Path-Dependence Properties of Leveraged Exchange-Traded Funds: Compounding, Volatility and Option Pricing

by

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Marco Avellaneda-Advisor

To my parents.

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Abstract

This thesis studies leveraged exchange-traded funds (ETF) and options written on them.

In the first part, we give an exact formula linking the price evolution of a leveraged ETF (LETF) with the price of its underlying ETF. We test the formula empirically on historical data for 56 leveraged funds (44 double-leveraged, 12 triple-leveraged) using daily closing prices. The results indicate excellent agreement between the formula and the empirical data. The formula shows that the evolution of the price of an LETF is sensitive to the realized volatility of the underlying product. The relationship between an LETF and its underlying asset is "path-dependent."

The second part of the study focuses on the relations between options on LETFs and options on the underlying ETFs. The main result shows that an option on an LETF can be replicated by a basket of options on the underlying ETF after a suitable choice of strikes and notionals. In particular, we obtain a new, relative-value, model for pricing LETF options. The derivation makes strong use of the path-dependency result of Part I. As a consequence, we derive a simple non-parametric formula which links the volatility skew of an LETF with the volatility skew of the underlying ETF.

We validate the theory empirically by showing that the model prices for options on LETFs are in excellent agreement with actual mid-market prices observed in markets. The empirical study was carried out on two LETFs linked to the S&P 500 index (one double-leveraged, one reverse-double-leveraged). The issue of vega-hedging options on LETFs with options on the underlying ETFs is also examined from this viewpoint.

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Chapter 1

Introduction to ETFs and Leveraged ETFs

1.1 Exchange Traded Funds (ETFs)

An exchange-traded fund (or ETF) is an investment fund traded on stock exchanges, much like stocks. An ETF holds assets, such as stocks or bonds, and trades at approximately the same price as the net asset value of its underlying assets over the course of the trading day. ETFs have been available in the US since 1993 and in Europe since 1996. ETFs traditionally have been index funds. In January 1993, SPY was introduced to track the S&P 500 index. In 1998, the "Dow Diamonds" (DIA) were introduced, tracking the notable Dow Jones Industrials Average. In 1999, the influential "cubes" (NASDAQ: QQQQ) were launched, attempting to replicate the movement of the NASDAQ-100. As of February 2008, index ETFs in the United States included 415 domestic equity ETFs, with assets of $350 billion; 160 global/international equity ETFs, with assets of $169 billion; and 53 bond ETFs, with assets of $40 billion.

Unlike traditional mutual funds, ETFs do not sell or redeem their individual shares at net asset value, or NAV. Instead, so-called authorized participants (usually large financial institutions), purchase and redeem shares directly from the ETF fund manager, but only in large blocks. Purchases and redemptions of the creation units generally are in kind, with the institutional investor contributing or receiving a basket of securities of the same type and proportion held by the ETF. Authorized participants usually act as market makers in the open market, using their ability to exchange creation units with their underlying securities to help ensure that their intraday market price approximates the net asset value of the underlying securities.

ETFs generally provide the easy diversification, low expense ratios, and tax efficiency of index funds, while still maintaining all the features of ordinary stock, such as limit orders, short selling, and options. Because ETFs can be economically acquired, held, and disposed of, some investors invest in ETF shares as a long-term investment for asset allocation purposes, while other investors trade ETF shares frequently to implement market timing investment strategies.

1.2 Leverage is the Key

In March 2008, the U.S. Securities and Exchange Commission began to authorize the creation of actively managed ETFs. Among them, leveraged ETFs have been drawing considerable amount of interest from investors, active traders, and portfolio managers. Leveraged exchange-traded funds (LETFs), or simply leveraged ETFs, are a special type of ETFs that attempt to achieve returns that are more sensitive to market movements than non-leveraged ETFs. There are two types of leveraged ETFs, bull and bear. A leveraged bull ETF attempts to achieve a daily return of 2 or 3 times of the daily return of the underlying index. For example, ProShares Ultra Financial ETF (UYG) offers investors double of the Dow Jones U.S. Financials index, while Direxion Daily Financial Bull 3X Shares (FAS) replicates triple of the same index. A leveraged inverse (bear) ETF, on the other hand, may attempt to achieve the return that is -2 or -3 times of the daily index return, meaning that it will gain double or triple the *loss* of the market. For instance, ProShares UltraShort Financial ETF (SKF) offers -2 times the Dow Jones Financials Index, while Direxion Daily Financial Bear 3X Shares (FAZ) tracks -3 times the same index. From the point of view of traders who are subjected to Reg T margin, these instruments provide a simple way of doubling or tripling exposure to an index while using the same amount of capital. Also, active traders can use an inverse leveraged ETF as a substitute for short-selling the underlying assets while the latter are hard-to-borrow. For instance, many traders took long positions in SKF, a financial bearish fund, toward the end of 2008, when financial stocks were difficult or even impossible to short.

Classical ETFs track an index or basket in a one-for-one fashion; they are essentially passively managed. In contrast, LETFs require active management: this involves borrowing funds to purchase additional shares (bullish LETFs), or short-selling (bearish LETFs) and rebalancing the position on a daily basis. Managers sometimes simplify the hedging of LETFs by entering into a daily resetting total-return swap with qualified counterparties [1].

Throughout 2009, issues have been raised in the marketplace regarding the suitability of leveraged ETFs for long-term investors seeking to replicate a multiple

[1]The description of the hedging mechanism given here is not intended to be exact, but rather to illustrate the general approach used by ETF managers to achieve the targeted leveraged long and short exposures. For instance, managers can trade the stocks that compose the ETFs or indices, or enter into total-return swaps to synthetically replicate the returns of the index that they track. The fact that the returns are adjusted daily is important for our discussion. Recently Direxion Funds, an LETF manager, has announced the launch of products with monthly rebalancing.

of an index performance. UBS AG announced, in July 2009, that it would cease marketing LETFs "because such products do not conform to its emphasis on long-term investing"(Bloomberg News, July 2009). "Leveraged products are complex, carry substantial risks and are intended for short-term trading," a warning to customers on Fidelity's Web site said on Aug, 2009. "Most reset daily and seek to achieve their objectives on a daily basis. Due to compounding, performance over longer periods can differ significantly from the performance of the underlying index." The move was followed by other major U.S. brokers, like Morgan Stanley. The Financial Industry Regulatory Authority (FINRA) in June, 2009 issued a reminder to brokers and advisers, urging them to use care in selling inverse and leveraged ETFs, and disclose fully the risks inherent in these products (see [1][2] for more information). Subsequently, in August, the Securities and Exchange Commission issued an Alert notice [3].

The problem is that the fund manager incurs trading losses because he needs to buy when the index goes up and sell when the index goes down in order to maintain a fixed leverage ratio. This means the manager is "short Gamma", or short convexity.

It has been empirically established that if we consider investments over extended periods of time (e.g, three months, one year, or more), there are significant discrepancies between LETF returns and the returns of the corresponding leveraged buy-and-hold portfolios composed of index ETFs and cash (see [4]). To better understand the suitability issue, let's consider, for instance, UYG and its underlying ETF, IYF. Figure 1.1 compares the price history of UYG and that of a static double-leveraged position in IYF, with both funds starting with the same investment in Jan 2008.

Clearly, the two charts do not coincide. In particular, UYG clearly underper-

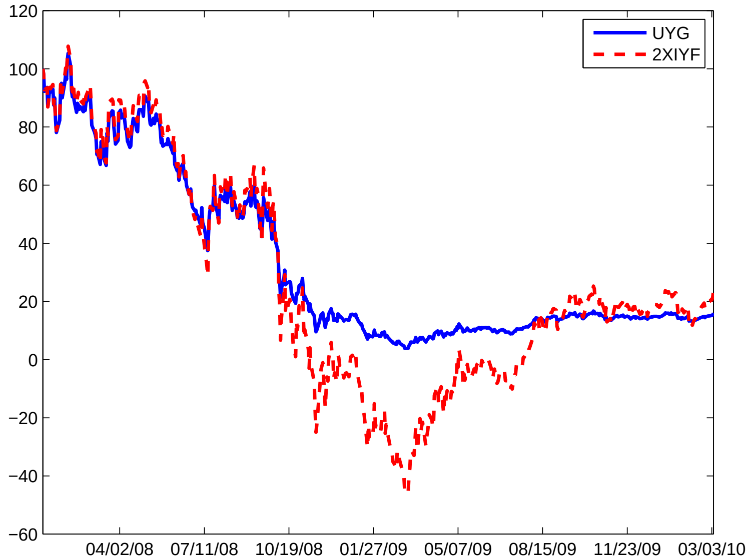

Figure 1.1: depicts the difference between the performance of UYG and a static double-leveraged position in IYF. Notice, in particular, that UYG does not grow like the double-leveraged position in the March 2009 rally.

forms the double leveraged IYF during the rally of Spring 2009. The corresponding chart for the reverse double leveraged fund (SKF versus -2 times IYF), shown in Figure 1.2, is even more dramatic.

Another interesting observation can be made by considering the price history of a pair consisting of a leveraged ETF and the corresponding reverse product with same leverage (e.g., UYG and SKF). Typically, the price charts should be "mirror images" of each other, at least over short periods of time. However, as we increase the time horizon, we see clearly that the graphs are no longer mirror images, and the correlation between the two products breaks down as time passes (see figures 1.3, 1.4 and 1.5). Mathematically speaking, the returns have correlation -1 for short periods of time, but the correlation diminishes as the horizon increases. We will discuss this in section 2.3.

Tracking daily returns is not the same as tracking long-term returns. As we

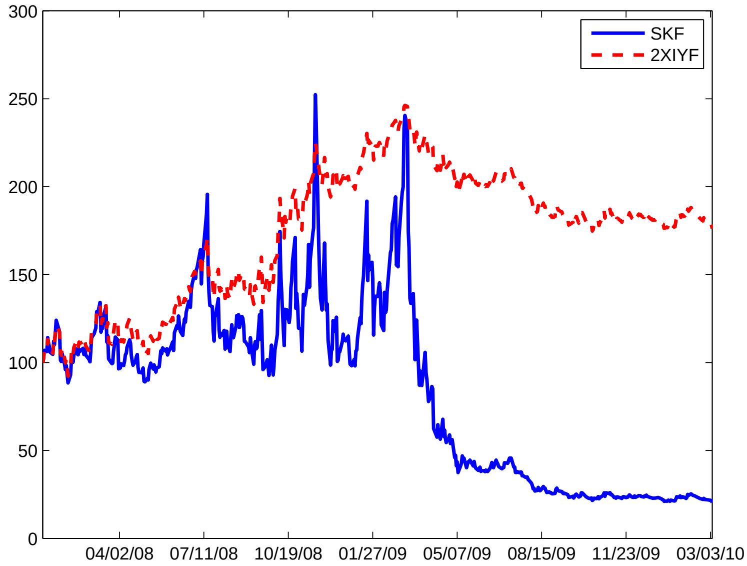

Figure 1.2: shows a comparison between the performance of SKF and a static double-leveraged short position in IYF. The discrepancy is remarkable.

shall see, there are two primary differences. First, there is an issue of compounding. We all know the difference between daily compounded and annually compounded interest rates. This effect is sometimes referred as convexity. In fixed income, compounding is tantamount to investing at higher returns more frequently, which implies a faster growth of capital. In the case of LETFs, there is a second important reason for the discrepancies seen in the charts: volatility. The more volatile the assets are, the larger the tracking error, as we shall see in the next section.

