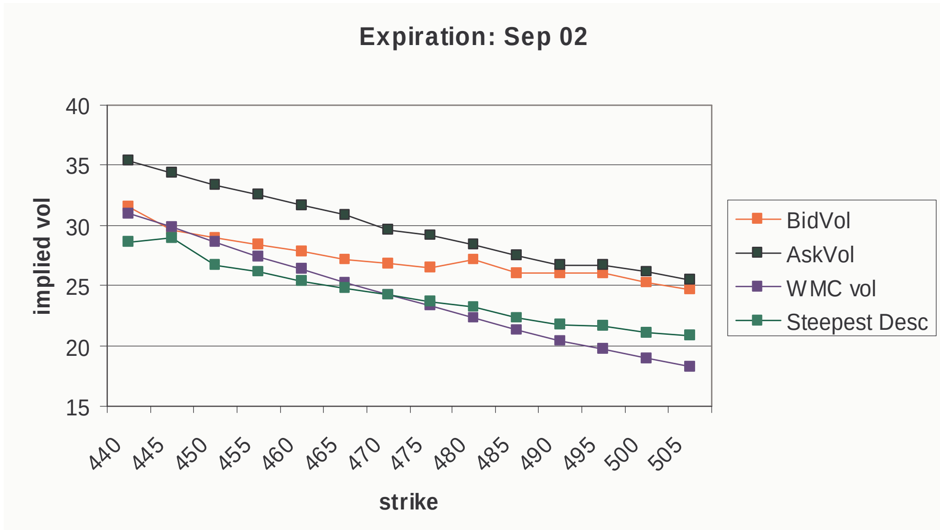
T = 6 months



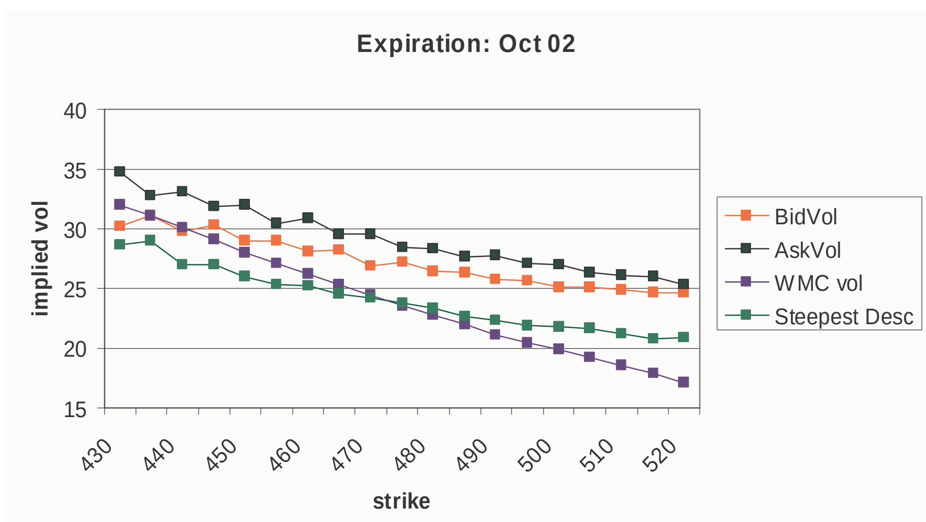
Is dimensionless time is too long? (Error bars: Juyoung Lim)

Is correlation causing the discrepancy?

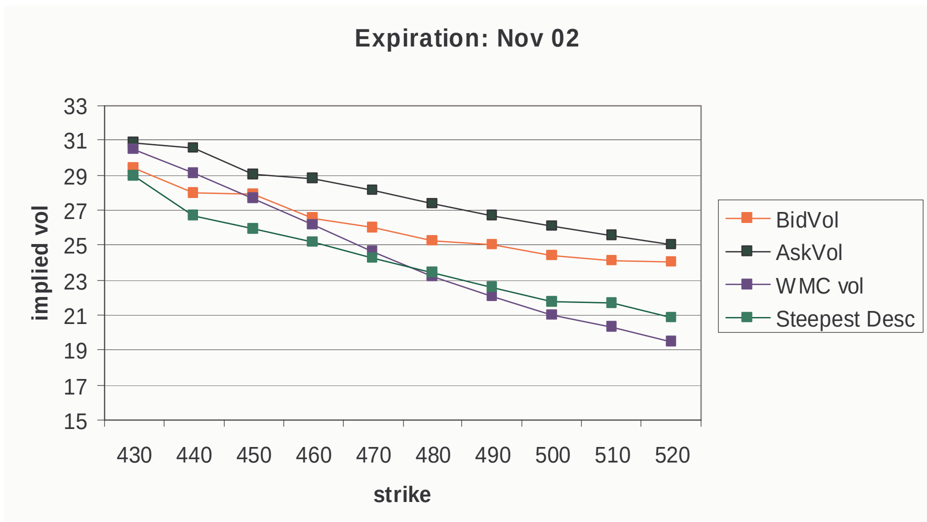
S&P 100 Index Options
(Quote date: Aug 20, 2002)



