Mathematical Models for Stock Pinning Near Option Expiration Dates

Marco Avellaneda*, Gennady Kasyan †

and

Michael D. Lipkin‡

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1 Introduction

This paper discusses mathematical models in Finance related to feedback between options trading and the dynamics of stock prices. Specifically, we consider the phenomenon of "pinning" of stock prices at option strikes around expiration dates. Pinning at the strike refers to the likelihood that the price of a stock coincides with the strike price of an option written on it immediately before the expiration date of the latter. (See Figure 1 for a diagrammatic description of pinning).

Conclusive evidence of stock pinning near option expiration dates was given by Ni, Pearson and Poteshman (2005) [8] based on empirical studies. Theoretical work was done by Krishnan and Nelken (2001) [5], who proposed a model to explain pinning based on the Brownian bridge. Later, Avellaneda and

*Courant Institute and Finance Concepts LLC
†Courant Institute
‡Columbia University and Katama Trading LLC

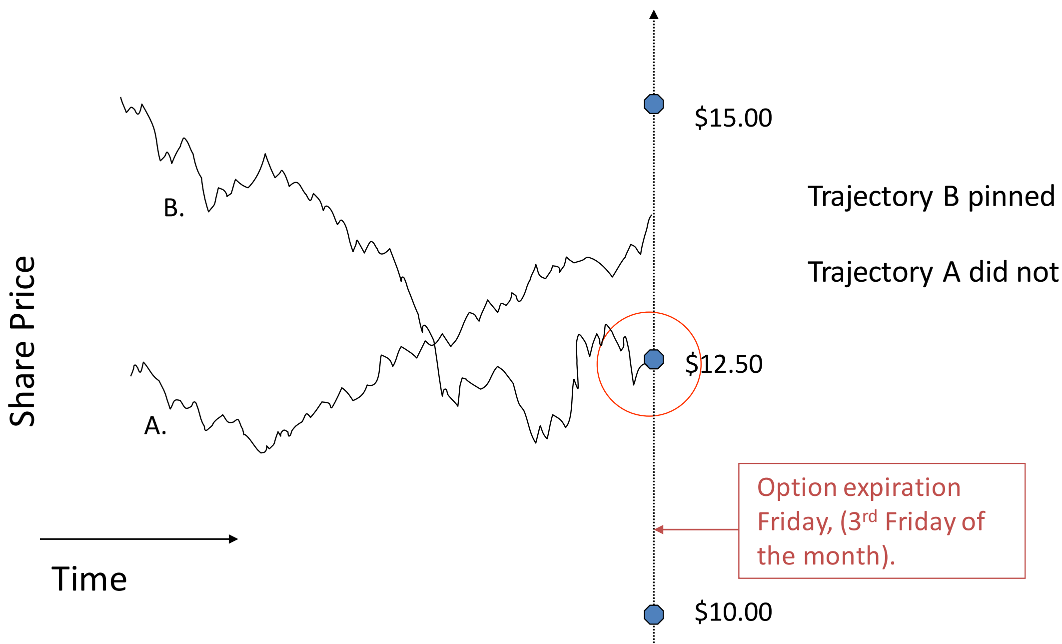

Figure 1: Stock price pinning around option expiration dates refers to the trajectory B which finishes exactly at an option strike price on an expiration date.

Lipkin (2003)[1] (henceforth AL) formulated a model based on the behavior of option market-makers which impact the underlying stock price by hedging their positions. AL consider a linear price-impact model namely,

$$\frac{\Delta S}{S} \sim E \cdot Q$$

where S is the price, E is a constant (elasticity of demand), and Q is the quantity of stocks demanded. According to AL, pinning is a consequence of the demand for Deltas by market-makers in the case when the open interest on a particular strike/expiration is unusually high. In this paper, we consider more general non-linear impact functions which follow power-laws, *i.e.*, we shall assume that

$$\frac{\Delta S}{S} \sim E \cdot Q^p \tag{1}$$

where p is a positive number. Such impact models have been investigated by many authors in Econophysics; see, among others, Lillo *et al.*[6], Gabaix[2] and Potters and Bouchaud [9].[1] In the particular context of pinning around option expiration dates, Jeannin *et al.* [4] suggested that the results of AL would be qualitatively different in the presence of non-linear price elasticity and, specifically, that pinning would be mitigated or would even disappear altogether for sufficiently low values of p.

The goal of this paper is twofold: first, we review the issue of pinning around option expiration dates, both from the point of view of the AL model and from empirical data, and, second, we analyze rigorously the non-linear model (2), expanding on the work of AL along the lines of Jeannin *et. al.* We find, in particular, that there exists a "phase transition" of sorts – in the sense of Statistical Physics – associated with the model's behavior in a neighborhood of

[1] To our knowledge, there is not yet a clear consensus for the correct value of the exponent p, as price impact is difficult to measure in practice.

$p = 1/2$. In fact, for $p \leq 1/2$, there is no stock pinning around option expiration dates.

The case $p > 1/2$ is first analyzed numerically by Monte Carlo simulation. We show that the probability of pinning at a strike based on model (1) satisfies

$$P_{pinning} = c_1 e^{-\frac{c_2}{(p-1/2)^+}} \left(1 + o(1)\right), \tag{2}$$

where $P_{pinning}$ is the probability that the stock price coincides with a strike level at expiration, for some constants c_1, c_2. This suggests that that the behavior of the pinning probability is C^∞ around $p = 1/2$, but not analytic. In other words, there is an infinite-order phase-transition in the vicinity of $p = 1/2$, according to the value of the exponent in (1). For $p \leq 1/2$ price trajectories behave like "free" random walks; for $p > 1/2$, there is a non-zero probability that they converge to an option strike level.

The outline of the paper is as follows: first, we review empirical results on the existence of pinning. Then, we discuss the AL case, $p = 1$, for which we have a complete analytical solution. Then, we consider general exponents p. We present numerical evidence of equation (2) and give a rigorous justification of (2) for all values of the exponent p, $0 \leq p \leq 1$ in the form of a theorem.

The mathematical techniques used in the proof consist of Large Deviation estimates for small-noise perturbation of dynamical systems (a.k.a. Ventsel-Freidlin theory) and a rigorous version of the real-space Renormalization Group (RG) technique, which is the key element in deriving (3) and, in particular, the behavior of the pinning probability around the critical point $p = 1/2$.

2 Empirical evidence of pinning

In a comprehensive empirical study on the behavior of prices around option expirations, Ni, Pearson and Poteshman (2003) (henceforth NPP)[8] considered two datasets:

- *IVY Optionmetrics*, which contains daily closing prices and volumes for stocks and equity options traded in U.S. exchanges from January 1996 to September 2002

- Data from the Chicago Board of Options Exchange (CBOE) from January 1996 to December 1001 providing a breakdown of option positions among different categories of traders for each product. This dataset divides the option traders into 4 categories: market-makers, firm proprietary traders, large firm clients and discount firm clients. After each option expiration, the data reveals the aggregate positions (long, short, quantity) for each trader category.

NPP separated stocks into *optionable stocks* (stocks on which options had been written on the date of interest) and *non-optionable stocks*. The data analyzed by NPP consists of at least 80 expiration dates. There were 4,395 optionable stocks on at least one date and 184,449 optionable stocks/expiration pairs. There were 12,001 non-optionable stocks on at least one date and 417,007 non-optionable stock/expiration pairs.

The NPP experiments consisted in studying the frequency of observations of closing stock prices which coincide with strike prices or with multiples of $2.5, or $5 (which are the standardized strike levels for U.S. equity options) on each day of the month. By separating stocks into optionable and non-optionable and looking at the frequency with which the price closed near such discrete levels, NPP established statistically that stocks are more likely to close near a strike level on option expiration dates than on other days. They also showed that pinning is definitely associated with optionable stocks (see Figures 2 and 3). NPP also compared the cases of non-optionable stocks which later became optionable and optionable stocks that were previously non-optionable, The empirical evidence being that the former category is not associated with pinning and the latter is (Figures 4 and 5).

