Lecture 6: The Greeks and Basic Hedging

Sources

Avellaneda and Laurence

J Hull

A general approach to hedging

- Suppose that an investor has a security A (or portfolio of securities) and wishes to determine the relation between the returns of security A and the returns of a given asset B (not equal to A).

- We consider a stochastic linear model for the returns of A and B. The simplest model looks like

$$\boxed{r_A = \alpha + \beta \cdot r_B + \varepsilon}$$

$r_A =$ return of asset A

$r_B =$ return of asset B

$\varepsilon =$ residual, uncorrelated with r_B, normalized to have mean zero.

The meaning of α, β

$$E(r_A) = \alpha + \beta E(r_B) \quad \therefore \qquad \alpha = E(r_A) - \beta E(r_B)$$

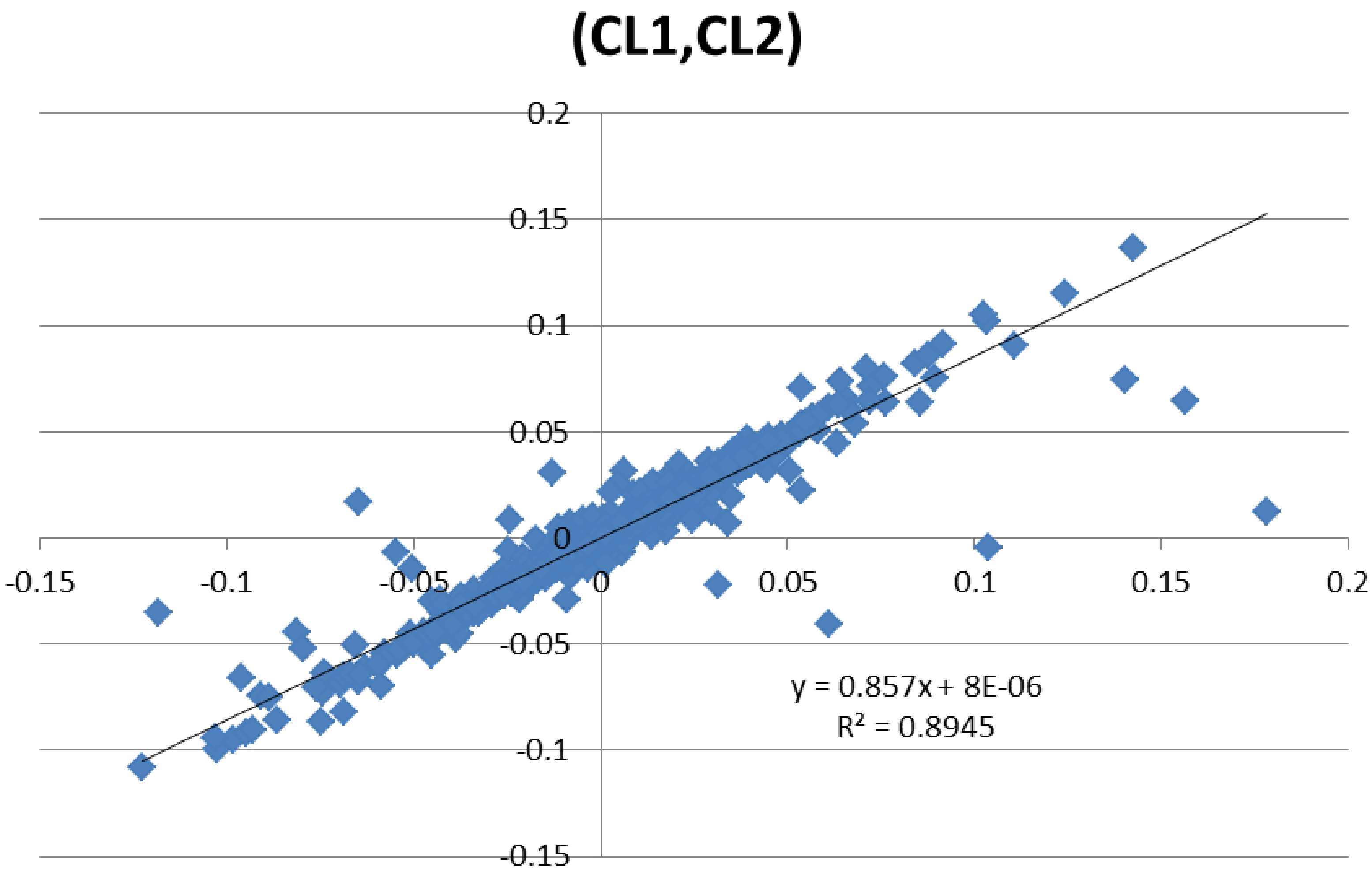
$$E(r_A r_B) = \alpha E(r_B) + \beta E(r_B^2) = E(r_A)E(r_B) - \beta E(r_B)^2 + \beta E(r_B^2)$$