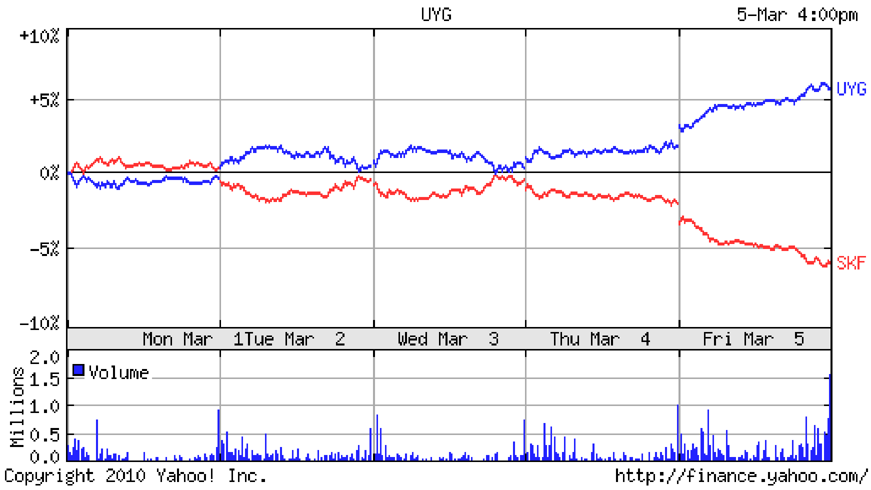

Figure 1.3: depicts a comparison of the charts of UYG and SKF for a period of 5 days, ending on March 5, 2010. Notice that the two charts are perfect mirror images of one another.

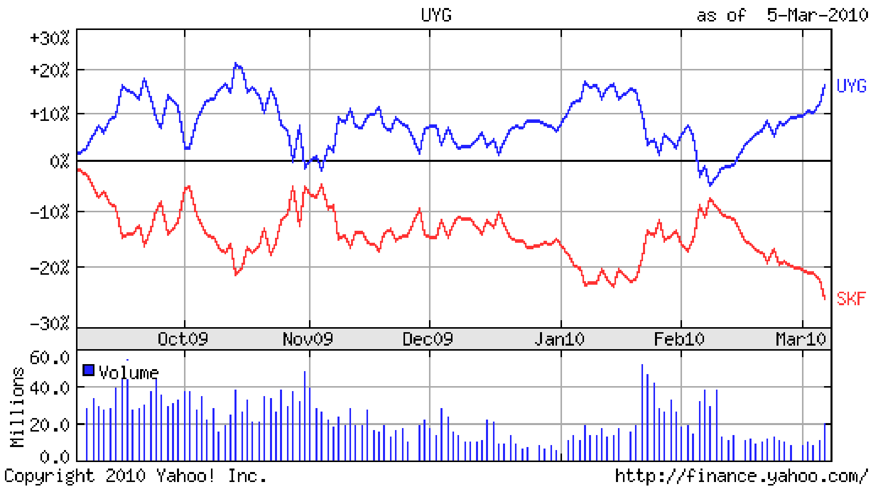

Figure 1.4: compares the charts of UYG and SKF for a period of 6 months, prior to March 5, 2010. Notice that the two charts are no longer perfect mirror images, with a slight under-performance by SKF.

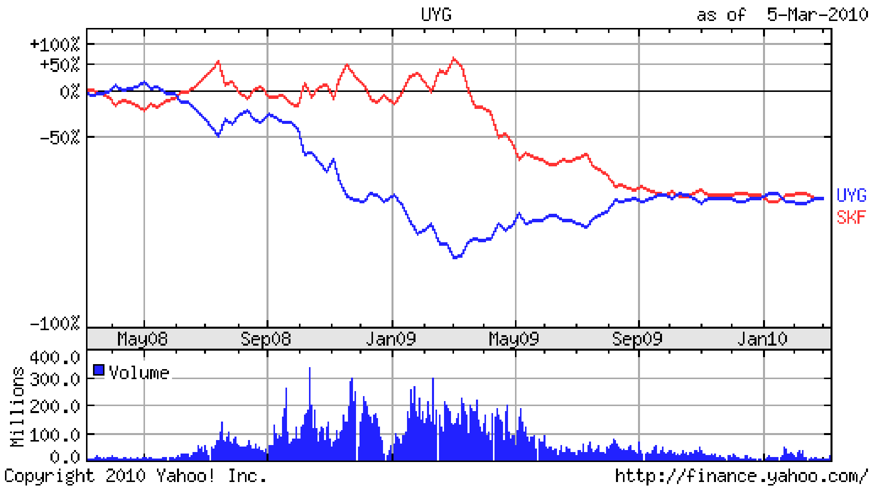

Figure 1.5: compares the charts of UYG and SKF for a period of 2 years, prior to March 5, 2010. Notice both funds actually have negative returns over the two-year period.

Chapter 2

Modeling Leveraged ETFs

To model the dynamics of LETFs, we denote the spot price of the underlying index or the ETF which tracks the underlying index by S_t, the leveraged ETF price by L_t, and the leverage ratio by β. For example, UYG corresponds to $\beta = 2$ and SKF will correspond to $\beta = -2$.

2.1 Discrete-time Model

Suppose there are N trading days, and denote the one-day return of the underlying ETF by R_i^S, and the one-day return of the LETF by R_i^L, where $i = 1, 2, 3, \ldots, N$. The LETF provides a daily exposure of β dollars of the underlying securities before fees and expenses. Accordingly, there is a link between R_i^S and R_i^L. If the LETF is bullish ($\beta > 0$)

$$R_i^L = \beta R_i^S - \beta r \Delta t - f \Delta t + r \Delta t \tag{2.1.1}$$

where r is the reference interest rate (for example, fed fund rate or 3-month LIBOR), $\Delta t = 1/252$ represents one trading day, and f is the expense ratio of the

LETF. Typical value of f ranges from 75 bps to 150 bps.

If the LETF is bearish ($\beta < 0$)

$$R_i^L = \beta R_i^S - \beta(r - \lambda_i)\Delta t - f\Delta t + r\Delta t \tag{2.1.2}$$

where the extra term $\lambda_i \Delta t$ represents the cost of borrowing components of the underlying index or the underlying ETF on day i. This cost is the difference between reference interest rate and rate applied to the cash proceeds from short-sales of the components of the underlying index or ETF. If the components of the underlying index or ETF are widely available for lending, the cost of borrowing is negligible. This cost of borrowing is important for calibrating the model to fit the time series of index or ETFs which are hard-to-borrow since 2008 [1].

Compounding the return of the LETF, we have

$$L_N = L_0 \prod_{i=1}^{N} (1 + R_i^L) \tag{2.1.3}$$

Substituting R_i^L in equation (2.1.1) and (2.1.2) (according to sign of β), we obtain a relationship between the price of the LETF and the underlying index or ETF. As shown in the Appendix (section 2.7.1), under mild conditions, we have

$$\frac{L_t}{L_0} \simeq \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(\frac{\beta - \beta^2}{2} V_t + \beta H_t + ((1-\beta)f + r)t\right) \tag{2.1.4}$$

where

$$V_t = \sum_{i=1}^{t} (R_i^S - \bar{R}^S)^2 \text{ with } \bar{R}^S = \frac{\sum_{i=1}^{t} R_i^S}{N}$$

[1] We emphasize the cost of borrowing, since we are interested in LETFs which track financial indices. The latter have often been hard-to-borrow since July 2008. Moreover, broad market ETFs such as SPY have also been sporadically hard-to-borrow in the last quarter of 2008; see Avellaneda and Lipkin [5].

i.e. V_t is the accumulative realized variance over the time t, and where

$$H_t = \sum_{i=1}^{t} \lambda_i \Delta t$$

represents the accumulative cost of borrowing the components of the underlying index or ETF. This cost, in practice, can be obtained by subtracting the average applicable "short rate" from the reference interest rate each day and accumulating it over the time period of interest. Notice, in addition to these two factors, formula (2.1.4) also shows dependence on the reference interest rate and the expense ratio of the LETF. We will show in the following paragraph, under the assumption that the price of the underlying ETF follows an Itô process, formula (2.1.4) is exact, meaning $\simeq$ can be replaced by $=$.

2.2 Continuous-time Model

Let's begin by assuming the price of the underlying ETF, S_t, follows a stochastic differential equation

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \tag{2.2.1}$$

where W_t is a standard Wiener process and σ_t, μ_t are the instantaneous volatility and drift respectively. Also, σ_t and μ_t are non-anticipative functions with respect to W_t and they can be random.

Following the same logic of (2.1.1), (2.1.2), if the LETF is bullish ($\beta > 0$), we have

$$\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} - ((\beta - 1)r + f)dt, \tag{2.2.2}$$

and if the LETF is bearish ($\beta < 0$), the equation becomes

$$\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} - ((\beta - 1)r - \beta\lambda + f)dt. \tag{2.2.3}$$

In the Appendix (section 2.7.2), we show the following formula holds:

$$\boxed{\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(((1-\beta)r - f)\,t + \beta \int_0^t \lambda_t ds + \frac{\beta - \beta^2}{2} \int_0^t \sigma_s^2 ds\right)} \tag{2.2.4}$$

where we assume $\lambda_t = 0$ if $\beta > 0$. Formula (2.1.4) and (2.2.4) are essentially the same if we define

$$V_t = \int_0^t \sigma_t^2 ds, \text{ and } H_t = \int_0^t \lambda_t ds.$$

The only difference is that equation (2.1.4) is only valid for $\Delta t \ll 1$, whereas formula (2.2.4) is exact, assuming the price process is an Itô process. These two equations show that the relationship between the LETF and its underlying index or underlying ETF depends on

1. the **realized variance** over the period of interest
2. the **leverage ratio** β
3. the **funding rate** or **reference interest rate**
4. the **expense ratio** of the LETF
5. the **cost of borrowing** in the case of bearish LETF

Items 3 and 4 do not require further explanation. Item 5 comes from the fact that the fund manager of the bearish ETF may have to incur additional financing cost to maintain short positions if the underlying components of the index are hard-to-borrow. Items 1 and 2 are the most interesting.