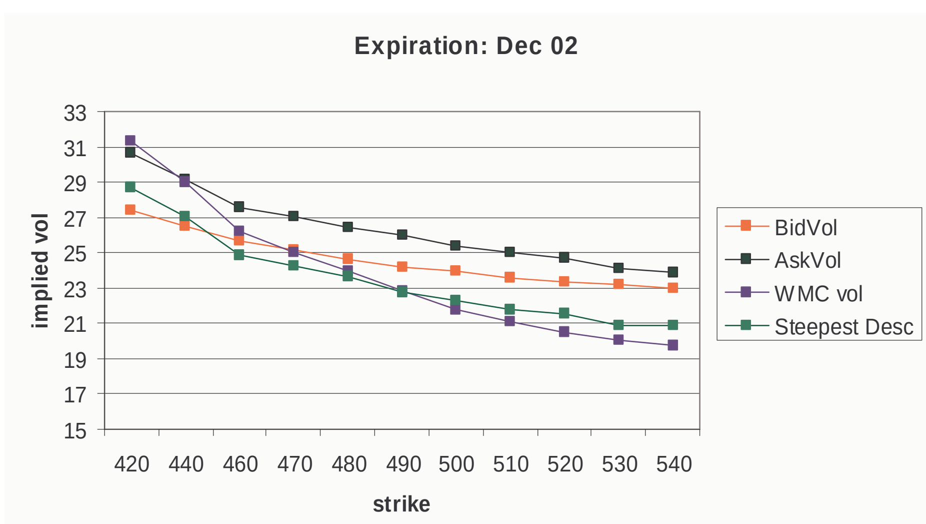
S&P 100 Index Options
(Quote date: Aug 20, 2002)



S&P 100 Index Options
(Quote date: Aug 20, 2002)



S&P 100 Index Options
(Quote date: Aug 20, 2002)



Implied Correlation: a single correlation coefficient consistent with index vol

$$\left(\sigma_I^{\text{impl}}\right)^2 = \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2 + \bar{\rho} \sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}$$

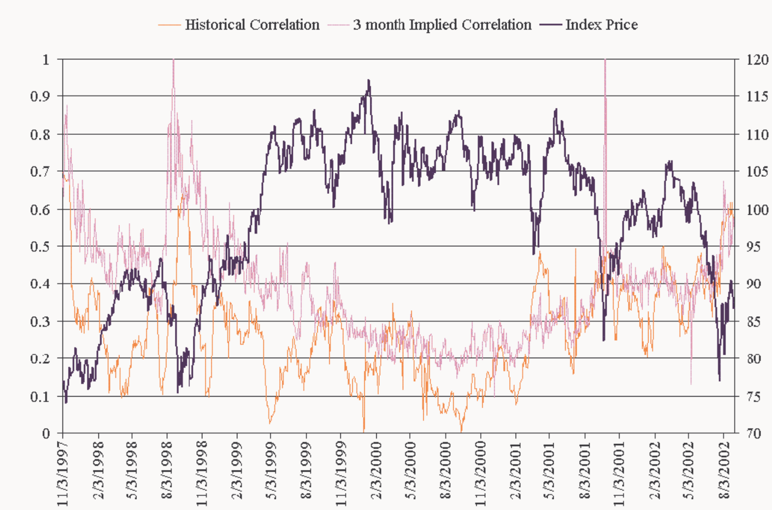
$$\therefore \quad \bar{\rho} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i \left(\sigma_i^{\text{impl}}\right)^2}{\sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}{\left(\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}$$

Approximate formula:

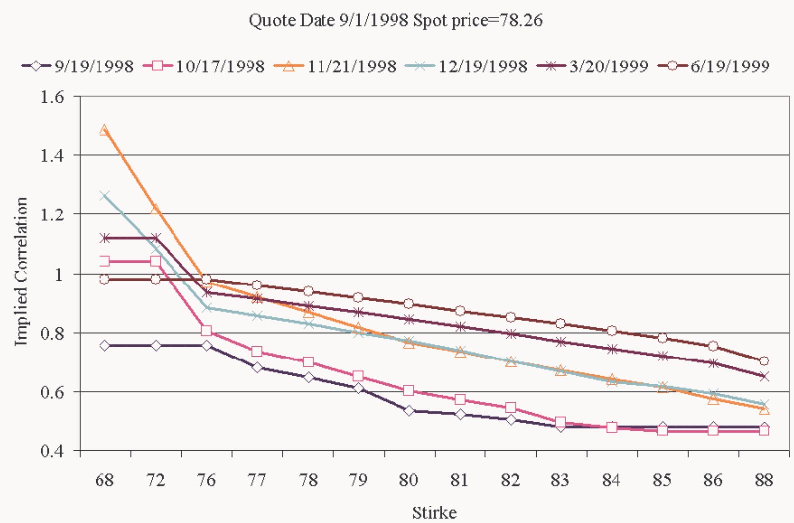
$$\bar{\rho} \approx \left(\frac{\sigma_I^{\text{impl}}}{\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}} \right)^2$$

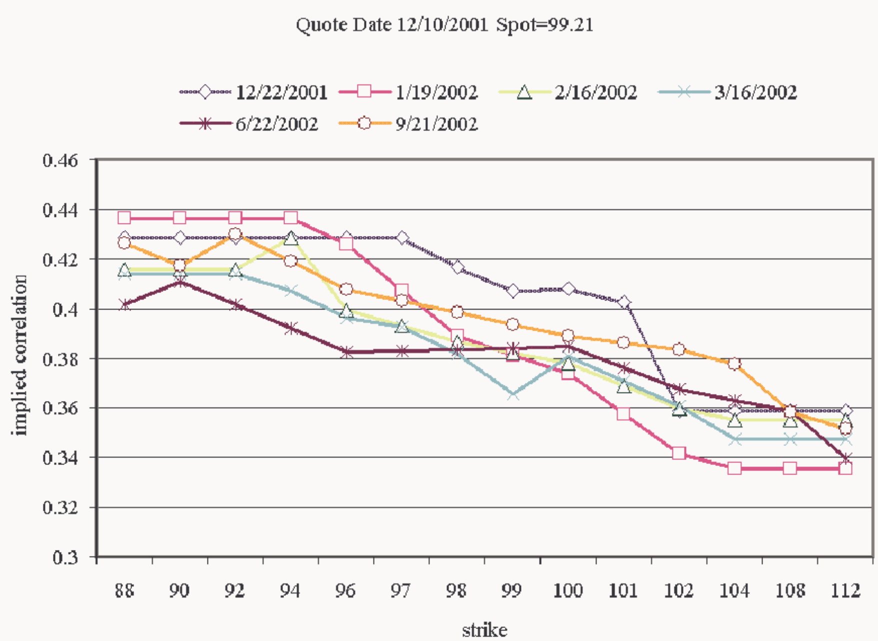
Implied correlation can be defined for different strikes, using SDA