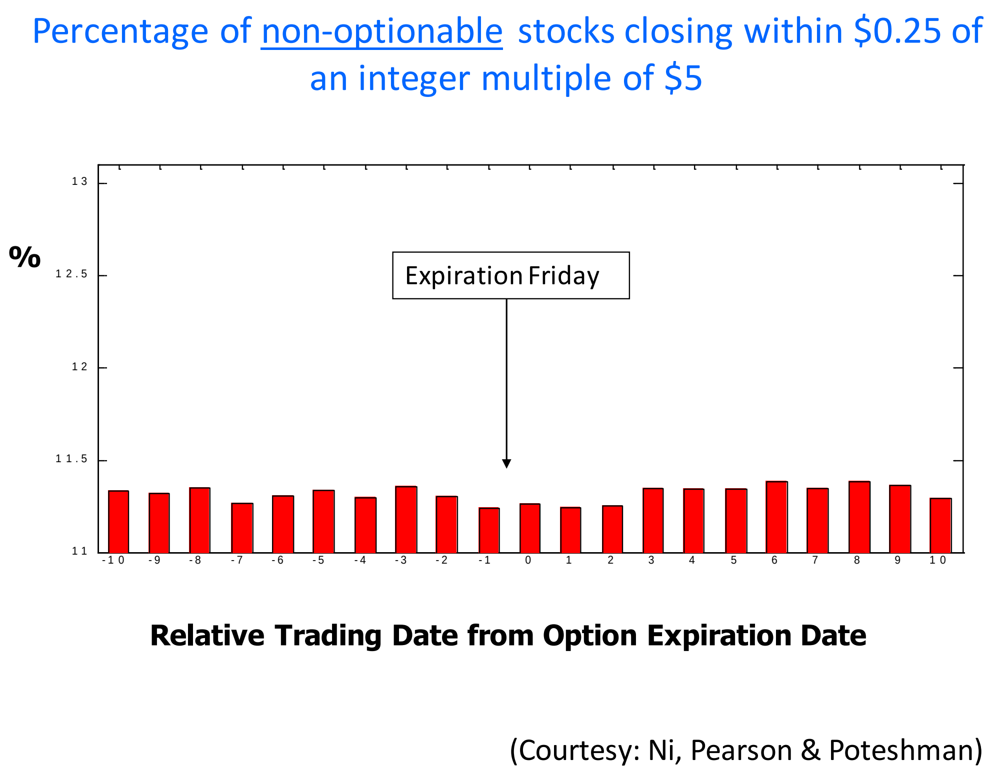

Figure 2: The different bins correspond to frequencies of instances for which the closing price of a non-optionable stocks is within \$0.25 of a multiple of \$5. Each trading day of the month is labeled with an integer between -10 and +10, and expiration Friday corresponds to the label 0. Notice that there is no appreciable difference between the frequencies associated with different days of the month, suggesting that closing near a level which is a multiple of 5 dollars is equally probable for different days of the month for non-optionable stocks.

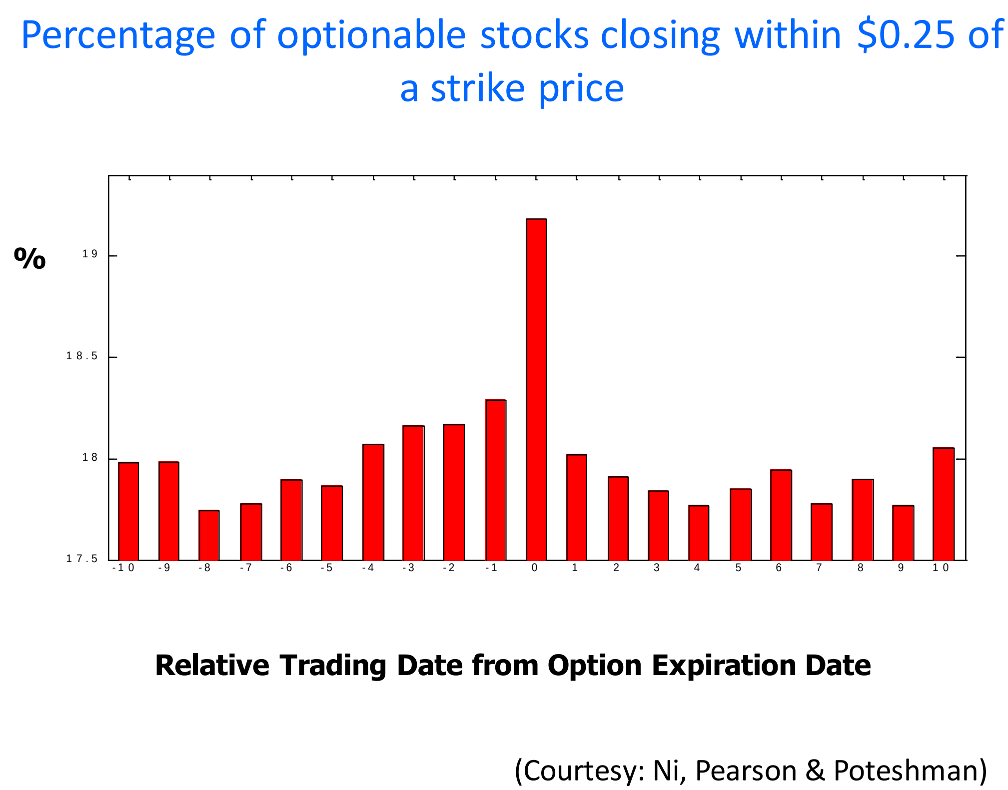

Figure 3: Frequencies of observations of prices of optionable stocks closing within \$0.125 of a strike price. The data shows that the likelihood that a price ends near an option strike price is significantly greater on expiration Friday, compared to other days.

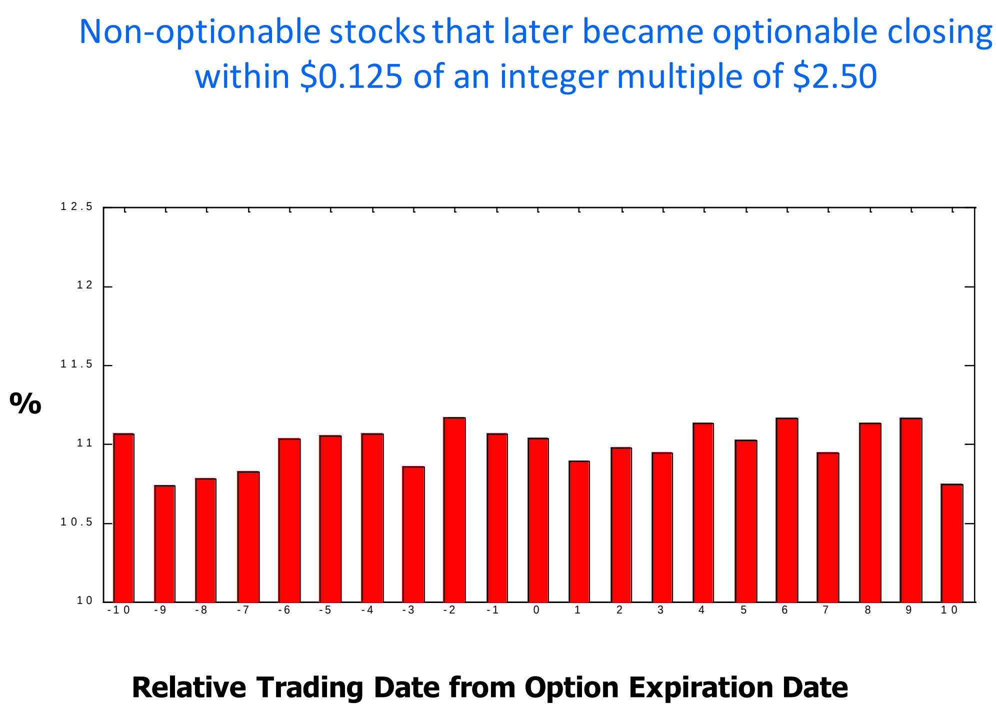

Figure 4: Same as in Figure 2 for non-optionable stocks which later became optionable. There is no evidence of pinning.

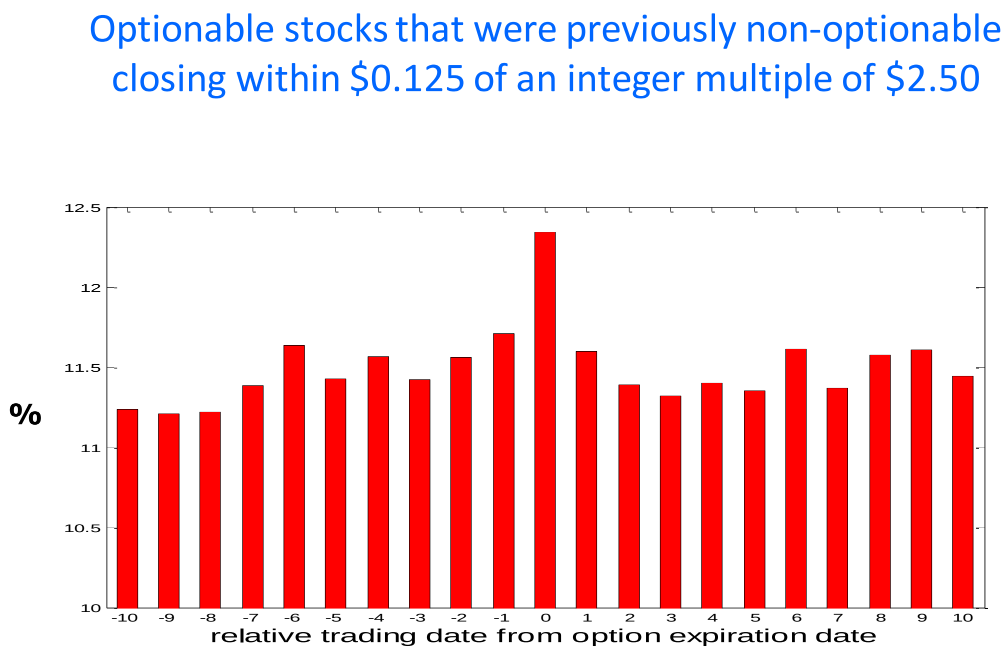

Figure 5: Same as in Figure 3 for optionable stocks which were previously non-optionable. Notice the peak at bin 0 which is associated with pinning.

The conclusions of NPP are that, based on frequencies of observations, optionables are more likely to end near a strike level (which is a multiple of \$2.5) on expiration dates, whereas non-optionables have the same likelihood of closing near a multiple of \$2.5, regardless of whether the day corresponds to the third Friday of the month or not.