The dependence on the realized variance might be surprising at first[2]. It means that the holder of the LETF has a negative exposure to the realized variance of the underlying index or the underlying ETF [3]. This holds for all $\beta > 1$ or $\beta < 0$, which means both bullish and bearish LETFs suffer from realized variance. For instance, if $\beta = 2$, which means it's a double-long LETF, the term corresponds to the realized variance in formula (2.2.4) reads

$$-\frac{2^2 - 2}{2} \int_0^t \sigma_s^2 ds = -\int_0^t \sigma_s^2 ds.$$

In the case of $\beta = -2$, which means it's a double-short LETF, the corresponding term is

$$\frac{(-2)^2 - (-2)}{2} \int_0^t \sigma_s^2 ds = -3 \int_0^t \sigma_s^2 ds.$$

We notice, in particular, the dependence on the realized variance is stronger on bearish LETF, which means the bearish LETF with the same $|\beta|$ as bullish LETF suffers more from the accumulated realized variance.

2.3 Correlation Decay

Because of the common "time decay" for the both bullish and bearish LETF, we have a way to explain figures 1.3, 1.4 and 1.5. Let's consider a pair of bullish and bearish LETFs, which track the same underlying ETF. For example, ProShares Ultra Financial ETF (UYG) and ProShares UltraShort Financial ETF (SKF). The

[2] We find out that analogous results were obtained independently in a note issued by Barclays Global Investors [6], which contains a formula similar to (2.2.4) without including finance and expense ratios.

[3] Interestingly, similar considerations about volatility exposure apply to a class of over-the-counter (OTC) products known as Constant Proportion Portfolio Insurance (CPPI). These products are typically embedded in structured notes and other derivatives marketed by insurance companies and banks (see, for instance, Black and Perold [7], Bertrand and Prigent [8] and Zhu [9]). Although not directly relevant to this study, the parallel between the behavior of listed LETFs and OTC products may be of independent interest.

daily return of UYG is the inverse of the daily return of SKF. Denote the price of the bullish LETF and the bearish LETF by L_t^+, L_t^- respectively. For simplicity, we assume the reference interest rate, the expense ratio and the borrowing cost are all zero. Following the same argument in section (2.2), we have a formula which relates L_t^+ and L_t^-.

$$\frac{dL_t^+}{L_t^+} = -\frac{dL_t^-}{L_t^-}. \tag{2.3.1}$$

It follows that

$$\frac{L_t^+}{L_0^+} = \left(\frac{L_t^-}{L_0^-}\right)^{-1} e^{-\int_0^t \sigma_s^2 ds} \tag{2.3.2}$$

Notice equation (2.3.2) is the same as equation (2.2.4) if we take $\beta = -1$ and f, r, λ to be zero. We are interested in the correlation $\rho(t)$ between the return of L_t^+ and the return of L_t^- over a time period t. Mathematically,

$$\rho(t) = \frac{E\left[(\frac{L_t^+}{L_0^+})(\frac{L_t^-}{L_0^-})\right] - E\left[\frac{L_t^+}{L_0^+}\right] E\left[\frac{L_t^-}{L_0^-}\right]}{\sqrt{E\left[(\frac{L_t^+}{L_0^+} - E[\frac{L_t^+}{L_0^+}])^2\right] E\left[(\frac{L_t^-}{L_0^-} - E[\frac{L_t^-}{L_0^-}])^2\right]}} \tag{2.3.3}$$

If L_t^- follows an Itô process $\frac{dL_t^-}{L_t^-} = \sigma_t dW_t$, we have $\frac{L_t^-}{L_0^-} = e^{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds}$

$$E\left[(\frac{L_t^-}{L_0^-})^2\right] = E\left[e^{2\int_0^t \sigma_s dW_s - \int_0^t \sigma_s^2 ds}\right] \tag{2.3.4}$$

$$E\left[(\frac{L_t^+}{L_0^+})^2\right] = E\left[e^{-2\int_0^t \sigma_s dW_s + \int_0^t \sigma_s^2 ds} e^{-2\int_0^t \sigma_s^2 ds}\right] = E\left[e^{-2\int_0^t \sigma_s dW_s - \int_0^t \sigma_s^2 ds}\right] \tag{2.3.5}$$

$$E\left[(\frac{L_t^+}{L_0^+})(\frac{L_t^-}{L_0^-})\right] = E\left[e^{-\int_0^t \sigma_s^2 ds}\right] \tag{2.3.6}$$

$$E[\frac{L_t^-}{L_0^-}] = E[\frac{L_t^+}{L_0^+}] = 1 \tag{2.3.7}$$

If volatility is deterministic, (2.3.4) and (2.3.5) become $e^{\int_0^t \sigma_s^2 ds}$ and

$$\begin{aligned}
\rho(t) &= \frac{E\left[(\frac{L_t^+}{L_0^+})(\frac{L_t^-}{L_0^-})\right] - E\left[\frac{L_t^+}{L_0^+}\right] E\left[\frac{L_t^-}{L_0^-}\right]}{\sqrt{E\left[(\frac{L_t^+}{L_0^+} - E[\frac{L_t^+}{L_0^+}])^2\right] E\left[(\frac{L_t^-}{L_0^-} - E[\frac{L_t^-}{L_0^-}])^2\right]}} \\
&= \frac{e^{-\int_0^t \sigma_s^2 ds} - 1}{e^{\int_0^t \sigma_s^2 ds} - 1} \\
&= -e^{-\int_0^t \sigma_s^2 ds}. \qquad (2.3.8)
\end{aligned}$$

If volatility is stochastic, but uncorrelated with W_t, equations (2.3.4) and (2.3.5) become $E[e^{\int_0^t \sigma_s^2 ds}]$. We have

$$\rho(t) = \frac{E[e^{-\int_0^t \sigma_s^2 ds}] - 1}{E[e^{\int_0^t \sigma_s^2 ds}] - 1} \qquad (2.3.9)$$

As we can see, ρ is always negative and it goes from -1 to 0 monotonically as $t \to \infty$. In fact, the correlation of the L_t^+ and L_t^- decays exponentially as a function of the accumulated realized variance. The same analysis applies to any pair of ETFs which are perfectly inversely correlated at small time interval. This is consistent with the empirical observation in section 1.2, figures 1.3, 1.4 and 1.5.

2.4 Consequences for Buy-and-hold Investors

2.4.1 Comparison with Buy-and-hold: Break Even Levels

Formula (2.2.4) shows that the value of an LETF, regardless of whether it is bullish or bearish, has a "time decay" associated with the realized variance of the underlying index or underlying ETF. In other words, if there is no significant change in S_t over the investment horizon, but the realized variance is large, the investor who holds the LETF will under-perform the corresponding leveraged return of the underlying ETF.

Let's consider an investor who buys one dollar of leveraged ETF and simultaneously shorts β dollars of underlying ETF (where, if $\beta < 0$, he buys $-\beta$ dollars). For simplicity, let's assume interest rate, expense ratio, and cost of borrowing are all zeros.

Using formula (2.2.4), the equity of the investor follows

$$E(t) = \frac{L_t}{L_0} - \beta\frac{S_t}{S_0} - (1-\beta) = \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(\frac{\beta-\beta^2}{2}V_t\right) - \beta\frac{S_t}{S_0} - (1-\beta) \quad (2.4.1)$$

where $1-\beta$ is the cash or credit from the initial transaction. To be precise, we will consider the cases of a double-long and a double-short separately. Let $Y = E(t)$ and $X = \frac{S_t}{S_0}$, we obtain

$$\begin{aligned} Y &= e^{-V_t}X^2 - 2X + 1,\ \beta = 2 \\ Y &= e^{-3V_t}\frac{1}{X^2} + 2X + 3,\ \beta = -2 \end{aligned} \quad (2.4.2)$$

In the case of double-long ETFs ($\beta = 2$), the equity in the portfolio behaves like a parabola in $X = \frac{S_t}{S_0}$ with the curvature tending to zero exponentially as a function of the accumulative realized variance. The investor is long convexity (Gamma), short variance, so he incurs time decay (Theta). The break-even levels for X as a function of V_t are

- Double-long

$$e^{-V_t}X^2 - 2X + 1 = 0$$

$$X_+ = e^{V_t}(1+\sqrt{1-e^{-V_t}}),\ \ X_- = e^{V_t}(1-\sqrt{1-e^{-V_t}}) \quad (2.4.3)$$

- Double-short

$$2X^3 - 3X^2 + e^{-V_t} = 0 \quad (2.4.4)$$

X_+,X_- are the two positive roots of the cubic equation (2.4.4).

As a result, the equity of the investor is positive if $X_t > X_+$ or $X_t < X_-$. For small time $t = \Delta t$ and $\beta > 0$

$$\begin{aligned} V_t &= \int_0^t \sigma_s^2 ds \simeq \sigma_0^2 \Delta t \\ X_+ &= e^{V_t}(1 + \sqrt{1 - e^{-V_t}}) = (1 + V_t)(1 + \sqrt{V_t}) \simeq 1 + \sqrt{V_t} = 1 + \sigma_0 \sqrt{\Delta t} \\ X_- &= e^{V_t}(1 - \sqrt{1 - e^{-V_t}}) = (1 + V_t)(1 - \sqrt{V_t}) \simeq 1 - \sqrt{V_t} = 1 - \sigma_0 \sqrt{\Delta t} \end{aligned}$$

The main observation is that no matter whether the LETF is bullish or bearish, they underperform the static leveraged strategy unless the returns of the underlying ETFs overcome the above volatility-dependent "break-even levels."

2.4.2 Targeting An Investment Return Using Dynamic Replication with LETFs

Let's assume the investor wants to replicate m times the return of the underlying ETF over a given investment horizon. We know, from previous the discussion, that merely holding the leveraged ETF with $\beta = m$ can not guarantee the desired return due to convexity and "time decay". We seek to achieve this objective by dynamically adjusting the holdings in the LETF.

Suppose the agent is long n shares of underlying ETF and short Δ shares of the corresponding $\beta-$leveraged ETF. The one period return of this portfolio is

$$\begin{aligned} d\Pi &= ndS - \Delta dL \\ &= ndS - \Delta L \frac{dL}{L} \\ &= ndS - \Delta L \beta \frac{dS}{S} + (\text{carry terms}) \end{aligned}$$

It follows that the choice of $\Delta = \frac{n}{L\beta}$ should eliminate the market risk. Based on this, consider an investor with an initial endowment of Π_0 dollars and a dynamic strategy which invests

$$\delta_t = \Pi_0 \frac{1}{\beta} \frac{S_t}{S_0} \tag{2.4.5}$$

dollars in LETF with multiplier β. For simplicity, we assume $\lambda = f = 0$. Let us denote the value of his position at time t by Π_t. The change in value of the position across time, funded at rate r, satisfies

$$\begin{aligned} d\Pi_t &= r\Pi_t dt + \delta_t \frac{dL_t}{L_t} - r\delta_t dt \\ &= r\Pi_t dt + \delta_t \left(\beta \frac{dS_t}{S_t} + (1-\beta) r dt \right) - r\delta_t dt \\ &= r\Pi_t dt + \Pi_0 \frac{1}{\beta} \frac{S_t}{S_0} \left(\beta \frac{dS_t}{S_t} + (1-\beta) r dt \right) - r\delta_t dt \\ &= r\Pi_t dt + \Pi_0 \frac{dS_t}{S_0} + \Pi_0 \frac{1}{\beta} \frac{S_t}{S_0} r dt - \Pi_0 \frac{S_t}{S_0} r dt - r\Pi_0 \frac{1}{\beta} \frac{S_t}{S_0} dt \\ &= r\Pi_t dt + \Pi_0 \frac{dS_t}{S_0} - \Pi_0 \frac{S_t}{S_0} r dt \end{aligned} \tag{2.4.6}$$

Integrating this stochastic differential equation from $t = 0$ to T, it follows that

$$\frac{\Pi_T}{\Pi_0} = \frac{S_T}{S_0}$$

for all T.