$$\beta = \frac{E(r_A r_B) - E(r_A)E(r_B)}{E(r_B^2) - E(r_B)^2} = \frac{Corr(r_A, r_B)}{Var(r_B)}$$

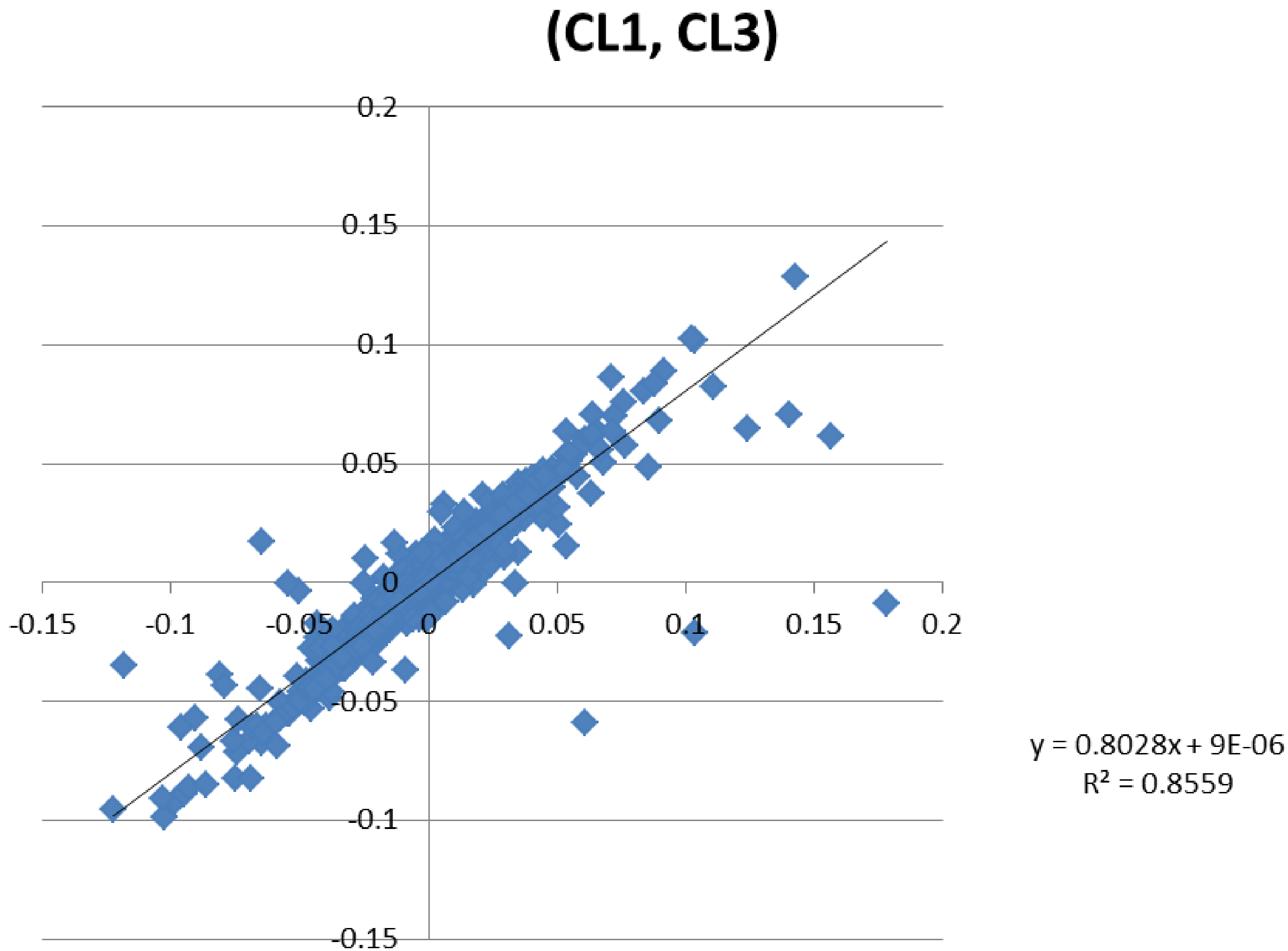
- Beta is the regression coefficient (with respect to the model used for the joint returns) of the returns of A on the returns of B

- Hedging with a linear model corresponds to taking an offsetting position using the beta (regression coefficient).