3 A model based on market microstructure

Consider the case of the stock of J.D. Edwards (JDEC) during February and March 2001. This stock experienced an unusual volume in options with March expiration during the last days of February as shown in Figure 6. Following a trade of 4,000 contracts on February 17, a very large volume of March options with strike price \$10 were traded on February 27, bringing the total open interest for puts and calls on the \$10 line to 56,000 contracts. Recalling that the equity option contracts correspond to 100 shares, the total notional shares corresponding to the options is 5.6 million shares. On the other hand, the average traded volume in stocks was approximately 1 million shares.

The existence of this large open interest in the 10-strike options is important, since the large increase in open interest will potentially increase the trading volume. Figure 7 shows the chart of the stock during the same period of time. We notice from Figure 7 that, after the large option trade on February 27, the stock price became less volatile and converged to the price of \$10 which is the strike price of the options with large open interest.

To connect the stock price dynamics to the increase in open interest in options, we posit that there is a "feedback effect" due to the demand for Deltas for hedging the options. To be more precise, we make the following assumptions.

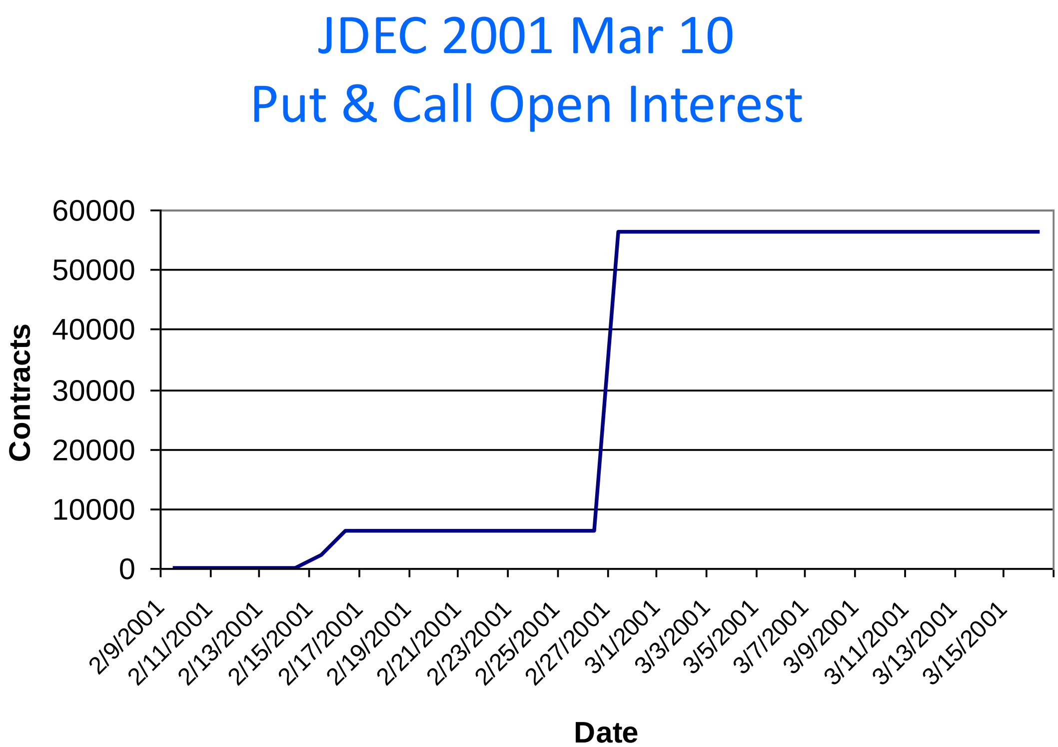

Figure 6: Evolution of the open interest in March options on JDEC with strike \$10, with a very volume transacted on February 27.

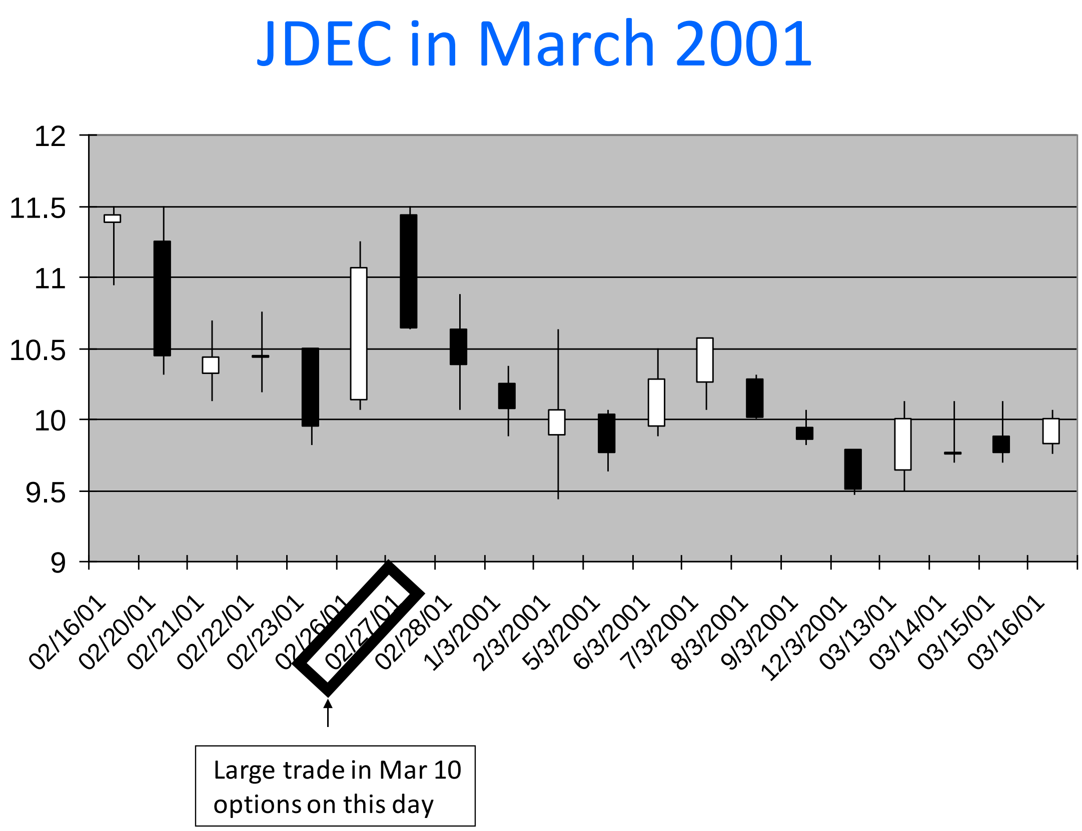

Figure 7: Evolution of the price of JDEC during the same period. We observe that the volatility of the stock diminishes after February 27 and the stock price converges to $10 as the March expiration approaches.

After the increase in open interest

1. The open interest becomes unusually large relative to normal volumes
2. A significant fraction of market-makers is long options (*i.e.* they *bought* the block of options that traded).

The two assumptions have the following consequences: first, since the open interest on the particular strike/maturity is large, the notional number of underlying Deltas (in the sense of Black-Scholes) is large compared with typical trading volumes. In particular, hedging the options – if one were to hedge – would imply trading relatively large quantities of stocks in relation to normal trading volume.

Second, the fact that market-makers are long options means that they are long Gamma. Delta-hedging implies that they will sell the stock when the price rises and buy the stock if the price drops. Delta-hedging in large amounts may affect the underlying stock price and drive it to the strike level.

What happens if we assume that assumption 1 holds but not assumption 2? If market-makers are only marginally long, then the demand for stock in the pattern described above may not be present and there is not price pressure pushing the stock to the strike price. Also if market-makers are predominantly short options, they may choose not to hedge or to hedge only partially. This is due to the fact that delta-hedging a short-gamma position implies buying high and selling low. On the other hand, if market-makers are long options they earn money by hedging frequently and thus may indeed impact the stock price. This is the essence of the AL model.

In order to formulate a quantitative model, we consider the following price-impact relation:

$$\frac{\Delta S}{S} \propto E \cdot \left| \frac{D}{< V >} \right|^p sign(D) \qquad \frac{D}{< V >} \gg 1, \tag{3}$$

where S is the stock price, D is the demand, $< V >$ is the average daily trading volume and p is an exponent. The choice of the parameter p is a fundamental question in Econophysics, with different authors proposing different values: $p = 0.22$ in [6], $p = 1/2$ in [2], and p=1.5 in [9].