Accordingly, the agent can "replicate" the returns of the underlying stock over any maturity by dynamic hedging with the LETF, with essentially, $\frac{1}{|\beta|}$ of the capital required to do the same trade in underlying ETF.

In section 2.5.2, we study this method empirically to show it is feasible to replicate multiple return of the underlying for longer horizons.

2.5 Empirical Validation

2.5.1 Validation of Equation (2.2.4)

To validate equation (2.2.4), we consider 56 LETFs that currently trade in the US equity market. Of these, we consider 44 LETFs issued by ProShares, consisting of 22 Ultra long ($\beta = 2$) and 22 Ultra short ($\beta = -2$) LETFs. Table 2.1 gives a list of ProShares LETFs[4], their tickers, together with corresponding sectors and their underlying ETFs. We consider the evolution of the 44 LETFs from January 2, 2008 to March 20, 2009, a period of 308 business days.

We also include 12 triple-leveraged ETFs issued by Direxion Funds. Direxion's LETFs[5] were issued later than the ProShares funds, in November 2008; they provide a shorter historical record to test our formula. Nevertheless, we include the 3X Direxion funds for sake of completeness and their triple leverage.

We define the tracking error

$$\epsilon(t) = \frac{L_t}{L_0} - \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(((1-\beta)r - f)\, t + \beta H_t + \frac{\beta - \beta^2}{2} V_t\right) \tag{2.5.1}$$

where V_t is the accumulated realized variance, H_t is the accumulated borrowing cost, r is the reference interest rate and f is the expense ratio for the LETF. To

[4]For information about ProShares, see http://www.proshares.com

[5]For information about Direxion, see http://www.direxionfunds.com

Table 2.1: Double-Leveraged ETFs considered in the study

Underlying ETF	ProShares Ultra ($\beta = 2$)	ProShares Ultra ($\beta = -2$)	Short Index/Sector
QQQQ	QLD	QID	Nasdaq 100
DIA	DDM	DXD	Dow 30
SPY	SSO	SDS	S&P500 Index
IJH	MVV	MZZ	S&P MidCap 400
IJR	SAA	SDD	S&P Small Cap 600
IWM	UWM	TWM	Russell 2000
IWD	UVG	SJF	Russell 1000
IWF	UKF	SFK	Russell 1000 Growth
IWS	UVU	SJL	Russell MidCap Value
IWP	UKW	SDK	Russell MidCap Growth
IWN	UVT	SJH	Russell 2000 Value
IWO	UKK	SKK	Russell 2000 Growth
IYM	UYM	SMN	Basic Materials
IYK	UGE	SZK	Consumer Goods
IYC	UCC	SCC	Consumer Services
IYF	UYG	SKF	Financials
IYH	RXL	RXD	Health Care
IYJ	UXI	SIJ	Industrials
IYE	DIG	DUG	Oil & Gas
IYR	URE	SRS	Real Estate
IYW	ROM	REW	Technology
IDU	UPW	SDP	Utilities

Table 2.2: Triple-Leveraged ETFs considered in the study

Underlying ETF or Index	Direxion 3X Bull ($\beta = 3$)	Direxion 3X Bear ($\beta = 3$)	Index/Sector
IWB	BGU	BGZ	Russell 1000
IWM	TNA	TZA	Russell 2000
RIFIN.X	FAS	FAZ	Russell 1000 Financial
RIENG.X	ERX	ERY	Russell 1000 Energy
EFA	DZK	DPK	MSCI EAFE Index
EEM	EDC	EDZ	MSCI Emerging Markets

estimate the daily volatility, we use the standard deviation of the returns of the underlying ETF sampled over a period of 5 days preceding each trading day:

$$\hat{\sigma_t}^2 = \frac{1}{5}\sum_{i=1}^{5}\left(R^S_{t/\Delta t-i}\right)^2 - \left(\frac{1}{5}\sum_{i=1}^{5}R^S_{t/\Delta t-i}\right)^2 \qquad (2.5.2)$$

For interest rates and expense ratios, we use the 3-month LIBOR rate published by federal Reverse Bank (H.15 Report), and the expense ratio for the ProShares LETFs published in the prospectus[6]. In all cases, we set $\lambda_t = 0$, *i.e.* we do not take into account stock-borrowing costs explicitly.

The empirical results for ProShares are summarized in Table 2.3 and Table 2.5. Graphical comparisons of the tracking error for some of the major indices are also displayed. See Figure 2.1 and 2.2.

In the case of long LETFs, we find that the average of the tracking error $\epsilon(t)$ over the simulation period is typically less than 100 basis points. The standard deviation is also on the order of 100 basis points, with a few slightly higher observations for financial sector LETFs. This suggests that the formula (2.2.4), using the model for stochastic volatility in (2.5.2), gives a reliable model for the relation between the leveraged and the underlying ETFs across time.

In the case of short LETFs, we also assume that $\lambda_t = 0$ but expect a slightly higher tracking error, particularly between July and November of 2008, when restrictions for short-selling in U.S. stocks were put in place. We observe higher levels for the mean and the standard deviation of the tracking error and some significant departures from the exact formula during the period of October and November 2008, especially in the financial sector, which we attribute to short-selling constraints. The tracking errors for the Direxion triple-leveraged ETFs have higher standard deviations, which is not surprising given that they have higher leverage.

[6]For more information, see http://www.proshares.com/resources/litcenter/

Double-Leveraged Bullish ETFs

Underlying ETF	Tracking Error average,%	Standard Deviation %	Leveraged ETF
QQQQ	0.04	0.47	QLD
DIA	0	0.78	DDM
SPY	-0.06	0.4	SSO
IJH	-0.06	0.38	MVV
IJR	1.26	0.71	SAA
IWM	1.26	0.88	UWM
IWD	1	0.98	UVG
IWF	0.5	0.59	UKF
IWS	-0.33	1.2	UVU
IWP	-0.02	0.61	UKW
IWN	2.15	1.29	UVT
IWO	0.5	0.74	UKK
IYM	1.44	1.21	UYM
IYK	1.2	0.75	UGE
IYC	1.56	1.04	UCC
IYF	-0.22	0.74	UYG
IYH	0.4	0.42	RXL
IYJ	1.05	0.74	UXI
IYE	-0.73	1.71	DIG
IYR	1.64	1.86	URE
IYW	0.51	0.55	ROM
IDU	0.25	0.55	UPW

Table 2.3: Average tracking error (2.5.1) and standard deviation obtained by applying formula (2.2.4) to the ProShares long LETFs from January 2, 2008 to March 20 2009. Notice that the average tracking error is for the most part below 100bps and the standard deviation is comparable. In particular the standard deviation is inferior to the daily volatility of these assets, which often exceeds 100 basis points. This suggests that formula (10) gives the correct relationship between the NAV of the LETFs and their underlying ETFs.

Triple-Leveraged Bullish ETFs

Underlying ETF/Index	Tracking Error average,%	Standard Deviation %	Leveraged ETF
IWB	0.44	0.55	BGU
IWM	0.81	0.75	TNA
RIFIN.X	3.67	2.08	FAS
RIENG.X	2.57	0.7	ERX
EFA	1.26	2.32	DZK
EEM	1.41	1.21	EDC

Table 2.4: Average tracking errors and standard deviations for triple-leveraged long ETFs analyzed here, since their inception in November 2008.

Double-Leveraged Bearish ETFs

Underlying ETF	Tracking Error average,%	Standard Deviation %	Leveraged ETF
QQQQ	0.22	0.8	QID
DIA	-2.01	3.24	DXD
SPY	-1.4	2.66	SDS
IJH	0.69	0.64	MZZ
IJR	-0.55	0.86	SDD
IWM	0.94	0.91	TWM
IWD	0.32	1.4	SJF
IWF	-0.3	1.34	SFK
IWS	-2.06	3.03	SJL
IWP	0.93	0.92	SDK
IWN	-2.21	1.8	SJH
IWO	-0.19	0.79	SKK
IYM	1.82	0.99	SMN
IYK	-0.76	1.98	SZK
IYC	0.79	0.92	SCC
IYF	3.3	3.03	SKF
IYH	1.04	0.91	RXD
IYJ	0.32	0.74	SIJ
IYE	0.43	3.09	DUG
IYR	2	2.07	SRS
IYW	0.01	0.8	REW
IDU	1.75	1.06	SDP

Table 2.5: Same as in Table 2.3, for double-short LETFs. Notice that the tracking error is relatively small, but there are a few funds where the tracking error is superior to 200 basis points. We attribute these errors to the fact that many ETFs, particularly in the Financial and Energy sectors, and the stocks in their holdings were hard-to-borrow from July to November 2008.

Triple-Leveraged Bearish ETFs			
Underlying ETF/Index	Tracking Error average,%	Standard Deviation %	Leveraged ETF
IWB	-0.08	0.64	BGZ
IWM	0.65	0.76	TZA
RIFIN.X	-1.63	4.04	FAZ
RIENG.X	-1.41	1.01	ERY
EFA	-1.54	1.86	DPK
EEM	0.49	1.43	EDZ

Table 2.6: Average tracking errors and standard deviations for triple-leveraged short ETFs analyzed here, since their inception in November 2008. Notice again that the errors for financial and energy are slightly higher than the rest.