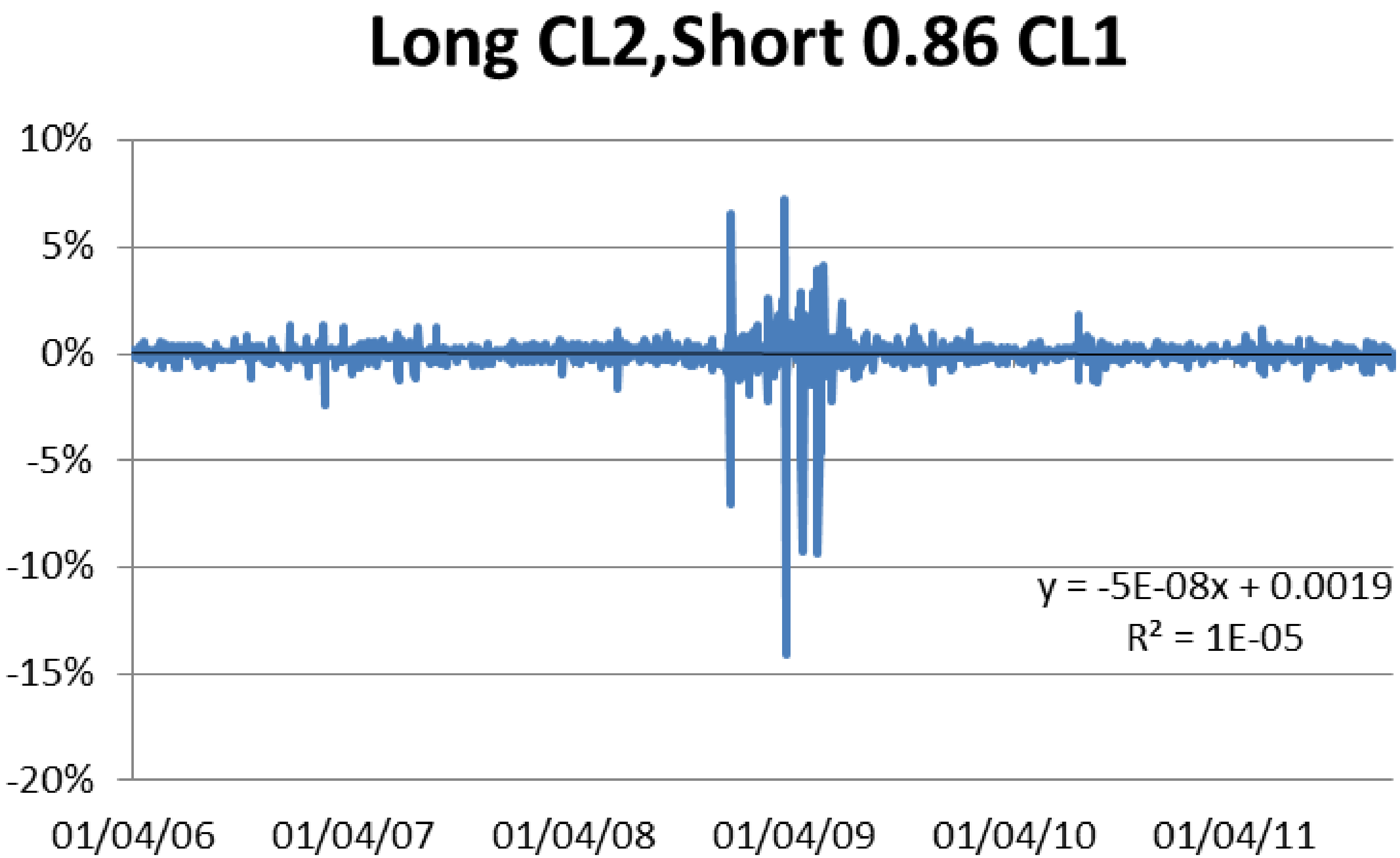
Regression of CL2 returns on CL1 returns (2003-2010)