4 AL model

We assume that $p = 1$ and that

$$D = -OI \, \frac{\partial \delta(S,t)}{\partial t} \, dt \tag{4}$$

where OI represents the open interest on the strike of interest δ is the Black-Scholes delta, or hedge-ratio for an option in terms of number of shares of the underlying asset. According to the Black-Scholes formula,

$$\delta = N(d_1) = \int_{-\infty}^{d_1} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \ , \quad d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(ln(\frac{Se^{\mu\tau}}{K}) + \frac{\sigma^2\tau}{2} \right),$$

where σ is the implied volatility, μ is the carry rate, S is the stock price, K is the strike price and $\tau = T - t$ is the time left before the option expires. For simplicity, we focus the analysis on the strike price with largest open interest and consider only one potential pinning point.[2] From the above considerations, it can be shown that the stochastic differential equation describing the phenomenon of stock pinning to leading order is [1],

[2]In practice, the analysis might involve more than one strike price if the open interest is large in several contracts.

$$dy = -\frac{E \cdot OI}{< V >} \frac{y - a(T-t)}{\sqrt{2\pi\sigma^2(T-t)^3}} e^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}} dt + \sigma dW, \tag{5}$$

where $y = ln(S/K)$ and $a = \mu + \frac{\sigma^2}{2}$. Since we expect the system to be driven by the drift's singularity, we assume that $a = 0$ and introduce the dimensionless variables

$$\begin{aligned} z &= \frac{y}{\sigma\sqrt{T}}, \\ z_0 &= \frac{y_0}{\sigma\sqrt{T}} = \frac{1}{\sigma\sqrt{T}} ln\left(\frac{S_0}{K}\right) \\ \beta &= \frac{E \cdot OI}{< V > \sqrt{2\pi\sigma^2 T}} \\ s &= t/T. \end{aligned} \tag{6}$$

With these new variables the SDE in (6) becomes

$$dz = -\frac{\beta z}{(1-s)^{3/2}} e^{-\frac{z^2}{2(1-s)}} ds + dW. \tag{7}$$

4.1 Solution of the model

We set $\tau = 1 - s$ and seek positive solutions of the Fokker-Planck equation

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{\beta z}{\tau^{3/2}} e^{-\frac{z^2}{2\tau}} \frac{\partial F}{\partial z}, \tag{8}$$

of the form

$$F(z, \tau) = \exp\left[\frac{1}{\sqrt{\tau}}\phi\left(\frac{z}{\sqrt{\tau}}\right)\right]. \tag{9}$$

Substituting this form in equation (9), we find that $\phi = \phi(\zeta)$ satisfies the SDE

$$\frac{\phi + \zeta\phi' + \phi''}{2\tau^{3/2}} + \frac{(\phi')^2 - 2\beta\zeta e^{-\frac{\zeta^2}{2}}\phi'}{\tau^2} = 0. \tag{10}$$

Since, as $\tau \to 0$ the second term in the equation is formally the dominant one, we consider the eikonal equation

$$(\phi')^2 - 2\beta\zeta e^{-\frac{\zeta^2}{2}}\phi' = 0,$$

which has the explicit solution

$$\phi(\zeta) = -2\beta e^{-\frac{\zeta^2}{2}}.$$

As it turns out, this function also makes the $O(\tau^{-3/2})$ term vanish. Therefore

$$F(z,\tau) = e^{-\frac{2\beta}{\sqrt{\tau}}\, e^{-\frac{z^2}{2\tau}}}. \tag{11}$$

is an exact solution of equation (8). In particular, the function

$$G_1(z,\tau) = 1 - F(z,\tau) = 1 - e^{-\frac{2\beta}{\sqrt{\tau}} e^{-\frac{z^2}{2\tau}}} \tag{12}$$

satisfies the Fokker-Planck equation (8), with initial condition

$$\begin{aligned} \lim_{\tau\to 0} G_1(z,\tau) &= 0 \quad z \neq 0 \\ &= 1 \quad z = 0. \end{aligned} \tag{13}$$

Hence, the analytical formula for the pinning probability is

$$P_{pinning} = \text{Prob.}\left\{\lim_{s\to 1}|z(s) = 0| \mid z(0) = z_0\right\} = 1 - e^{-2\beta\, e^{-\frac{z_0^2}{2}}}. \tag{14}$$

We note that this formula contains two adjustable parameters: z_0, the log-distance from the current price to the option's strike price measured in standard deviations, and β, the coupling constant, which is proportional to the dimensionless open interest $(OI/ < V >)$ and inversely proportional to the stock volatility and the time-to-expiration. In particular, it suggests that the presence of a large open interest gives rise to a large probability of pinning, as shown in Figure 8.

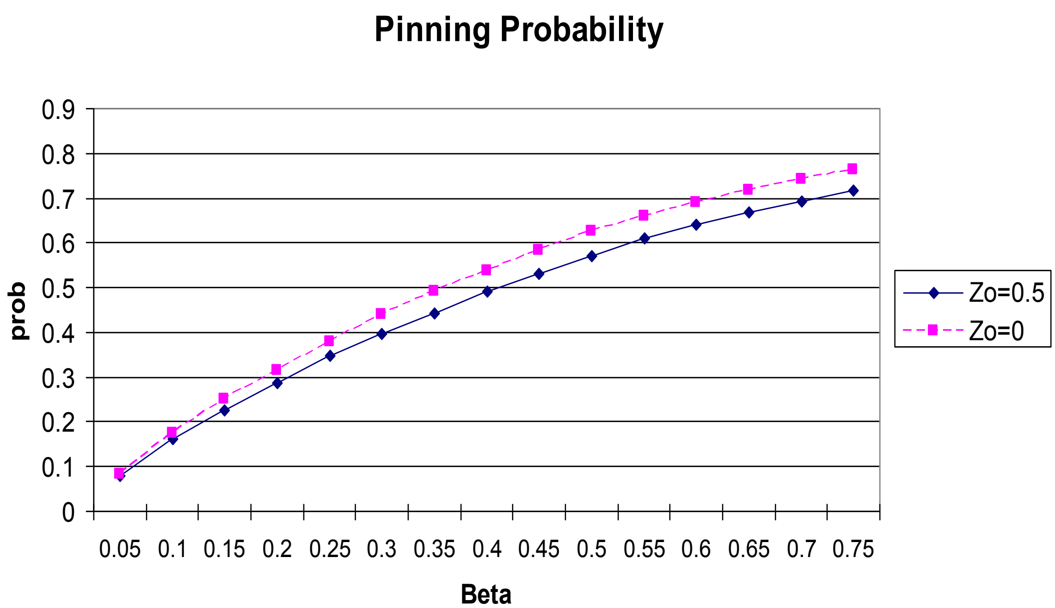

Figure 8: Pinning probability for the AL model (equation (14)) as a function of the dimensionless parameter $\beta = \frac{E \cdot OI}{<V>\sqrt{2\pi\sigma^2 T}}$. The curves show the function for $z_0 = 0$ and if $z_0 = 0.5$.

5 Empirical evidence in favor of the AL model

We know from NPP that pinning is associated with option expirations; and this is consistent with our model. However, we made a strong additional assumption to justify pinning: namely that market-makers are net long options near the expiration date. But is this actually the case?

We asked Ni, Pearson and Poteshman to analyze pinning along the lines of their empirical study taking into account the positions of market-makers, which is feasible to do using CBOE data. Their results, show in Figures 9 and 10, confirm our second hypothesis: if market-makers are net long options, the frequency of pinning at option expiration dates is much higher than if market-makers are net short.

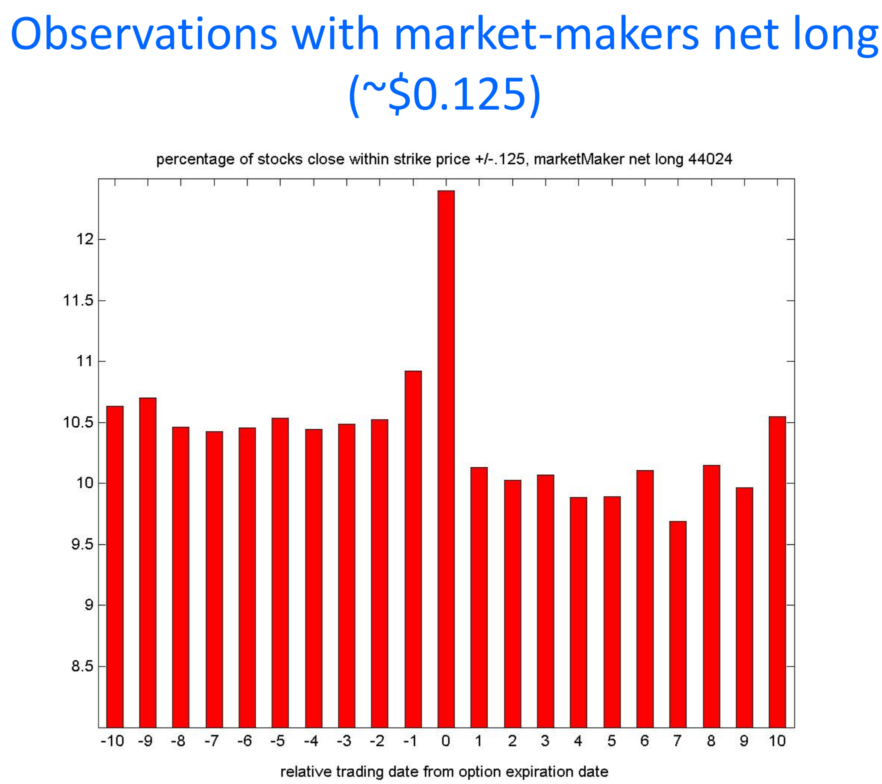

Figure 9: Frequency of pinning at the strike for expirations in which market-makers are net long options. (Courtesy of Ni, Pearson and Poteshman (2003)).