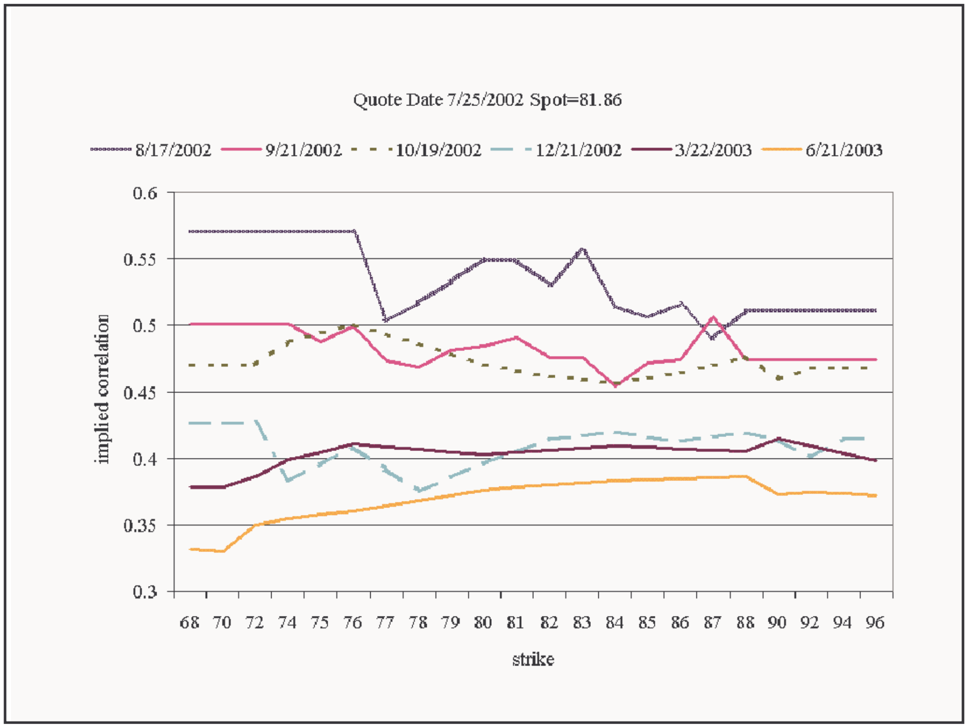
Dow Jones Index



Dow Jones Index: Correlation Skew







A model for ``Correlation skew'': Stochastic Volatility Systems

$$\frac{dS_i}{S_i} = \sigma_i dW_i \qquad \frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i$$

$$E\left(dW_i dW_j\right) = \rho_{ij} dt \qquad E\left(dW_i dZ_j\right) = r_{ij} dt$$

$$\bar{x} = \frac{dI}{I}, \qquad x_i = \frac{dS_i}{S_i} \qquad y_i = \frac{d\sigma_i}{\sigma_i}$$

Look for most likely configuration of stocks and vols $(x_1,...,x_n,y_1,...,y_n)$ corresponding to a given index displacement $\bar{x}$

Most likely configuration for Stochastic Volatility Systems

$$x_i^{\ *} = \beta_i \bar{x} \qquad \beta_i = \frac{\sigma_i \rho_{iI}}{\sigma_I}$$

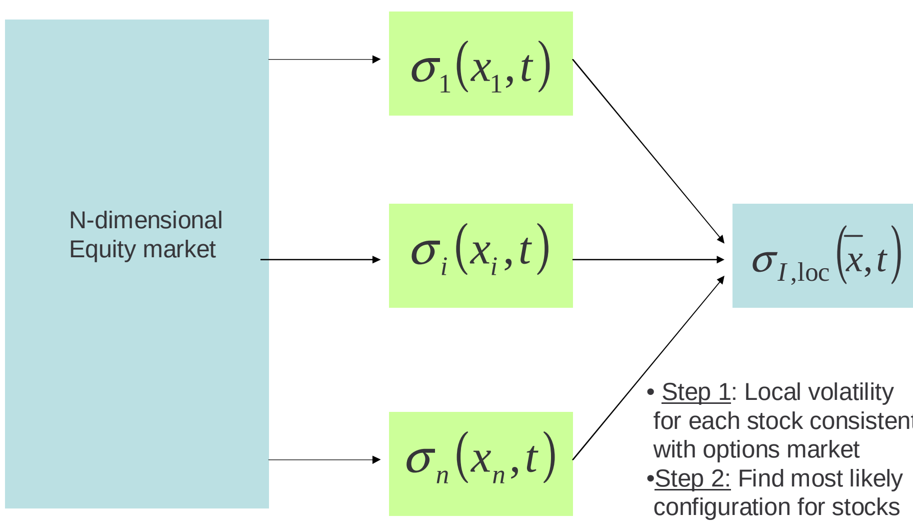
$$y_i^{\ *} = \gamma_i \bar{x} \qquad \gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Most likely configuration for stocks moves and volatility moves, given the index move

$$\sigma_{I,\text{loc}}^2(\bar{x},t) \cong \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t)\sigma_j(0,t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij}$$