Linear model applied to (CL1,CL3)



Example: Long CL2, short CL1



$$\sigma_{CL2} = 2.4\%$$

$$\sigma_{CL2-0.86CL1} = 0.8\%$$

Hedging option exposure against the underlying asset

- Assume that you are long 1 call option on XYZ, with strike K, maturity T.
- Assume that the dividend yield and interest rate are known.

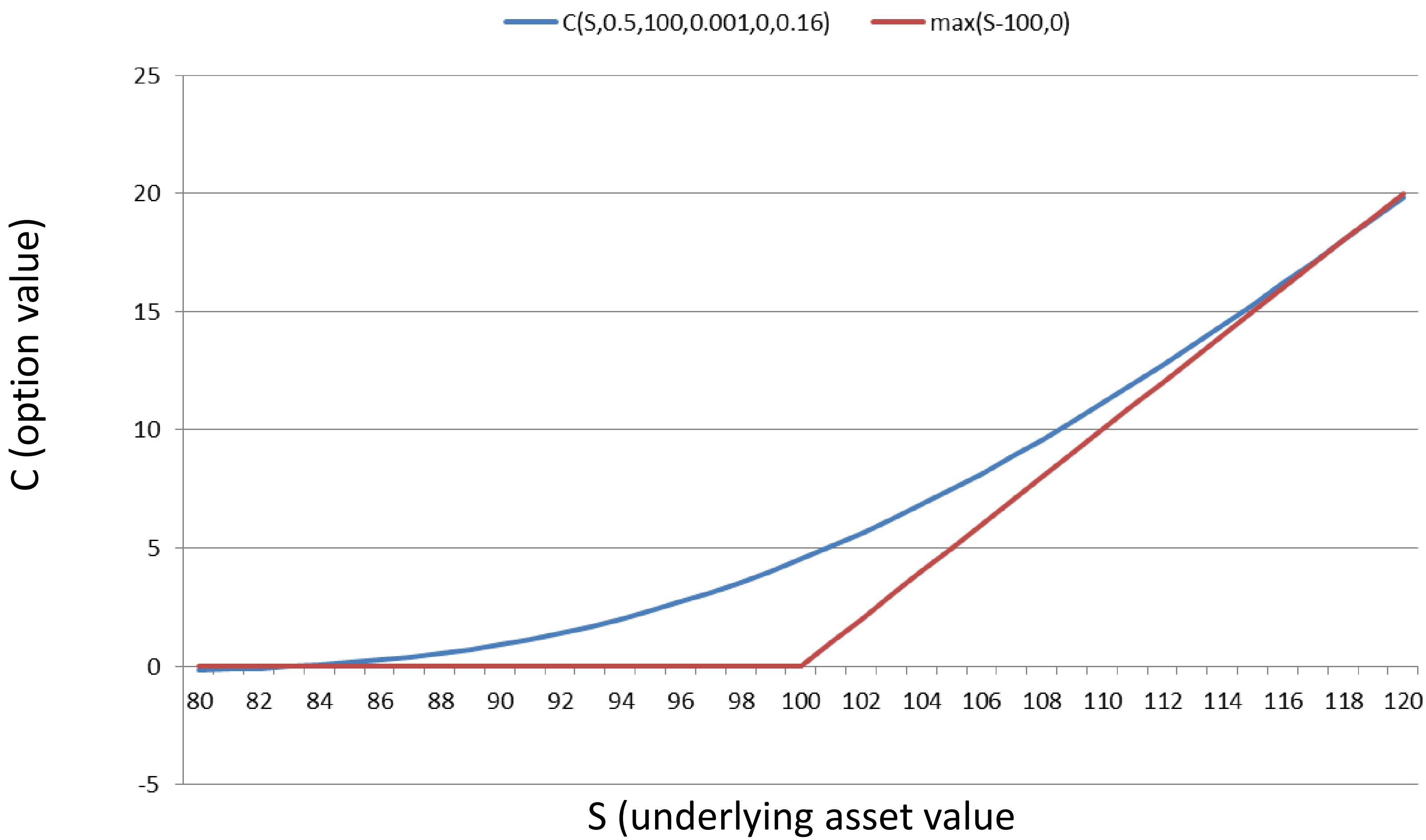
-- compute the **implied volatility** σ

$$C = BSCall(S,T,K,r,q,\sigma) = C(S,T,K,r,q,\sigma)$$

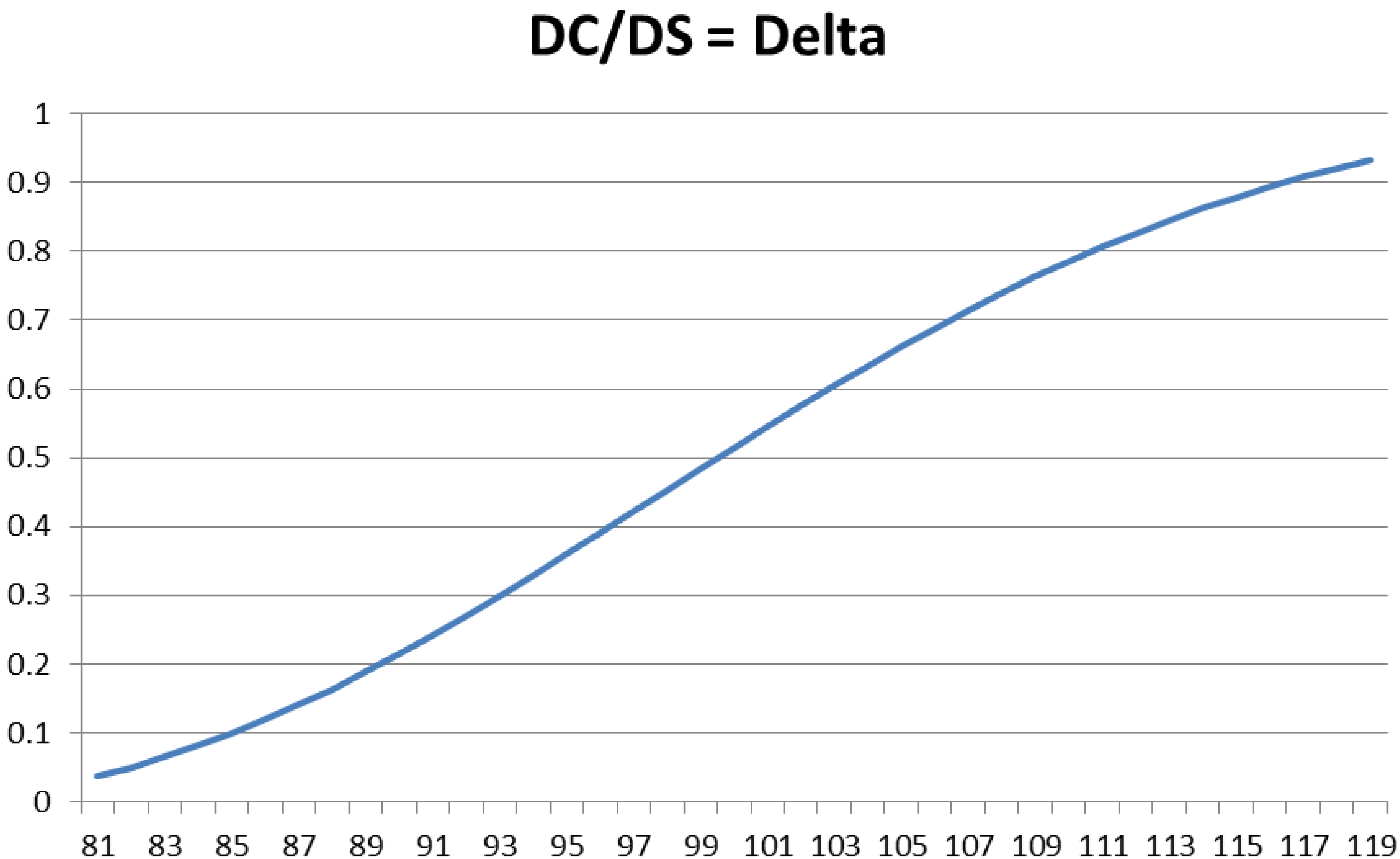
$$\Delta C \cong \underline{\frac{\partial C}{\partial S}} \Delta S + \frac{\partial C}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + o(\Delta t)$$