Additional empirical validation of the model was done by Lipkin and Stan-

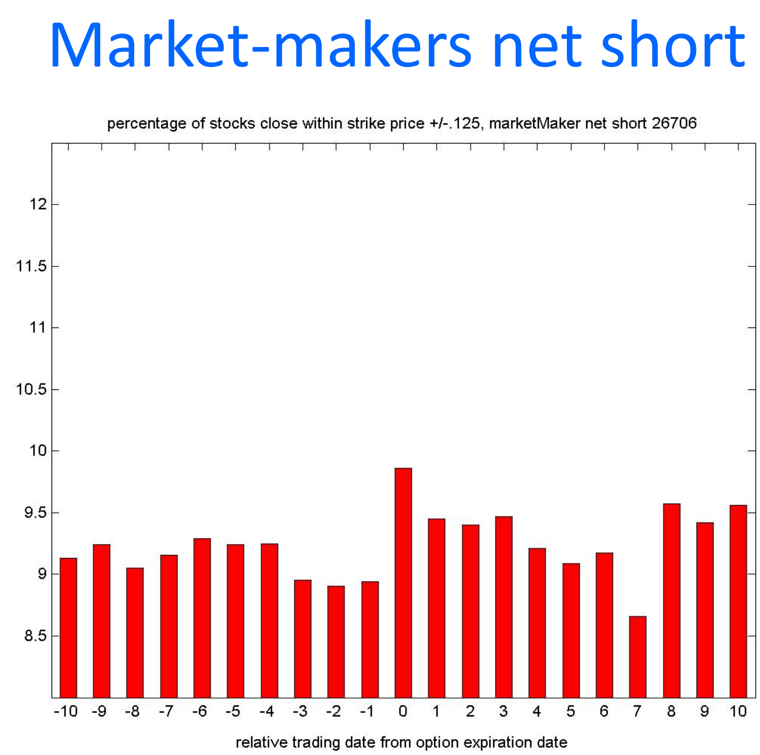

Figure 10: Frequency of pinning at the strike for expirations in which market-makers are net short options. (Courtesy of Ni, Pearson and Poteshman (2003)).

ton (2006) [7] in unpublished work. Using the *IVY OptionMetrics* data, they obtained clear empirical evidence of monotonicity of the pinning probability as a function of $OI/(<V>\sigma)$, consistently with Figure 8 and equation (14); see Figure 11.

6 Power-law model

We turn to the case in which price/demand elasticity is non-linear and follows a power law. Based on the previous considerations, we propose the generalization of the AL model:

$$\frac{dS}{S} = -E\,OI\left(\frac{1}{<V>}\frac{\partial\delta(S,t)}{\partial t}\right)^p sign\left(\frac{\partial\delta(S,t)}{\partial t}\right)dt + \sigma dW \tag{15}$$

or, in dimensionless variables,

$$dz = -\frac{\beta|z|^p sign(z)}{(1-s)^{3p/2}}e^{-\frac{pz^2}{2(1-s)}}ds + dW, \tag{16}$$

the coupling constant being $\beta = \frac{E\,OI}{<V>^p(2\pi\sigma^2 T)^{p/2}}$.

In (16), the drift of the SDE corresponds to a "restoring force" that blows up as $s \to 1$, favoring pinning at $z = 0$ for $s = 1$. However, this force is localized in a small neighborhood of the origin, due to the presence of the Gaussian cutoff function $e^{-\frac{pz^2}{2(1-s)}}$. The behavior of $Z(s)$ as s approaches 1 is the result of a tradeoff between these two effects: the restoring force which favors pinning and the localization with diffusion, which favors not pinning.

To formulate a hypothesis about the model's behavior, we performed Monte Carlo simulations to calculate numerically the probability of pinning for a trajectory starting at $z_0 = 0$ for different values of p and for fixed $\beta = 0.2$. The results indicate that there is no pinning for $p \leq 0.5$ and that pinning occurs for $p > 0.5$ following equation (3). The functional form (2) is strikingly apparent

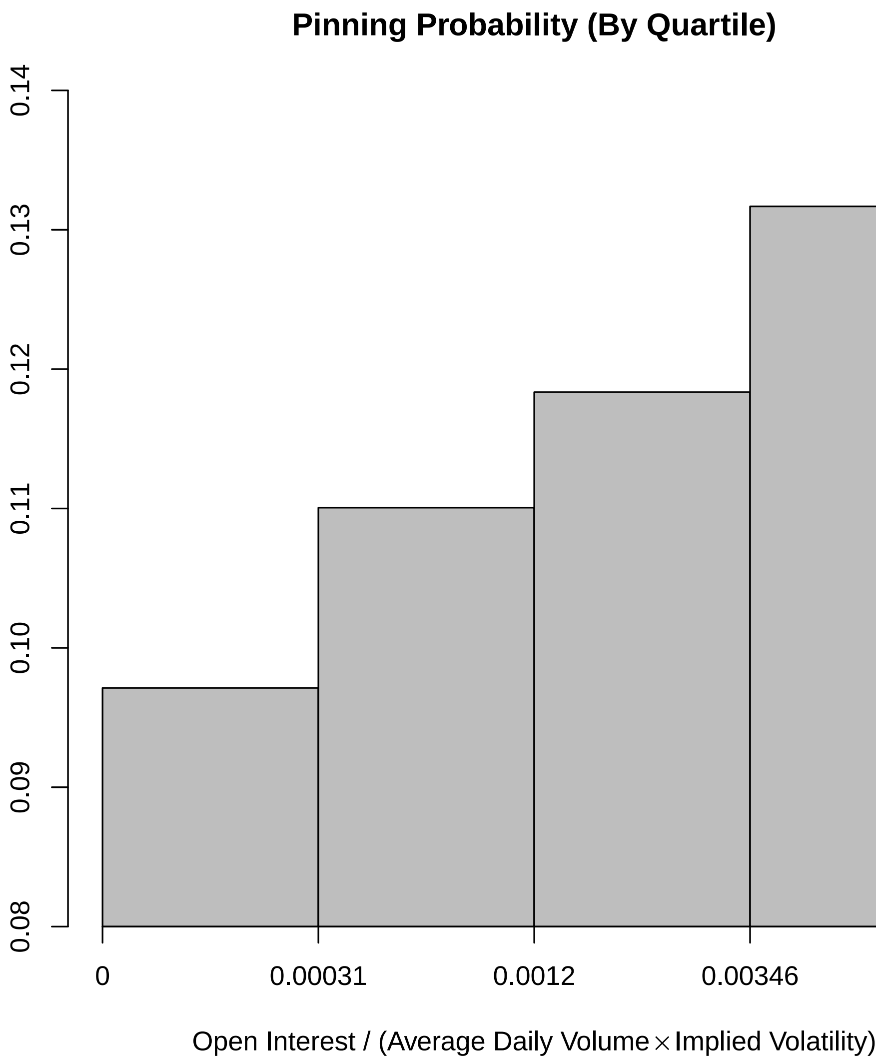

Figure 11: Empirical result showing the monotonicity of the pinning probability versus $\beta \propto \frac{OI}{<V>\sigma}$. (Courtesy of Lipkin and Stanton (2006) [7].

from the simulations, as seen in Figures 12, 13 and 14.

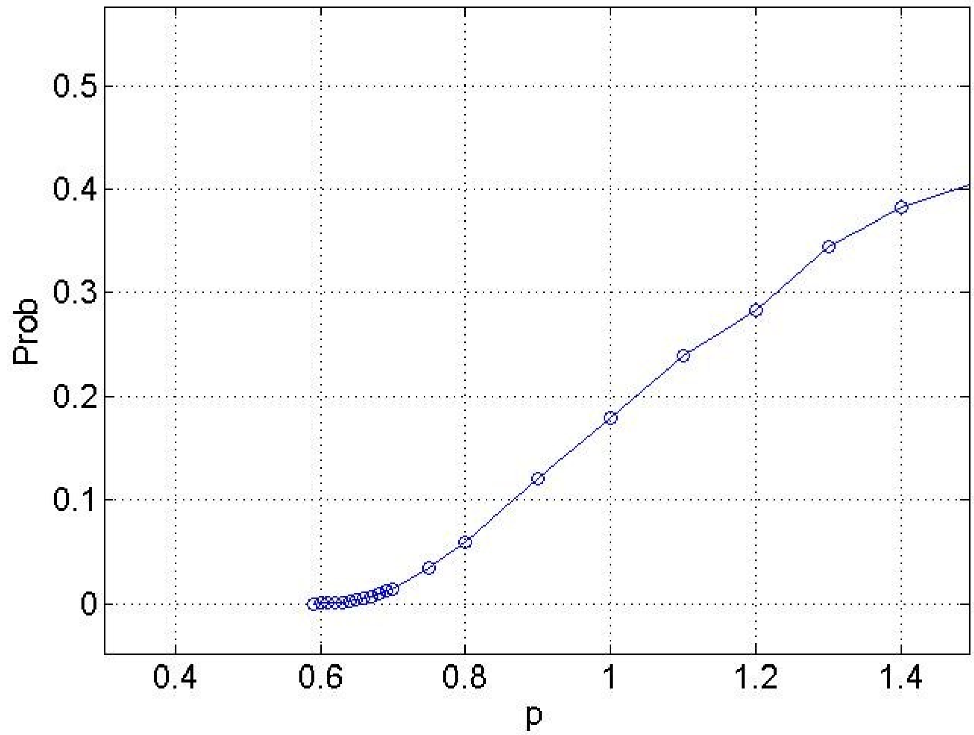

Figure 12: Pinning probability as a function of the parameter p for the power-law impact model. Each point corresponds to a simulation with a different value of p, with more points used near $p = 0.5$.