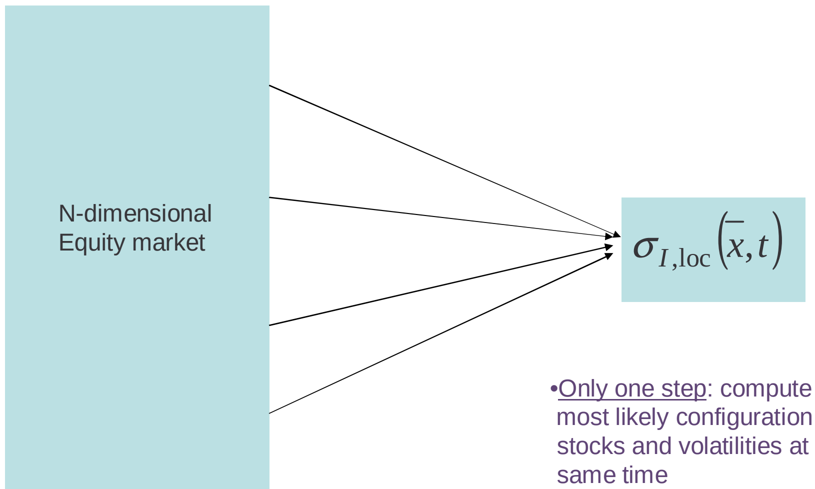
SDA

Method I: Dupire & Most Likely Configuration for Stock Moves



- Step 1: Local volatility for each stock consistent with options market
- Step 2: Find most likely configuration for stocks

Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities



- Only one step: compute the most likely configuration of stocks and volatilities at the same time

Methods I and II are not `equivalent'

Dupire local vol. for single names →

$$\sigma_{i,\mathrm{loc}}(x_i,t) \approx \sigma_i(0,t)e^{\varpi_i x_i} \qquad \varpi_i = \frac{\kappa_i r_{ii}}{\sigma_i}$$

Index vol., Method I →

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\varpi_i \beta_i \bar{x}} e^{\varpi_j \beta_j \bar{x}}$$

Index vol., Method II →

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

Stochastic Volatility Systems give rise to Index-dependent correlations

Method II

$$\sigma_{I,\mathrm{loc}}^2\left(\bar{x},t\right) \approx \sum_{ij} p_i p_j \sigma_i\left(0,t\right)\sigma_j\left(0,t\right)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

$$\approx \sum_{ij} p_i p_j \sigma_i\left(0,t\right) e^{\beta_i \varpi_i \bar{x}} \sigma_j\left(0,t\right) e^{\beta_j \varpi_j \bar{x}} \rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} e^{-\beta_i \varpi_i \bar{x}} e^{-\beta_i \varpi_i \bar{x}}$$

$$\approx \sum_{ij} p_i p_j \sigma_{i,\mathrm{loc}}\left(\beta_i \bar{x},t\right)\sigma_{i,\mathrm{loc}}\left(\beta_i \bar{x},t\right)\rho_{ij}\left(\bar{x}\right)$$

$$\rho_{ij}\left(\bar{x}\right) \equiv \rho_{ij} e^{\left(\gamma_i + \gamma_j - \beta_i \varpi_i - \beta_j \varpi_j\right)\bar{x}}$$

Equivalence holds only under additional assumptions on stock-volatility correlations

$$\varpi_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \frac{\sigma_i \rho_{iI}}{\sigma_I} = \frac{\kappa_i r_{ii} \rho_{iI}}{\sigma_I}$$

Method I

$$\gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Method II

$$r_{iI} = r_{ii}\rho_{iI}$$

$$r_{ij} = r_{ii}\rho_{ij}$$

Conditions under which both methods give equivalent valuations

Open (and very doable) problems

- Apply this technology for pricing swaptions based on the volatility skew of LIBOR rates or forward rates
- If we use a Local Volatility model (e.g. BGM with square-root volatility), the answer is <u>identical</u> to the previous formula
- The ``full'' SABR multi-asset model gives rise to a complicated Riemannian metric

$$dL^2 = \sum_{ij=1}^{n} g_{ij} \frac{d\eta_i}{\sigma_i}\frac{d\eta_j}{\sigma_j} + \sum_{i=1}^{n} \frac{(d\sigma_i)^2}{\kappa_i^2 \sigma_i^2}$$

- Credit default models for pricing CDOs are amenable to the same approach, especially copula-type models. I am not aware of any solutions

Epilogue: Structural Credit Model

$\mathrm{x} = (x_1, ..., x_n)$ vector of firm values

Firm i defaults before time T if $x_i(T) < \alpha_i$

Equal weighted CDO : loss of m dollars if

$$\mathrm{x}(T) \in \Omega_m = \bigcup_{\mathrm{card}(I) \geq m} \bigcap_{i \in I} \{\mathrm{x} : x_i < \alpha_i\}$$

Solve

$\inf\{L(0,\mathrm{x}) : \mathrm{x} \in \Omega_m\}$