- The exposure to the underlying asset is represented by the first derivative with respect to price

Options are non-linear functions of the underlying asset



Delta = $\frac{\partial C(S,T)}{\partial S}$



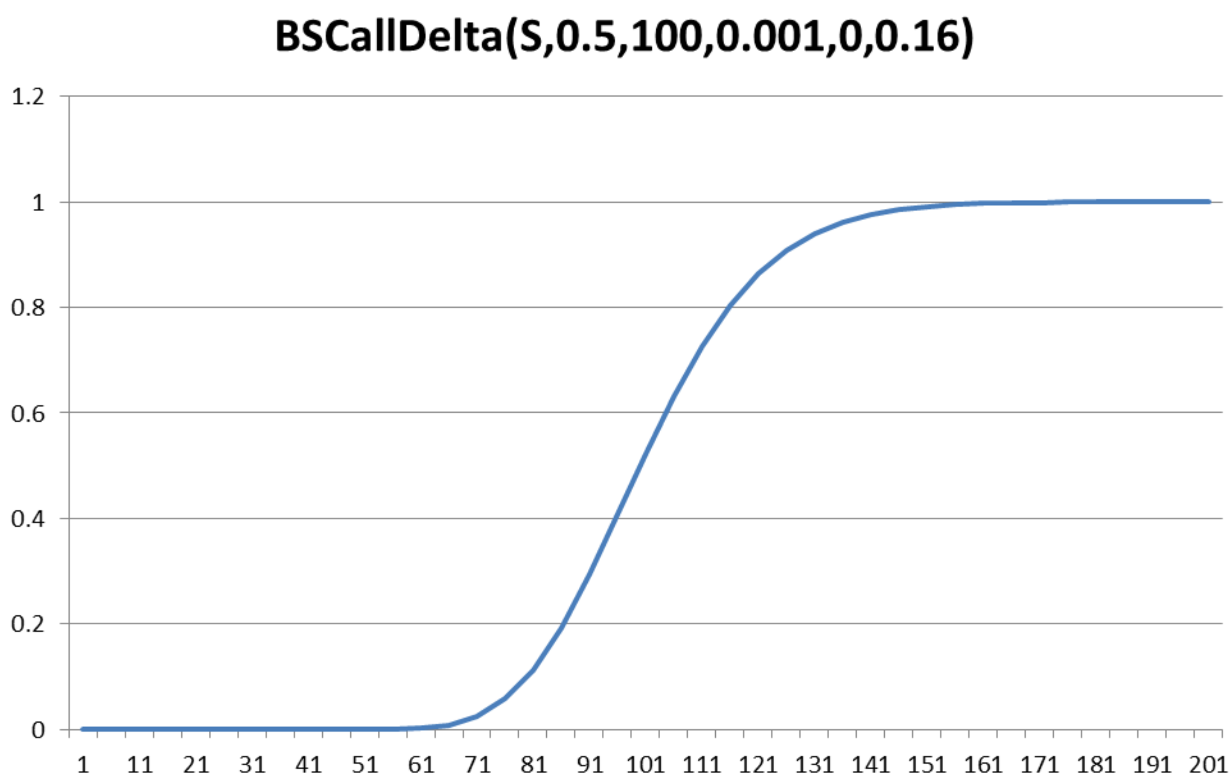
Delta for European Call (Black & Scholes)

$$C = e^{-qT} S N(d_1) - e^{-rT} K N(d_2)$$

$$\begin{aligned}
\frac{\partial C}{\partial S} &= e^{-qT} N(d_1) + e^{-qT} S N'(d_1) \frac{\partial d_1}{\partial S} - e^{-rT} K N'(d_2) \frac{\partial d_2}{\partial S} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} N'(d_1) - e^{-rT} K N'(d_2) \frac{1}{S\sigma\sqrt{T}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{\left(d_1 - \sigma\sqrt{T}\right)^2}{2}}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2} - \ln\left(\frac{S e^{(r-q)T}}{K}\right)}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1)
\end{aligned}$$