The associated Fokker-Planck equation for general values of p is given by

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{\beta |z|^p sign(z)}{\tau^{3p/2}} e^{-\frac{pz^2}{2\tau}} \frac{\partial F}{\partial z}. \tag{17}$$

A simple analytic solution of this equation such as (14) does not appear to exist for $p \neq 1$. Nevertheless, suppose that (17) admits a solution with the boundary condition (13), which we denote by $G_p\left(z, \tau, \beta\right)$. Dimensional analysis implies that $G_p(z, , \beta)$ satisfies the RG identity

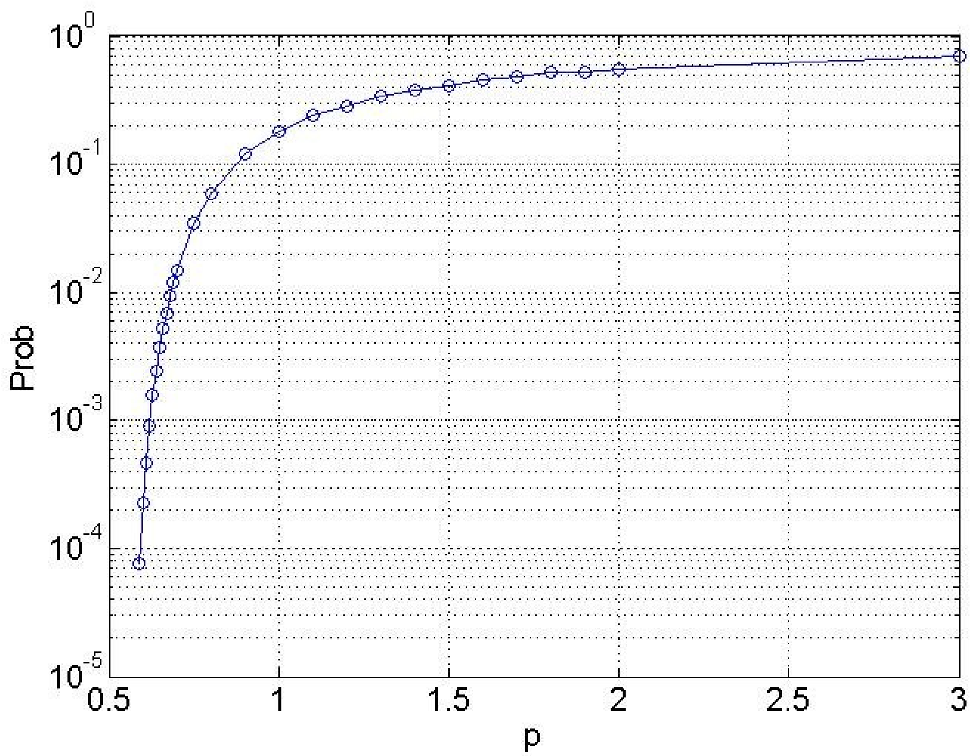

Figure 13: Same as Figure 12, but with pinning probability plotted on a log scale.

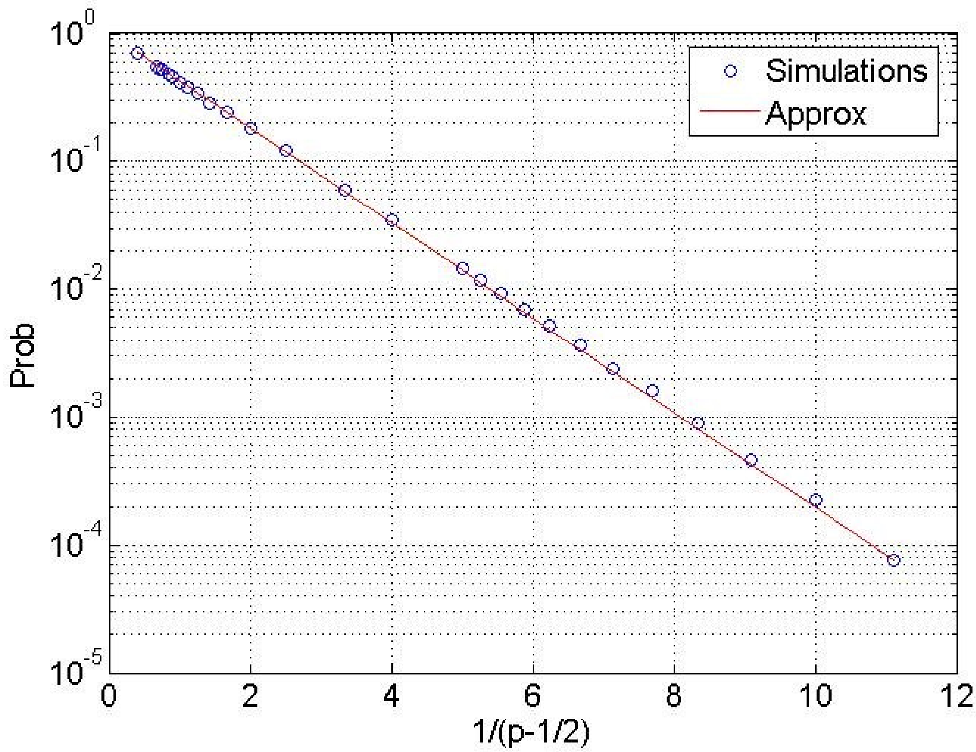

Figure 14: Same as Figure 12. Logarithm of the pinning probability plotted against $\frac{1}{2p-1}$.

$$G_p\left(\alpha z, \alpha^2\tau, \beta\right) = G_p\left(z, \tau, \frac{\beta}{\alpha^{2p-1}}\right). \tag{18}$$

Inspired by the numerical results, we shall use Large Deviations and RG analysis to study the model rigorously for $1/2 \leq p \leq 1$.

We shall prove the following result:

Theorem: *Let $z(s)$ be the solution of the stochastic differential equation*

$$dz = -\frac{\beta|z|^p sign(z)}{(1-s)^{3p/2}} e^{-\frac{pz^2}{2(1-s)}} ds + dW, \quad z(0) = z_0, \quad 0 \leq s < 1 \tag{19}$$

with $\beta > 0$.

(i) If $p < 1/2$, there is no pinning, i.e.,

$$Prob\left\{\lim_{s\to 1} |z(s)| = 0 \mid \ \ z(0) = z_0\right\} = 0$$

for all z_0.

(ii) (Lower bound). Let $0.5 < p$. There exists positive constants C_1 and C_2, depending only on β, such that

$$Prob.\left\{\lim_{s\to 1} |z(s)| = 0 \mid z(0) = 0\right\} > C_1 e^{-\frac{C_2}{(p-0.5)}}. \tag{20}$$

(iii) (Upper bound). Let $1/2 \leq p \leq 1$. There exist constants C_3, C_4 depending only on β but not on p such that

$$Prob.\left\{\lim_{s\to 1} |z(s)| = 0 \mid z(0) = 0\right\} < C_3 e^{-\frac{C_4}{(p-0.5)}}. \tag{21}$$

In particular, there is no pinning for $p = 1/2$.

6.1 Absence of pinning for $p < 1/2$

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The magnitude of the drift $V(z,s)$ of the SDE (19), satisfies

$$\beta \frac{|z|^p}{\tau^{3p/2}} e^{-\frac{pz^2}{2\tau}} \le \frac{\beta e^{-p/2}}{\tau^p} < \frac{\beta}{\tau^p}.$$

If $p < 1/2$, $V(z,s)$ is square-integrable in the interval $[0,1]$ and, furthermore,

$$\int_0^1 (V(z_s,s))^2 ds < \beta^2 \int_0^1 \frac{ds}{(1-s)^{2p}} = \frac{\beta^2}{1-2p}. \tag{22}$$

Therefore, for any constant $c > 1$, we have

$$E\left\{ e^{c\int_0^1 (V(z_s,s))^2 ds} \right\} < e^{c\,\frac{\beta^2}{1-2p}},$$

so the drift satisfies Novikov's condition [3] for absolute continuity of the process $z(\cdot)$ with respect to standard Brownian motion. This rules out pinning for $p < 0.5$.

6.2 Technical lemma for the lower bound

Our proof of Part (ii) of the Theorem makes use of a technical lemma which provides an upper bound for the exit probability of the process $z(s)$ from a "standardized" parabolic space-time region.