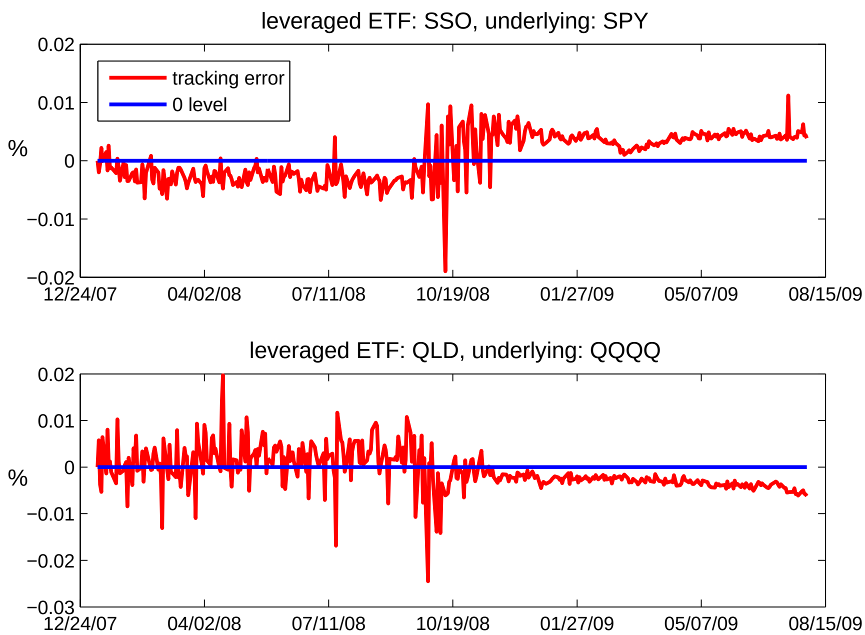

Figure 2.1: shows tracking error for SSO and QLD which track the double leveraged ($\beta = 2$) return of S&P 500 and NASDAQ 100 respectively. We see that the tracking error spikes during the crisis (Oct 2008-Nov 2008).

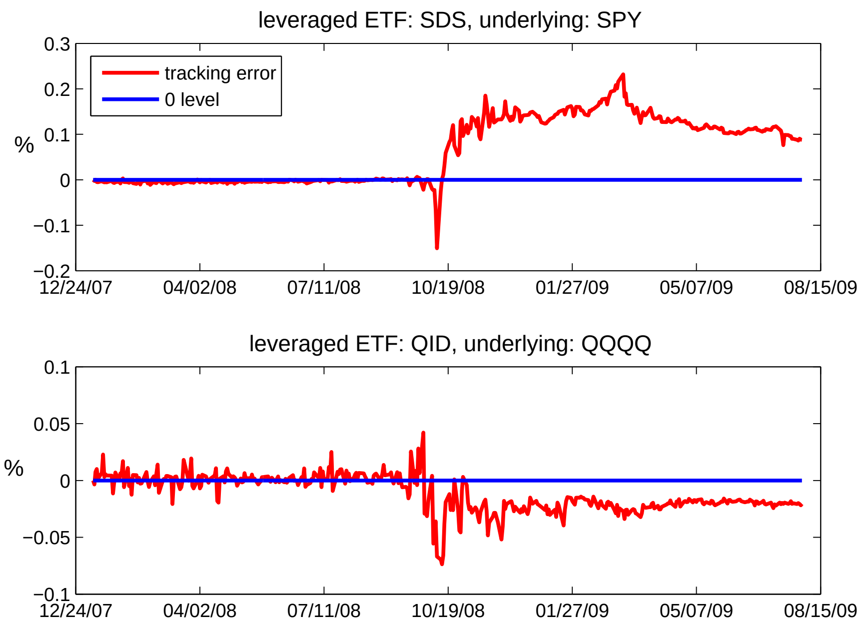

Figure 2.2: shows tracking error for SDS and QID which track the inverse double leveraged ($\beta = -2$) return of S&P 500 and NASDAQ 100 respectively. Notice during the crisis, the tracking error jumps and deviates from the 0 level. The reason is that we assume $\lambda_t = 0$. There are a significant number of stocks in S&P 500 and NASDAQ 100 which were hard-to-borrow from Oct 2008 to Nov 2008.

We note, in particular, that the errors for FAS and FAZ are the largest, which is consistent with the fact that they were the most volatile and hard-to-borrow LETFs during the crisis.

2.5.2 Validation of Dynamic Replication Strategy with LETFs

To validate the replicating strategy in section 2.4.2, we consider using all ProShares double-long ($\beta = 2$) LETFs to replicate double of the 6-month return of the underlying ETFs. Therefore, $m = \beta = 2$. We demonstrate the effectiveness of the dynamic replication method using different rebalancing techniques. We consider a dynamic hedging in which we rebalance if the Δ (equation (2.4.5)) exceeds a band of 1%, 2%, 5% and 10% and also hedging with fixed time-steps of 1, 2, 5 and 15 business days. The results are shown in Table 2.7 and 2.8. Table 2.9 indicates the expected number of days between rebalancing for Δ-band based strategy. The results indicate that rebalancing when the Δ exceeds 5% of notional gives reasonable tracking errors. The corresponding average intervals between rebalancing can be large, which means that, in practice, one can achieve reasonable tracking errors without necessarily having to rebalance the Δ daily or even weekly.

A strong motivation for using LETFs to target a given level of return is to take advantage of leverage. However, in order to achieve his target return over an extended investment period using LETFs, the investor needs to rebalance his portfolio according to his Δ exposure. As a result, dynamic replication with LETFs may not be suitable for many retail investors. On the other hand, this analysis will be useful to active traders, or traders who manage LETFs with longer investment horizons.

Average tracking error (%) for dynamic replication of 6-month returns using double-long LETFs

ETF	1%	2%	5%	20%	1 day	2 days	5 days	15 days
QQQQ	-0.29	-0.71	-1.05	-1.62	-0.56	-0.96	-1.45	-1.74
DIA	-0.99	-0.99	-1.37	-1.45	-0.47	-0.59	-0.84	-0.99
SPY	-0.97	-0.92	-1.19	-1.47	-0.92	-1.17	-1.55	-1.77
IJH	-0.39	-0.37	-0.79	-0.99	-0.53	-0.75	-1.05	-1.09
IJR	-0.56	-0.57	-1.07	-2.68	-0.66	-0.9	-1.44	-1.7
IWM	0.37	0.22	-0.49	-1.44	0.47	0.03	-0.7	-0.93
IWD	-0.03	-0.35	-0.3	-0.64	0	-0.15	-0.57	-0.79
IWF	-0.15	-0.25	-0.54	-1.08	-0.12	-0.37	-0.68	-0.81
IWS	0.87	0.22	0.81	0.14	0.69	0.71	0.54	0.24
IWP	-0.16	-0.14	-0.54	-1.41	-0.36	-0.4	-0.82	-0.89
IWN	0.94	0.4	0.56	-0.03	-0.91	0.86	0.36	0.14
IWO	0.23	0.03	-1	-1.63	-0.05	-0.44	-1.15	-1.45
IYM	-0.39	-0.51	-0.89	-2.35	-0.24	-0.67	-1.54	-1.91
IYK	0.24	0.13	-0.16	-0.06	0.37	0.34	0.1	0.04
IYC	0.58	0.57	-0.13	-0.76	0.71	0.7	0.04	-0.21
IYF	-0.36	-0.62	0.01	-0.54	-0.3	-0.35	-1.28	-2.17
IYH	0.22	-0.1	-0.14	0.27	0.3	0.19	0.03	0.07
IYJ	0.12	-0.09	-0.36	-0.92	0.14	-0.04	-0.3	-0.61
IYE	-1.44	-2.02	-1.9	-1.76	-1.19	-1.82	-2.21	-2.07
IYR	-0.43	0.58	-0.8	-0.95	-0.61	0.55	-0.74	-1.48
IYW	-0.5	-0.46	-1.67	-1.39	-0.37	-0.85	-1.41	-1.76
IDU	0.75	0.45	0.73	0.11	0.83	0.78	0.46	0.52

Table 2.7: Average tracking error, in % of notional, for the dynamic replication of ETF returns over 6 months with $m = \beta = 2$. The first four columns correspond to rebalancing when the Δ reaches the edge of a band of $\pm x\%$ around zero. The last four columns correspond to rebalancing at fixed time intervals. The data used to generate this table corresponds, for each ETF, to all overlapping 6-month returns in the year 2008.

Standard deviation of tracking error (%) for dynamic replication of 6-month returns using double-long LETFs

ETF	1%	2%	5%	20%	1 day	2 days	5 days	15 days
QQQQ	0.75	0.77	0.8	1.2	0.75	0.76	0.84	0.93
DIA	0.35	0.37	0.41	0.36	0.36	0.39	0.47	0.43
SPY	0.27	0.32	0.39	0.69	0.27	0.3	0.39	0.45
IJH	0.48	0.49	0.56	1.19	0.47	0.48	0.62	0.65
IJR	1.19	1.21	1.33	1.39	1.2	1.29	1.22	1.17
IWM	0.66	0.67	0.71	1.6	0.67	0.75	0.71	0.74
IWD	1.38	1.38	1.4	1.52	1.38	1.43	1.41	1.49
IWF	0.93	0.94	1.07	1.21	0.95	0.99	0.94	0.99
IWS	2.05	2.05	2.08	2.29	2.05	2.01	2.07	2.09
IWP	0.83	0.82	0.93	1.24	0.83	0.91	0.84	0.91
IWN	1.71	1.7	1.76	2.09	1.72	1.7	1.8	1.82
IWO	0.8	0.8	0.91	1.32	0.79	0.99	0.84	1
IYM	1.05	1.07	1.15	1.24	1.07	1.2	1.29	1.59
IYK	0.57	0.57	0.63	0.67	0.56	0.63	0.61	0.63
IYC	0.8	0.78	0.83	0.95	0.8	0.98	0.91	1.03
IYF	1.12	1.18	1.12	1.88	1.1	1.21	2.01	1.49
IYH	0.56	0.55	0.58	0.73	0.56	0.55	0.57	0.62
IYJ	0.71	0.75	0.82	0.84	0.7	0.7	0.79	0.87
IYE	0.64	0.66	0.77	1.26	0.64	0.65	1.02	1.49
IYR	1.47	1.47	1.62	2.08	1.47	1.6	1.88	1.83
IYW	1.45	1.45	1.59	2.05	1.45	1.39	1.49	1.42
IDU	0.53	0.52	0.54	0.72	0.51	0.53	0.54	0.6

Table 2.8: Same as the previous table, for the standard deviations of the tracking errors.