Delta: European Call- Black-Scholes model

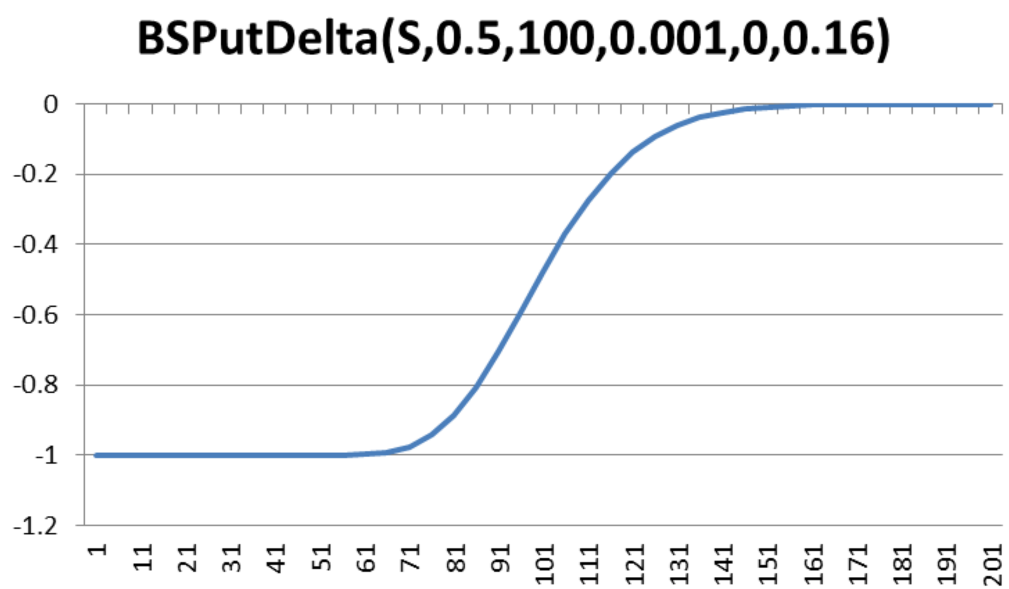
$$BSCallDelta(S,T,K,r,q,\sigma) = e^{-qT}N(d_1) = e^{-qT}N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{F}{K}\right) + \frac{\sigma\sqrt{T}}{2}\right)$$



Delta For European Put (Black-Scholes)

$$P = C - Se^{-qT} + Ke^{-rT} \qquad \delta_{put} = \delta_{call} - e^{-qT}$$ (Put-call parity)

$$\delta_{put} = e^{-qT} N(d_1) - e^{-qT} = -e^{-qT} N\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{K}{F}\right) - \frac{\sigma\sqrt{T}}{2} \right)$$



Gamma – the change in Delta as the stock price moves

- Options are non-linear financial instruments, in the sense that they do not have a constant Delta with respect to the underlying instrument

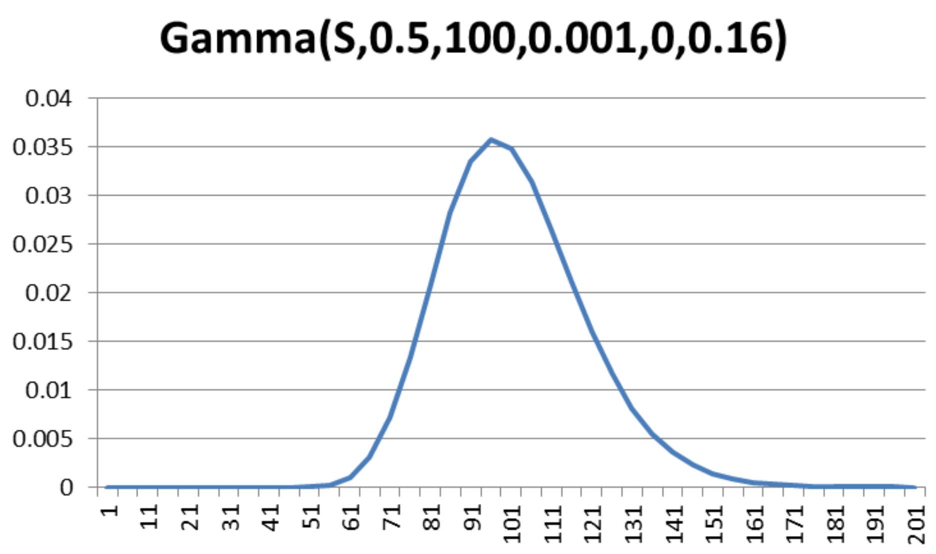
- The second derivative of the option value with respect to the underlying price is called Gamma. It represents the rate of change of Delta as the price moves. In the European B-S model, we have

$$\Gamma\left(S,T,K,r,q,\sigma\right)=e^{-qT}\,\frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{2\pi T}}$$

- The Gammas of a call and put with the same parameters are identical.