Lemma 1: *Let Ω denote the region in the z,s-plane defined by*

(i) $0 \le s \le 3/4$,

(ii) $|z| \le 2\sqrt{1-s}$,

(see Figure 15) and let θ be the first exit time of $z(\cdot)$ from Ω. Then

$$\limsup_{\beta\to\infty} \frac{1}{\beta} \ln Prob.\{\theta < 3/4 \text{ or } |z(3/4)| > 1/2 \ \big|\, |z(0)| < 1\} = -I \tag{23}$$

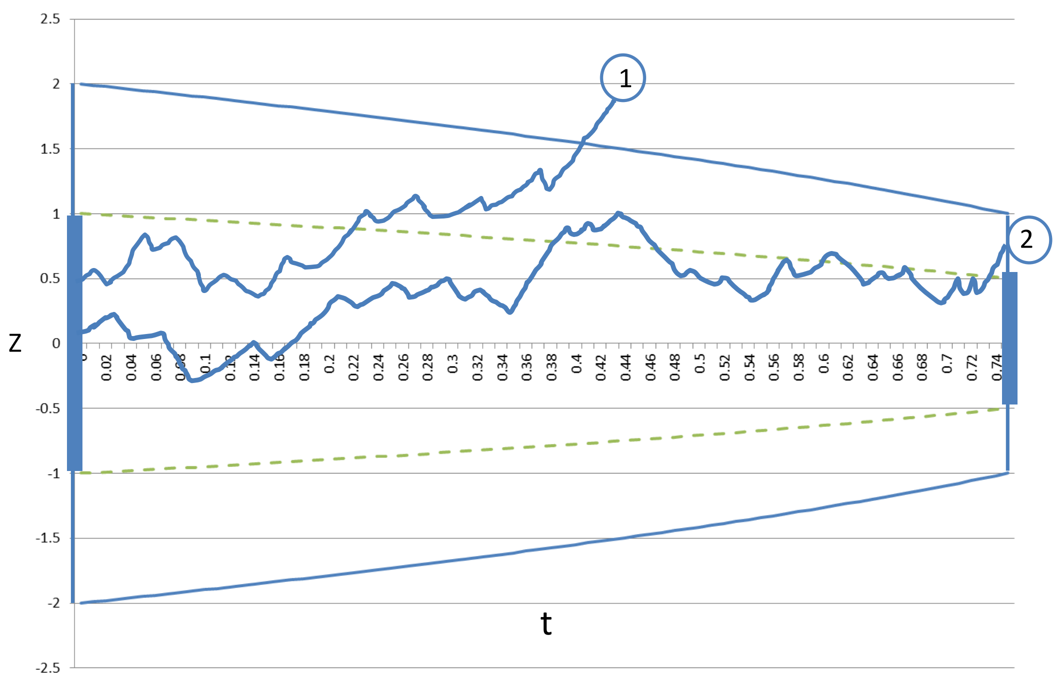

Figure 15: The parabolic region Ω used in the proof of Lemma 1. The main statement of the lemma is that, for large values of β, paths which start at $|z| < 1$ are most likely to end at $|z(3/4)| < 1/2$ without exiting Ω. In particular, paths which either (1) exit before time $t = 3/4$, or (2) end outside $|z(3/4)| < 1/2$ have exponentially small probability of the order of $exp(-\beta A)$, where A is the Ventsel-Freidlin action. This action is uniformly bounded away from zero.

where I is a constant independent of p and β.

Proof: We set

$$\xi(t) = z(t/\beta), \quad 0 \le t \ \le 3\beta/4.$$

This process satisfies the stochastic differential equation

$$\begin{aligned} d\xi(t) &= -U(\xi(t),t)dt + dW(t/\beta) \\ &= -U(\xi(t),t)dt + \frac{1}{\sqrt{\beta}}dZ(t), \qquad 0 \le t \le 3\beta/4. \end{aligned} \tag{24}$$

where $U(\xi,t) = \frac{|\xi|^p sign(\xi) e^{-\frac{\xi^2}{2(1-t/\beta)}}}{(1-t/\beta)^{3p/2}}$ and $Z(t)$ is a Wiener process. If we consider the region $\Omega_\beta = \left\{(\xi,t) : \xi < 2\sqrt{1-t/\beta}, 0 \le t \le \frac{3\beta}{4}\right\}$, the estimate that we seek corresponds to the first-exit time of $(\xi(t),t)$ from this region, where $\xi(t)$ is a diffusion process with small diffusion constant $\frac{1}{\sqrt{\beta}}$.

According to Ventsel-Freidlin (1970) [10], the probability that a trajectory $\xi(\cdot)$ remains in a tube-like neighborhood of a given path $\gamma(t), 0 \le t \le \infty$ until time $t = \frac{3\beta}{4}$ is given, for $\beta \gg 1$, by the "action asymptotics"

$$P\left\{\text{tube around } \gamma(\cdot)\right\} \approx e^{-\beta A(\gamma)}$$

with

$$A(\gamma) = \frac{1}{2}\int_0^\infty \left(\gamma'(t) - U(\gamma(t),t)\right)^2 dt.$$

We claim that the actions corresponding to the event of interest are bounded uniformly bounded away from zero. To see this, we note that U is uniformly bounded in the region of interest and satisfies

$$U(\xi) \ge |\xi|^p e^{-2}.$$

In particular, the characteristic paths $\xi' = -U(\xi, t)$ are such that $|\xi(t)| < |\omega(t)|$ where $\omega' = -|\omega|^p e^{-2}$. The latter ODE has the explicit solution

$$\omega(t) = (1-p)^{\frac{1}{1-p}} \left[\frac{(\omega(0))^{1-p}}{1-p} - e^{-2} t \right]^{\frac{1}{1-p}} . \tag{25}$$

Notice that as $t \to \infty$, the latter trajectory hits $\omega = 0$ in finite time $t < \frac{3\beta}{4}$. Therefore, the characteristic ("zero-diffusion") paths starting in the interval $|\xi(0)| < 1$ also reach zero in finite time.[3] For β sufficiently large, they exit the region through $\xi = 0$ at time $t = \frac{3\beta}{4}$. This shows that the action $A(\gamma)$ of paths which exits Ω_β before $t = 3\beta/4$, or with an absolute value greater than 1/2 for $t = 3\beta/4$, is bounded from below by a positive constant, I. We leave it to the reader to verify that the constant can be chosen independently of p and β, thus establishing the Lemma.

6.3 Proof of the lower bound

We consider the parabolic region

$$\Gamma = \left\{ (z, s) : |z| \leq 2\sqrt{1-s},\ 0 \leq s \leq 1 \right\}, \tag{26}$$

which is shown in Figure 16. The proof of the lower bound is based on estimating the probability that the path $z(\cdot)$ remains inside the region Γ, which clearly implies pinning.

Let D be the event that the space-time process $(z(s), s)$ remains inside Γ, i.e.,

$$D = \{ (z(\cdot), \cdot) \in \Gamma \},$$

and let $t_n = 1 - (1/4)^n,\ n = 0, 1, 2...$ We denote the probability measure asso-

[3] A similar comparison argument can be made for $p \geq 1$.

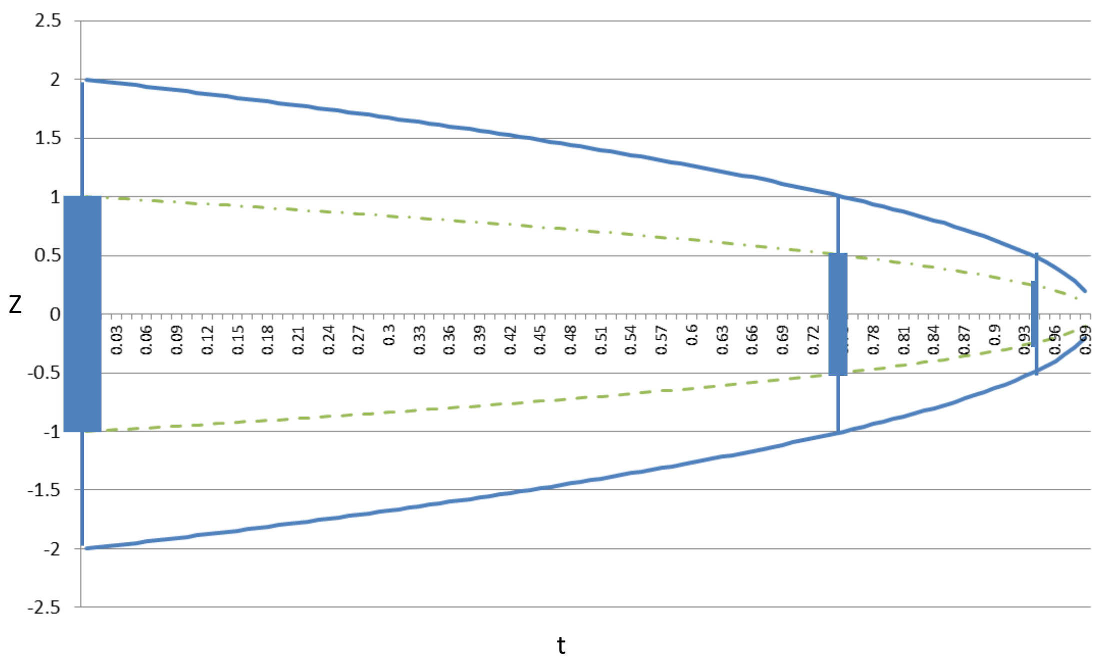

Figure 16: Representation of the region Γ used in the proof of the lower bound. The strategy of the proof is to estimate the probability that a path $x(s),\ 0 \leq s \leq 1$ remains confined to the region and also passes through the highlighted segments. The region can be viewed as an infinite union of parabolically "homothetic" regions which map to the standardized region Ω after the scaling transformation (28).