Average number of business days between portfolio rebalancing for the 6-month dynamic hedging strategy: the effect of changing the Δ band

ETF	1%	2%	5%	10%
QQQQ	2.03	4.14	24	60
DIA	2.5	5.22	30	120
SPY	2.73	5.22	40	NR
IJH	2.26	4.62	24	NR
IJR	2.03	4.29	20	NR
IWM	1.85	4.62	30	NR
IWD	2.18	5	30	120
IWF	2.26	5	30	NR
IWS	2.26	6.67	17	NR
IWP	1.85	4.14	30	NR
IWN	2.26	4.62	24	NR
IWO	1.85	4	20	60
IYM	1.74	3.08	9	40
IYK	3.16	8.57	30	120
IYC	1.9	3.87	30	NR
IYF	1.45	2.93	9	30
IYH	2.79	10.91	30	120
IYJ	2.35	4.8	17	NR
IYE	1.79	3.43	12	40
IYR	1.62	3.16	17	30
IYW	2	3.53	40	NR
IDU	2.67	6.32	20	120

Table 2.9: Each column shows the average number of days between rebalancing the portfolio, assuming different Δ-bandwidth for portfolio rebalancing. For instance, the column with heading of 1% corresponds to a strategy that rebalances the portfolio each time the net delta exposure exceeds 1% of the notional amount. The expected number of days between rebalancing increases as the bandwidth increases. NR means the band is never reached during the time period of interest

2.6 Conclusion

In this chapter, we proposed a formula linking the evolution of an LETF with the price of the underlying index or ETF and its realized volatility. The formula is validated empirically using end-of-day data on 56 LETFs, of which 44 are double-leveraged and 12 are triple leveraged. This formula validates the fact that on log-scale LETFs will underperform the nominal returns by a contribution that is primarily due to the realized volatility of the underlying ETF. The formula also takes into account the financing costs and shows that for short ETFs, the cost of borrowing the underlying stock may play a role as well, as observed in Avellaneda and Lipkin [5].

We also demonstrate that LETFs can be used for hedging and replicating non-leveraged ETFs, provided that traders engage in dynamic hedging. The path-dependence of LETFs makes them interesting for the professional investors, since they are linked to the realized variance and the financing costs. However, they may not be suitable for buy-and-hold investors who aim at replicating a particular index by taking advantage of the leveraged provided, for the reasons explained above.

2.7 Appendix to Chapter 2

2.7.1 Discrete-time Model (Equation (2.1.4))

Let $a_i = (\beta - 1)r + f - \beta\lambda_i$, where λ_i is the cost of borrowing the underlying asset on day i (λ is 0 if $\beta > 0$). We assume the daily return of the underlying index or ETF is

$$R_i^S = \xi_i\sqrt{\Delta t} + \mu\Delta t$$

where $\xi_i, i = 1, 2, 3, \ldots, N$ is a stationary process such that $E[\xi_i] = 0$ and $E[|\xi_i|^3] < \infty$. Notice we don't assume the returns are uncorrelated.

From equation (2.1.1), (2.1.2) and (2.1.3), we use Taylor expansion and find

$$
\begin{aligned}
\ln\left(\frac{L_t}{L_0}\right) &= \sum_i \ln(1 + R_i^L) \\
&= \sum_i \ln(1 + \beta R_i^S - a_i \Delta t) \\
&= \sum_i (\beta R_i^S - a_i \Delta t) - \frac{1}{2} \sum_i (\beta R_i^S - a_i \Delta t)^2 + \sum_i \Theta((R_i^S)^3) \\
&= \sum_i \left(\beta R_i^S - a_i \Delta t - \frac{1}{2}\beta^2 (R_i^S)^2\right) \\
&\quad + \sum_i \left(\Theta((R_i^S)^3) + \Theta(R_i^S \Delta t)\right) \qquad (2.7.1)
\end{aligned}
$$

Using the same method, we have

$$
\begin{aligned}
\beta \ln\left(\frac{S_t}{S_0}\right) &= \beta \sum_i \ln(1 + R_i^S) \\
&= \beta \sum_i \left(R_i^S - \frac{1}{2}(R_i^S)^2\right) + \sum_i \Theta((R_i^S)^3) \qquad (2.7.2)
\end{aligned}
$$

Subtracting equation (2.7.2) from (2.7.1), we get

$$
\begin{aligned}
&\ln\left(\tfrac{L_t}{L_0}\right) - \beta \ln\left(\tfrac{S_t}{S_0}\right) \\
&= \sum_i \left(-a_i \Delta t + \frac{\beta - \beta^2}{2}(R_i^S)^2\right) + \sum_i \left(\Theta((R_i^S)^3) + \Theta(R_i^S \Delta t)\right) \\
&= \sum_i \left(-a_i \Delta t + \frac{\beta - \beta^2}{2}((R_i^S)^2 - \mu^2 \Delta t^2)\right) \\
&\quad + \sum_i (\Theta((R_i^S)^3) + \Theta(R_i^S \Delta t) + \Theta(\Delta t^2)) \\
&= -\sum_i a_i \Delta t - \frac{\beta^2 - \beta}{2} V_t + \sum_i \left(\Theta((R_i^S)^3) + \Theta(R_i^S \Delta t)\right) \quad (2.7.3)
\end{aligned}
$$

We need to show the remainder in (2.7.3) is negligible. In fact

$$
\begin{aligned}
\sum_i (R_i^S)^3 &= \sum_i (\xi_i^3 \Delta t^{\frac{3}{2}} + O(\Delta t^2)) \\
&\simeq \sum_i \xi_i^3 \Delta t \sqrt{\Delta t} = \sum_i \xi_i^3 \frac{t}{N} \sqrt{\Delta t} = E[\xi^3] t \sqrt{\Delta t}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\sum_i R_i^S \Delta t &= \sum_i (\xi_i \Delta t^{\frac{3}{2}} + O(\Delta t^2)) \\
&\simeq \sum_i \xi_i \Delta t \sqrt{\Delta t} = \sum_i \xi_i \frac{t}{N} \sqrt{\Delta t} = E[\xi] t \sqrt{\Delta t} = 0
\end{aligned}
$$

As a result, the remainder is bounded by the third moment of ξ_i, multiplied by the investment horizon t and by $\sqrt{\Delta t}$. This means that, for reasonable size of t, say less than a year, the error is of the order of the third moment of the daily return of the underlying ETF, which is negligible.

2.7.2 Continuous-time Model (Equation (2.2.4))

From equation (2.2.1), (2.2.2) and (2.2.3), we use Itô's lemma to obtain

$$d \ln S_t = \frac{dS_t}{S_t} - \frac{\sigma_t^2}{2} dt \tag{2.7.4}$$

$$\begin{aligned} d \ln L_t &= \frac{dL_t}{L_t} - \frac{\beta^2 \sigma_t^2}{2} dt \\ &= \beta \frac{dS_t}{S_t} - ((\beta - 1)r - \beta\lambda + f)dt - \frac{\beta^2 \sigma_t^2}{2} dt \end{aligned} \tag{2.7.5}$$

Multiplying (2.7.4) by β and subtracting it from (2.7.5), we obtain

$$d \ln L_t - \beta \ln S_t = \frac{\beta - \beta^2}{2} \sigma_t^2 dt - ((\beta - 1)r - \beta\lambda + f)dt, \tag{2.7.6}$$

which gives the desired result (2.2.4).

Chapter 3

Options on Leveraged ETFs

3.1 Replicating Options on LETFs with Options on Underlying ETF

In this chapter, we consider options on LETFs, which are of great interest to both academics and practitioners. The two simple equations ((2.2.2) and (2.2.3)), which link the underlying ETF return with its leveraged bull and bear ETFs return, provide us a way to price the options on LETFs with respect to options on the underlying ETF. From now on, we will use S_t, L_t^+ and L_t^- to denote the price of the underlying ETF, leveraged bullish ETF and leveraged bearish ETF respectively. Because of formula (2.2.4), a contingent claim on L_t^+ or L_t^- can be treated as a contingent claim on S_t. It is well-known that for any payoff function $g(S_t)$ and any scalar $c \in \mathbb{R}$ (see Carr-Madan [10]):

$$g(S) = g(c)+g'(c)(S-c)+\int_{-\infty}^{c} g''(K)(K-S)^+dK+\int_{c}^{\infty} g''(K)(S-K)^+dK \quad (3.1.1)$$

No arbitrage condition implies that the value of the option $V_t[g]$ can be expressed in terms of the price of puts $P_t(K)$ and calls $C_t(K)$ respectively:

$$V_t[g] = g(c) + g'(c)(C_t(c) - P_t(c)) + \int_0^c g''(K)P_t(K)dK + \int_c^\infty g''(K)C_t(K)dK \tag{3.1.2}$$

If we assume the interest rate, expense ratio and cost of borrowing are all zero for simplicity, $L_t = L_0 \left(\frac{S_t}{S_0}\right)^\beta e^{\frac{\beta-\beta^2}{2}V_t}$, where $V_t = \int_0^t \sigma_s^2 ds$. This is the payoff of a power option on S_t with "time decay" due to the realized variance. Let's further assume that the volatility is deterministic first. We will discuss how to extend this to a stochastic volatility environment later. If we consider the contingent claim on S_t to be just L_t, that is $g(S_t) = L_t = L_0 \left(\frac{S_t}{S_0}\right)^\beta e^{\frac{\beta-\beta^2}{2}V_t}$. Substituting into equation (3.1.2), we get

$$g'(S_t) = L_0 \frac{\beta}{S_0}\left(\frac{S_t}{S_0}\right)^{\beta-1} e^{\frac{\beta-\beta^2}{2}V_t} \quad \text{and} \quad g''(S_t) = L_0 \frac{\beta(\beta-1)}{S_0^2}\left(\frac{S_t}{S_0}\right)^{\beta-2} e^{\frac{\beta-\beta^2}{2}V_t}.$$

If we pick a double-long LETF ($\beta = 2$), $g'(S_t) = \frac{2L_0}{S_0}\left(\frac{S_t}{S_0}\right) e^{-V_t}$ and $g''(S_t) = \frac{2L_0}{S_0^2} e^{-V_t}$. Equation (3.1.2) becomes

$$\begin{aligned} V_t[g] = \quad & g(c) + \left(\tfrac{2cL_0}{S_0^2}\right) e^{-V_t}\left(C_t(c) - P_t(c)\right) \\ & + \tfrac{2L_0}{S_0^2} e^{-V_t}\left(\textstyle\int_0^c P_t(K)dK + \int_c^\infty C_t(K)dK\right) \end{aligned} \tag{3.1.3}$$

Since c is arbitrary, we can pick $c = 0$. We have

$$V_t[g] = \frac{2L_0}{S_0^2} e^{-V_t} \int_0^\infty C_t(K)dK \tag{3.1.4}$$

This means that if we want to replicate L_t using options on the underlying

ETF, we should choose a basket of equally weighted calls if the L_t is bullish. This is the same as saying L_t is a power option on S_t with a time decay factor e^{-V_t}.