Properties of Gamma

- The option price is convex in S, so Gamma is positive for a long options position
- Gamma is mostly concentrated near the strike price, i.e. Gamma is the largest for at-the-money options. OTM and ITM options have less convexity



Delta and Gamma for American Options

- The derivatives can be computed by finite-differences (trinomial scheme)
- If we assume that the arrays are labeled C(-M to +M), S(-M to +M) for the option price and the stock price respectively, then we have

$$\begin{aligned}
&\text{Option price} = C_0^0 \\
&\text{Delta} \quad = \frac{C_0^1 - C_0^{-1}}{2\Delta x \cdot S_0^0} \\
&\text{Gamma} \quad = \frac{1}{(S_0^0)^2}\left[\frac{C_0^1 + C_0^{-1} - 2C_0^0}{\Delta x^2} - \frac{C_0^1 - C_0^{-1}}{2\Delta x}\right]
\end{aligned}$$

- These values are very close to the analytic expressions for European-style Greeks for ATM options

Example

- A trader has a position in SPY stock and options on SPY expiring in June 2012. He is long 10,000 SPY December 105 puts. He is also delta-neutral through SPY stock.

  SPY=$114.25
  Bid price=$3.91, Ask price=$4.00
  Implied Volatility=37.7%
  Delta=-0.29371
  Gamma= 0.02653

  Option market value = 10,000*100*3.955=$3,955,000
  SPY hedge= long 293,371 shares (MV= $ 33,517,337)

- If SPY increases by 1 dollar, New Delta ~ -0.29371+0.02653= -0.26718
  New theoretical hedge= long 267,180 shares
  Difference = 26,530 shares
  To be market-neutral, the trader would need to **sell 26,530 shares at $115.25**

- If SPY decreases by 1 dollar, in order to become delta-neutral, the trader would need to **buy 26,530 shares at $113.25**

Gamma and hedged portfolios

$$\Delta C = \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 + ...$$

$$\begin{aligned}
\Delta\Pi &= -rC\Delta t + \Delta C - \delta\left(\Delta S - rS\Delta t + qS\Delta t\right) + ... \\
&= \left(-rC + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 + S\frac{\partial C}{\partial S}(r-q)\right)\Delta t + ... \\
&= \left(-rC + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + S\frac{\partial C}{\partial S}(r-q)\right)\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 - \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\Delta t + ... \\
&= \frac{S^2}{2}\frac{\partial^2 C}{\partial S^2}\left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2\Delta t\right] + ...
\end{aligned}$$

Vega

- Vega is the sensitivity of an option price to changes in implied volatility

$$Vega = \frac{\partial}{\partial \sigma}\left(e^{-qT} SN(d_1) - e^{-rT} KN(d_2)\right)$$

$$= e^{-qT} SN'(d_1)\left(-\frac{\ln(F/K)}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}\right) - e^{-rT} KN'(d_2)\left(-\frac{\ln(F/K)}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)$$

$$= e^{-qT} SN'(d_1)\sqrt{T}$$

$$\boxed{\text{Vega} = e^{-qT} S \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \sqrt{T}}$$

Time-dependence

- Option **premia above par** value decrease with time-to-maturity
- Gamma increases with time to maturity (for ATM options)
- Vega decreases with time-to-maturity (for ATM options)

Short-term options are mostly sensitive to Gamma
(frequent delta hedging needed to maintain market-neutrality)

Long-term options are mostly sensitive to Vega
(value is very sensitive to the implied volatility)

Theta (time decay rate)

- Theta is the derivative of the option value with respect to time to maturity

- To get intuition for Theta, assume that r and q are zero. Then, by the Black Scholes equation

$$\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = 0$$

$$\vartheta = \frac{\partial C}{\partial T} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = \frac{\sigma^2 S^2}{2}\Gamma$$

$$\boxed{\theta = \frac{\sigma S e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi T}}}$$