ciated with $z(\cdot)$ by P_β to emphasize the dependence on the coupling constant. Then

$$P_\beta(D) > P_\beta \left\{ D; |z(t_n)| < \frac{1}{2^n}, \forall n \right\},$$

$$\begin{aligned}
&= \int_{-1/2}^{1/2} P_\beta \left\{ (z(s), s) \in \Omega; z(t_1) = x \right\} \, P_\beta \left\{ D; |z(t_n)| < \frac{1}{2^n}, \forall n \geq 1 | z(t_1) = x \right\} dx \\
&> \left(1 - ce^{-\beta I} \right) \, P_\beta \left\{ |z(s)| \leq 2\sqrt{1-s}, s \geq 3/4, |z(3/4)| < 1/2 \right\} \\
&= \left(1 - ce^{-\beta I} \right) P_{\beta_1}(D), \qquad (27)
\end{aligned}$$

where $\beta_1 = \beta 2^{2p-1}$. The third line follows from Lemma 1, where c is a constant independent of β. The last line follows from the fact that the diffusion equation governing the process is invariant under the scaling transformation

$$\begin{aligned}
z_1 &= 2z \\
\tau_1 &= 4\tau \\
\beta_1 &= \beta 2^{2p-1}, \qquad (28)
\end{aligned}$$

(see equation (18)) . Iterating this last result, we obtain the lower bound

$$P_\beta(D) > \prod_{n=0}^{\infty} \left(1 - ce^{-I\beta(2^{2p-1})^n} \right). \qquad (29)$$

Let us evaluate this infinite product as a function of p. We have[4]

$$\begin{aligned}
\ln P_\beta(D) &> \sum_{n=0}^{\infty} \ln\left(1 - ce^{-I\beta(2^{2p-1})^n}\right) \\
&\geq -c\sum_{n=0}^{\infty} e^{-I\beta(2^{2p-1})^n} - \frac{c^2}{2}\sum_{n=0}^{\infty} e^{-2I\beta(2^{2p-1})^n} \\
&> -c\int_0^{\infty} e^{-I\beta(2^{2p-1})^x}dx - \frac{c^2}{2}\int_0^{\infty} e^{-2I\beta(2^{2p-1})^x}dx - (c+\frac{c^2}{2}) \\
&= -\frac{1}{(2p-1)}\frac{1}{\ln 2}\left[c\int_1^{\infty}\frac{e^{-I\beta u}}{u}du + c^2\int_1^{\infty}\frac{e^{-2I\beta u}}{u}du\right] - (c+\frac{c^2}{2}).
\end{aligned} \tag{30}$$

This establishes the desired lower bound for the pinning probability for $p > 1/2$. Notice that this implies that there is a “phase transition” at $p = 1/2$, since the lower bound implies that pinning occurs for p greater than $1/2$.

It remains to show that the exponential form associated with the lower bound also holds as an upper bound, as suggested by the numerical experiments.

6.4 Two more technical lemmas

We begin with:

Lemma 2: *Let $U_1(z,\tau)$ and $U_2(z,\tau)$ be two positive functions such that $U_1(z,\tau) < U_2(z,\tau)$ for all (z,τ), and let $\psi_0(z)$ be an even function which is decreasing for $z > 0$. Let $\psi_i,\ i = 1,2$ denote the corresponding solutions of the Cauchy problem*

$$\frac{\partial\psi_i}{\partial\tau} = \frac{1}{2}\frac{\partial^2\psi_i}{\partial z^2} - sign(z)U_1\frac{\partial\psi_i}{\partial z}, \quad z \in R \quad \tau > 0,$$

[4]We use the estimate $\sum_{n=1}^{\infty} f(n) \leq \int_0^{\infty} f(x)dx$ for non-negative decreasing functions $f(x)$.

with $\psi_i(z,0) = \psi_0(z), i = 1,2$. *Then,*

$$\psi_1(z,\tau) \ \leq \psi_2(z,\tau) \quad \forall z \ \forall \tau.$$

Proof: The proof follows immediately from the Maximum Principle applied to the PDE satisfied by the function $\psi_1(z,\tau) - \psi_2(z,\tau)$.

Lemma 2 is useful to formalize the intuition that, as p increases, the probability of pinning should increase as well. To show this, we introduce a "modified drift" which is always greater than unity (as opposed to the drift in the model, which may take small values).

Let $G_p(z,\tau,\beta)$ represent the solution of the Fokker-Planck equation (17) with initial condition

$$\begin{aligned} G_p(z,\tau,\beta) &= 1 \ \text{ if } \ z = 0 \\ &= 0 \ \text{ if } \ z \neq 0, \end{aligned} \tag{31}$$

and let $\hat{G}_p(z,\tau,\beta)$ be the solution of the auxiliary PDE

$$\frac{\partial \hat{G}_p}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \hat{G}_p}{\partial z^2} - \beta sign(z) U(z,\tau)^p \frac{\partial \hat{G}_p}{\partial z},$$

where

$$U(z,\tau) = 1 + \frac{|z|}{\tau^{\frac{3}{2}}} e^{-\frac{z^2}{2\tau}},$$

satisfying the same boundary conditions (31). (The modified drift alluded to above is $-\beta \, sign(z) \, U(z,\tau)^p$.) Lemma 2 implies that

$$G_p(z,\tau,\beta) \leq \hat{G}_p(z,\tau,\beta).$$

Moreover, since the function $1+U$ is greater than 1, $(1+U)^p$ is an increasing function of p. Hence, also by Lemma 2, we have

$$\hat{G}_p(z,\tau,\beta) \leq \hat{G}_1(z,\tau,\beta). \tag{32}$$

Let $E(\cdot)$ denote the expectation value with respect to the probability distribution of $z(s)$. To evaluate the right-hand side of (32), we use Girsanov's theorem and the Cauchy-Schwartz inequality:

$$\begin{aligned}
\hat{G}_1(z,\tau,\beta) &= E\left\{ e^{\int_{1-\tau}^{1} \beta sign(z(s))dW - \frac{\beta^2\tau}{2}} ; \lim_{s\to 1} |z(s)| = 0 \mid z(1-\tau) = z \right\} \\
&< \left[E\left\{ e^{2\int_{1-\tau}^{1} \beta sign(z(s))dW - \beta^2\tau} \right\} \right]^{1/2} \left[E\left\{ \lim_{s\to 1} |z(s)| = 0 \mid z(1-\tau) = z \right\} \right]^{1/2} \\
&= e^{\frac{\tau\beta^2}{2}} \left[G_1(z,\tau,\beta) \right]^{1/2}.
\end{aligned}$$

Therefore, using equation (14), we have

Lemma 3.

$$\hat{G}_p(z,\tau,\beta) < e^{\frac{\tau\beta^2}{2}} \left[G_1(z,\tau,\beta) \right]^{1/2} = e^{\frac{\tau\beta^2}{2}} \left[1 - \exp\left(-\frac{2\beta \; e^{-\frac{z^2}{2\tau}}}{\sqrt{\tau}} \right) \right]^{1/2} \tag{33}$$

6.5 Proof of the upper bound

We make use of the renormalization identity (18) with $\alpha = 2^{\frac{1}{2p-1}}$. Accordingly,

$$G_p\left(z\, 2^{\frac{1}{2p-1}}, \tau\, 2^{\frac{2}{2p-1}}, \beta \right) = G_p\left(z, \tau, \frac{\beta}{2} \right);$$

so

$$G_p\left(z\, 2^{\frac{1}{2p-1}}, \tau\, 2^{\frac{2}{2p-1}}, 2\beta \right) = G_p\left(z, \tau, \beta \right). \tag{34}$$

In particular, the probability of pinning starting at $z = 0$ satisfies

$$Prob.\{z(1) = 0|z(0) = 0\} = G_p\left(0, 1, \beta\right) = G_p\left(0, 2^{\frac{2}{2p-1}}, 2\beta\right). \tag{35}$$

We now make use of Lemma 3. Accordingly,

$$\begin{aligned}
Prob.\{z(1) = 0|z(0) = 0\} &= G_p\left(0, 2^{\frac{2}{2p-1}}, 2\beta\right) \\
&< \hat{G}_p\left(0, 2^{\frac{2}{2p-1}}, 2\beta\right) \\
&< e^{\beta^2/2}\left[1 - e^{-\frac{4\beta}{2^{\frac{1}{2p-1}}}}\right]^{1/2} \\
&< e^{\beta^2/2}\,\frac{2\sqrt{\beta}}{2^{\frac{1}{2(2p-1)}}} \\
&= 2\sqrt{\beta}\,e^{\beta^2/2}\,e^{-\frac{\ln 2}{2(2p-1)}},
\end{aligned} \tag{36}$$

which is what we wanted to show. This concludes the proof of the upper bound for $p > 1/2$. The absence of pinning at exactly $p = 1/2$ follows from similar considerations, since $G_{1/2}(z, \tau, \beta) < \hat{G}_p(z, \tau, \beta)$ for any $p > 1/2$.

7 Conclusions

The model for stock pinning near option expiration dates introduced in Avellaneda and Lipkin (2003) was generalized to the case of non-linear, power-law, price impact functions. The price trajectories of stocks are affected by Delta-hedging by market-makers which, in the case of large number of options, can impact the price of the stock, driving it to the strike price.

Mathematically, pinning is described by a stochastic differential equation with a drift similar to the SDE for the Brownian bridge, except for the fact that the drift is localized in a neighborhood of the strike price. The result is a competition between “pinning to the strike” due to the drift and “escaping” via diffusion. We show that the model has interesting behavior in terms of the parameter p and, in particular, pinning occurs only for values of p greater than 1/2. Furthermore, we were able to characterize precisely the behavior of the model in a neighborhood of the critical value $p = 1/2$ via Monte Carlo simulations and through rigorous upper and lower asymptotic estimates for the probability of pinning.

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