Now, let's use the same idea on a European call option on the bullish LETF ($\beta > 0$) with strike k. The payoff function is

$$g(S_t) = \left(L_0 \left(\frac{S_t}{S_0}\right)^{\beta} e^{\frac{\beta-\beta^2}{2}V_t} - k \right)^+ = L_0 e^{\frac{\beta-\beta^2}{2}V_t} \left((\frac{S_t}{S_0})^{\beta} - \frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t} \right)^+ \quad (3.1.5)$$

Let $x = \frac{S_t}{S_0}$ and $k^* = \left(\frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t}\right)^{\frac{1}{\beta}}$, substituting into 3.1.5

$$g(x) = L_0 e^{\frac{\beta-\beta^2}{2}V_t} \left(x^{\beta} - (k^*)^{\beta}\right)^+. \quad (3.1.6)$$

Its first and second derivatives are

$$g'(x) = H(x-k^*) L_0 e^{\frac{\beta-\beta^2}{2}V_t} \beta x^{\beta-1} \quad (3.1.7)$$

$$g''(x) = \begin{cases} L_0 e^{\frac{\beta-\beta^2}{2}V_t} \beta(\beta-1) x^{\beta-2} & \text{if } x > k^* \\ \delta(x-k^*) L_0 e^{\frac{\beta-\beta^2}{2}V_t} \beta (k^*)^{\beta-1} & \text{if } x = k^* \\ 0 & \text{if } x < k^* \end{cases} \quad (3.1.8)$$

where $H(x)$ is the Heaviside function and $\delta(x)$ is the Dirac delta function. Substituting (3.1.7) and (3.1.8) into (3.1.1), we obtain

$$g(x) = \underbrace{L_0 e^{\frac{\beta-\beta^2}{2}V_t} \beta (k^*)^{\beta-1} (x-k^*)^+}_{\text{tangent approximation}} + \underbrace{\int_{k^*}^{\infty} L_0 e^{\frac{\beta-\beta^2}{2}V_t} \beta(\beta-1) K^{\beta-2} (x-K)^+ dK}_{\text{tangent correction}} \quad (3.1.9)$$

Suppose the volatility is deterministic, taking expectation of both sides of equation

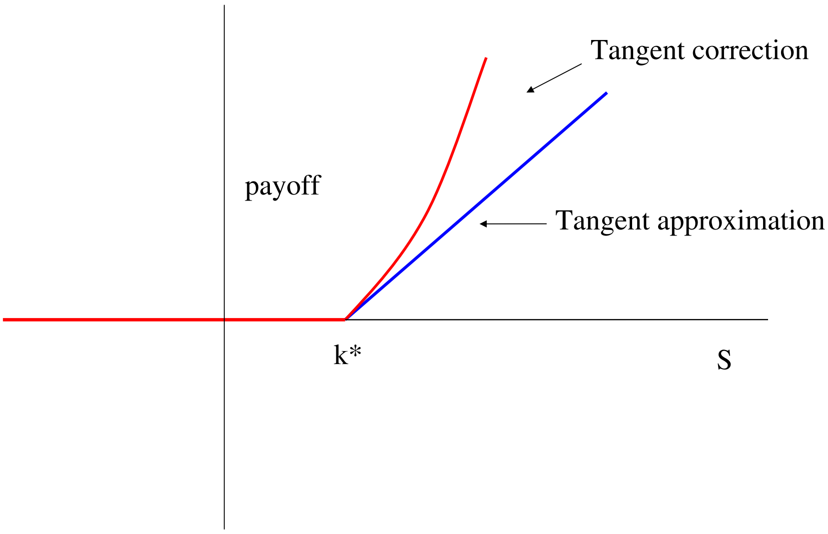

Figure 3.1: shows the value decomposition of a call on a bullish leveraged ETF: the tangent approximation and the tangent correction.

(3.1.11) and using the no arbitrage condition, we get

$$\boxed{C_L(k,t) = \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta \left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (\beta - 1) \int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t) dK \right),} \tag{3.1.10}$$

where $k^* = \left(\frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t} \right)^{\frac{1}{\beta}}$, $C_L(k,t)$ is the value of a call on L with strike k at time t and $C_S(S_0 k^*, t)$ is the value of a call on S with strike $S_0 k^*$ at time t.

If $k \neq 0$, we can do a change of variable in the tangent correction part, let $K = yk^*$,

$$\begin{aligned}
& \int_{k^*}^{\infty} \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta(\beta-1) K^{\beta-2} (x-K)^+ dK \\
= & \int_1^{\infty} \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta(\beta-1) (k^* y)^{\beta-2} (x - k^* y)^+ k^* dy \\
= & \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta(\beta-1) (k^*)^{\beta-1} \int_1^{\infty} y^{\beta-2} (x - k^* y)^+ dy.
\end{aligned}$$

Equation (3.1.9) becomes

$$\begin{aligned} g(x) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta \left((k^*)^{\beta-1}(x-k^*)^+ + (\beta-1)(k^*)^{\beta-1} \int_1^\infty y^{\beta-2}(x-k^*y)^+ dy \right) \\ &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta (k^*)^{\beta-1} \left((x-k^*)^+ + (\beta-1) \int_1^\infty y^{\beta-2}(x-k^*y)^+ dy \right), \end{aligned} \tag{3.1.11}$$

whence

$$\begin{aligned} C_L(k,t) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta (k^*)^{\beta-1} \left(C_S(S_0k^*,t) + (\beta-1) \int_1^\infty y^{\beta-2} C_S(S_0k^*y,t) dy \right) \\ &= \frac{1}{S_0} \beta L_0^{\frac{1}{\beta}} k^{1-\frac{1}{\beta}} e^{\frac{1-\beta}{2}V_t} \left(C_S(S_0k^*,t) + (\beta-1) \int_1^\infty y^{\beta-2} C_S(S_0k^*y,t) dy \right), \end{aligned}$$

In the case that $\beta = 2$, we have

$$C_L(k,t) = \frac{2}{S_0} \sqrt{L_0 k} e^{-\frac{1}{2}V_t} \left(C_S(S_0k^*,t) + \int_1^\infty C_S(S_0k^*y,t) dy \right) \tag{3.1.12}$$

We can derive similar equations for the case $\beta < 0$. Consider a call on a bearish LETF,

$$g(S_t) = \left(L_0 \left(\frac{S_t}{S_0} \right)^\beta e^{\frac{\beta-\beta^2}{2}V_t} - k \right)^+ = L_0 e^{\frac{\beta-\beta^2}{2}V_t} \left((\frac{S_t}{S_0})^\beta - \frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t} \right)^+ \tag{3.1.13}$$

Let $x = \frac{S_t}{S_0}$ and $k^* = \left(\frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t} \right)^{\frac{1}{\beta}}$, the payoff function becomes

$$g(x) = L_0 e^{\frac{\beta-\beta^2}{2}V_t} (x^\beta - (k^*)^\beta)^+ \tag{3.1.14}$$

Its first and second derivatives are

$$g'(x) = H(k^* - x) L_0 e^{\frac{\beta-\beta^2}{2}V_t} (\beta) x^{\beta-1} \tag{3.1.15}$$

$$g''(x) = \begin{cases} L_0 e^{\frac{\beta-\beta^2}{2}V_t}\beta(\beta-1)x^{\beta-2} & \text{if } x < k^* \\ \delta(k^*-x)L_0 e^{\frac{\beta-\beta^2}{2}V_t}|\beta|(k^*)^{\beta-1} & \text{if } x = k^* \\ 0 & \text{if } x > k^* \end{cases} \tag{3.1.16}$$

Substituting (3.1.15) and (3.1.16), we have

$$\begin{aligned} g(x) = \; & \underbrace{L_0 e^{\frac{\beta-\beta^2}{2}V_t}|\beta|(k^*)^{\beta-1}(k^*-x)^+}_{\text{tangent approximation}} \\ & + \underbrace{\int_0^{k^*} L_0 e^{\frac{\beta-\beta^2}{2}V_t}\beta(\beta-1)K^{\beta-2}(K-x)^+ dK}_{\text{tangent correction}} . \end{aligned} \tag{3.1.17}$$

No arbitrage condition gives us

$$\boxed{C_L(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1}P_S(S_0k^*,t) + (1-\beta)\int_0^{k^*} K^{\beta-2}P_S(S_0K,t)dK\right)} \tag{3.1.18}$$

where $k^* = \left(\frac{k}{L_0}e^{\frac{\beta^2-\beta}{2}V_t}\right)^{\frac{1}{\beta}}$. If $k \neq 0$, we can perform a change of variable $K = yk^*$,

$$\int_0^{k^*} K^{\beta-2}P_S(K,t)dK = (k^*)^{\beta-1}\int_0^1 y^{\beta-2}P_S(k^*y,t)dy. \tag{3.1.19}$$

Equation (3.1.17) becomes

$$\begin{aligned} C_L(k,t) &= \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|(k^*)^{\beta-1}\left(P_S(S_0k^*,t) + (1-\beta)\int_0^1 y^{\beta-2}P_S(S_0yk^*,t)dy\right) \\ &= \frac{1}{S_0}|\beta|L_0^{\frac{1}{\beta}}k^{1-\frac{1}{\beta}}e^{\frac{1-\beta}{2}V_t}\left(P_S(S_0k^*,t) + (1-\beta)\int_0^1 y^{\beta-2}P_S(S_0yk^*,t)dy\right) \end{aligned}$$

If we pick $\beta = -1$, which corresponds to an inverse ETF, we have

$$C_L(k,t) = \frac{k^2}{S_0L_0}e^{V_t}\left(P_S(S_0k^*,t) + 2\int_0^1 y^{-3}P_S(S_0yk^*,t)dy\right). \tag{3.1.20}$$

where $k^* = \left(\frac{k}{L_0}e^{V_t}\right)^{-1}$

Using the same argument, for puts on a bullish LETF we have

$$\boxed{P_L(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta\left((k^*)^{\beta-1}P_S(S_0k^*,t) + (\beta-1)\int_0^{k^*} K^{\beta-2}P_S(S_0K,t)dK\right)}. \tag{3.1.21}$$

and for bearish LETF, we have

$$\boxed{P_L(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1}C_S(S_0k^*,t) + (1-\beta)\int_{k^*}^{\infty} K^{\beta-2}C_S(S_0K,t)dK\right)}. \tag{3.1.22}$$

We summarize the results as follows:

$$\begin{aligned}
C_{L^+}(k,t) &= \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta\left((k^*)^{\beta-1}C_S(S_0k^*,t) + (\beta-1)\int_{k^*}^{\infty} K^{\beta-2}C_S(S_0K,t)dK\right)\\
C_{L^-}(k,t) &= \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1}P_S(S_0k^*,t) + (1-\beta)\int_0^{k^*} K^{\beta-2}P_S(S_0K,t)dK\right)\\
P_{L^+}(k,t) &= \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta\left((k^*)^{\beta-1}P_S(S_0k^*,t) + (\beta-1)\int_0^{k^*} K^{\beta-2}P_S(S_0K,t)dK\right)\\
P_{L^-}(k,t) &= \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1}C_S(S_0k^*,t) + (1-\beta)\int_{k^*}^{\infty} K^{\beta-2}C_S(S_0K,t)dK\right).
\end{aligned} \tag{3.1.23}$$

These four equations will be our major tools to price options on LETF with respect to options on underlying ETF. We call $S_0k^* = S_0\left(\frac{k}{L_0}e^{\frac{\beta^2-\beta}{2}V_t}\right)^{\frac{1}{\beta}}$ the "most likely" strike. The reason is that if L_t goes to k at time t, S_0k^* is the "most likely" price of S_t. The same idea is used in index option pricing, where given a basket of single name options, one would like to know the "most likely" level of the index if every stock in the index does go to a set of strikes at maturity (see [11]). As we can see, an option on the LETF could be decomposed into two parts: 1) a single contract on the underlying ETF with the "most likely" strike 2) a continuous strip of options on the underlying ETF. We will see in later sections that part I is what really matters. We will validate these four equations with numerical simulations and market data in later sections. So far, we derive the four equations under the assumption that volatility is deterministic. We will discuss how to extend it to the stochastic volatility world in Chapter 4.

3.2 Closed Form Solution under Heston Model

Before we validate equation (3.1.23) with market data, we investigate the existence of a closed-form solution for a European call on LETF under Heston model. Suppose σ is random, we can write the value of the option as

$$\psi(S,\sigma,V,t) = e^{-r(T-t)}E\left[\left(L_0\left(\frac{S_T}{S_0}\right)^{\beta}e^{V(0,T)}-k\right)^+\middle|S_t=S,\sigma_t=\sigma,V(0,t)=V\right] \tag{3.2.1}$$

where $V(0,t)=\int_0^t(\frac{\beta-\beta^2}{2}\sigma_s^2+((1-\beta)r-f)+\beta\lambda_s)ds$. We would like to write the value of the option as a conditional expectation of another measure. For simplicity, let's assume that interest rates, expense ratios and cost of borrowing are all zero. Thus $V(0,t)=\int_0^t\frac{\beta-\beta^2}{2}\sigma_s^2 ds$ and

$$\begin{aligned}
&\psi(S,\sigma,V,t)\\
&= E\left[\left(L_0\left(\frac{S_T}{S_0}\right)^{\beta}e^{V(0,T)}-k\right)^+\middle|S_t=S,\sigma_t=\sigma,V(0,t)=V\right]\\
&= E\left[\left(L_t\left(\frac{S_T}{S_t}\right)^{\beta}e^{V(t,T)}-k\right)^+\middle|S_t=S,\sigma_t=\sigma,V(0,t)=V\right]\\
&= E\left[e^{V(t,T)}\left(L_t\left(\frac{S_T}{S_t}\right)^{\beta}-ke^{-V(t,T)}\right)^+\middle|S_t=S,\sigma_t=\sigma,V(0,t)=V\right]\\
&= \xi(S,\sigma,V,t)\phi(S,\sigma,V,t),
\end{aligned} \tag{3.2.2}$$

where

$$\xi(S,\sigma,V,t) = E[e^{V(t,T)}|S_t = S, \sigma_t = \sigma, V(0,t) = V] \quad (3.2.3)$$

$$\phi(S,\sigma,V,t) = \frac{E\left[e^{V(t,T)}\left(L_t\left(\frac{S_T}{S_t}\right)^\beta - ke^{-V(t,T)}\right)^+ \middle| S_t = S, \sigma_t = \sigma, V(0,t) = V\right]}{E[e^{V(t,T)}|S_t = S, \sigma_t = \sigma, V(0,t) = V]} \quad (3.2.4)$$

We define a new probability measure

$$\tilde{E}[f(X_s, s \le T)] = \frac{E[e^{V(0,T)}f(X_s, s \le T)]}{E[e^{V(0,T)}]} \quad (3.2.5)$$

$\tilde{E}(\cdot)$ is associated with a diffusion process and $\tilde{E}[F|X_u, u \le s] = \frac{E[e^{V(t,T)}F|X_s]}{E[e^{V(t,T)}|X_s]}$. We want to find out the PDE for ϕ,

$$\begin{aligned} \xi(S,\sigma,V,t)\phi(S,\sigma,V,t) &= \psi(S,\sigma,V,t) \\ \phi_t\xi + \phi\xi_t &= \psi_t \\ \phi_t\xi &= \psi_t - \phi\xi_t \end{aligned}$$

For convenience, denote $\nu = \sigma^2$ and let $\mathcal{L}$ be the infinitesimal generator associated with the systems of stochastic differential equations (Heston Model)

$$\begin{cases} dS &= S\sqrt{\nu_t}dW_t \\ dV &= \frac{\beta-\beta^2}{2}\nu_t dt \\ d\nu_t &= \kappa(\theta - \nu_t)dt + c\sqrt{\nu_t}dZ_t \\ E[dW_t dZ_t] &= \rho dt \end{cases} \quad (3.2.6)$$

That is

$$\mathcal{L}f(S,\nu,V,t)=\frac{\beta-\beta^2}{2}\nu\frac{\partial f}{\partial V}+\kappa(\theta-\nu)\frac{\partial f}{\partial \nu}+\frac{1}{2}\nu S^2\frac{\partial^2 f}{\partial S^2}+\frac{1}{2}\nu c^2\frac{\partial^2 f}{\partial \nu^2}+cS\nu\rho\frac{\partial^2 f}{\partial S\partial \nu} \tag{3.2.7}$$

$$\begin{aligned}
\phi_t\xi &= \psi_t-\phi\xi_t\\
&= -\mathcal{L}\psi-\frac{\beta-\beta^2}{2}\nu\psi-\phi(-\mathcal{L}\xi-\frac{\beta-\beta^2}{2}\nu\xi)\\
&= -\mathcal{L}\psi+\phi\mathcal{L}\xi-\frac{\beta-\beta^2}{2}\nu(\phi\xi)+\frac{\beta-\beta^2}{2}\nu(\phi\xi)\\
&= -\mathcal{L}(\phi\xi)+\phi\mathcal{L}\xi\\
&= -(\xi\mathcal{L}\phi+\phi\mathcal{L}\xi+c^2\nu\frac{\partial\xi}{\partial\nu}\frac{\partial\phi}{\partial\nu}dt+cS\rho\nu\frac{\partial\xi}{\partial\nu}\frac{\partial\phi}{\partial S}dt)+\phi\mathcal{L}\xi\\
&= -\xi\mathcal{L}\phi-c^2\nu\frac{\partial\xi}{\partial\nu}\frac{\partial\phi}{\partial\nu}dt-cS\rho\nu\frac{\partial\xi}{\partial\nu}\frac{\partial\phi}{\partial S}dt
\end{aligned}$$

As a result, we have

$$\phi_t+\mathcal{L}\phi+c^2\nu\frac{\frac{\partial\xi}{\partial\nu}}{\xi}\frac{\partial\phi}{\partial\nu}+cS\rho\nu\frac{\frac{\partial\xi}{\partial\nu}}{\xi}\frac{\partial\phi}{\partial S}=0, \tag{3.2.8}$$

We find that, after changing of measure, the PDE has two extra terms: $c^2\nu\frac{\frac{\partial\xi}{\partial\nu}}{\xi}\frac{\partial\phi}{\partial\nu}$ and $cS\rho\nu\frac{\frac{\partial\xi}{\partial\nu}}{\xi}\frac{\partial\phi}{\partial S}$. That means there is a "drift" introduced by the change of measure. Notice that ξ only depends on σ, we can solve the following to find ξ

$$dx=\kappa(\theta-x)dt+c\sqrt{x}dZ_t$$
$$\xi(x,t,T)=E[e^{\frac{\beta-\beta^2}{2}\int_t^T x_s ds}|x_t=x]$$

The PDE for $\xi(x,t)$ is

$$\xi_t + \kappa(\theta - x)\xi_x + \frac{1}{2}c^2 x\xi_{xx} + \frac{\beta - \beta^2}{2}x\xi = 0 \tag{3.2.9}$$

Assuming the solution is of the form $\xi(x,t,T) = e^{a(t,T)x+b(t,T)}$ and substituting into the PDE, we get a Ricatti equation

$$\begin{cases} \dot{a} - a\kappa + \frac{1}{2}c^2a^2 + \frac{\beta-\beta^2}{2} &= 0 \\ \dot{b} + a\kappa\theta &= 0 \\ a(T) &= 0 \\ b(T) &= 0 \end{cases} \tag{3.2.10}$$

Assuming $a = \frac{p}{q}$, we get a system of ODE for p and q,

$$\begin{cases} \dot{p} &= \kappa p + \frac{\beta^2-\beta}{2}q \\ \dot{q} &= \frac{1}{2}c^2 p \end{cases} \tag{3.2.11}$$

Solving the above two systems of equations, we have

$$\begin{cases} a(t,T) &= -\frac{\beta^2-\beta}{2}\frac{\sinh\gamma\tau}{\gamma\cosh\gamma\tau + \frac{\kappa}{2}\sinh\gamma\tau} \\ b(t,T) &= \frac{\kappa\theta(\beta^2-\beta)}{c^2}\ln\left(\frac{\gamma e^{\frac{\kappa\tau}{2}}}{\gamma\cosh\gamma\tau+\frac{\kappa}{2}\sinh\gamma\tau}\right) \end{cases} \tag{3.2.12}$$

where $\gamma = \frac{\sqrt{\kappa^2+c^2(\beta^2-\beta)}}{2}$ and $\tau = T - t$. The "drift" introduced by the change of measure is $c^2\nu\frac{\xi_x}{\xi} = a(t,T)c^2\nu$ and $cS\rho\nu\frac{\xi_x}{\xi} = a(t,T)cS\rho$. Putting the "drift" back into the SDE, we have

$$\begin{cases} dS &= a(t,T)cS\rho dt + S\sqrt{\nu_t}dW_t \\ dV &= \frac{\beta-\beta^2}{2}\nu_t dt \\ d\nu_t &= (\kappa - a(t,T)c^2)(\frac{\kappa\theta}{\kappa-a(t,T)c^2} - \nu_t)dt + c\sqrt{\nu_t}dZ_t \\ E[dW_t dZ_t] &= \rho dt \end{cases} \tag{3.2.13}$$

This is a "Heston model" with a set of time-dependent parameters.

3.3 Conclusion

In this chapter, We find that the LETF can be treated as a power option on the underlying ETF with a "time decay" factor. A European call/put option on LETF can be decomposed into two parts: a single contract on the underlying ETF with the "most likely" strike (tangent approximation) and a continuous strip of options on the underlying ETF (tangent correction). We also derived a closed form solution for a European call option on LETF under Heston model by using a change of measure technique. In the next chapter, we seek a model-independent approach to price options on LETFs relative to options on underlying ETFs.

Chapter 4

Relative Value Pricing of Leveraged ETFs Options

4.1 Introduction

Following the discussion of Chapter 3, options on LETFs can be treated as options on the underlying ETFs. As a result, there are three different option markets on the underlying ETF: 1) options on the underlying ETF, 2) options on the bullish LETF, 3) options on the bearish LETF. Theoretically, all the options should be priced under the same risk-neutral measure. Recall formula (2.2.4)

$$L_t = L_0 \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left(((1-\beta)r - f)\, t + \beta \int_0^t \lambda_t ds + \frac{\beta - \beta^2}{2} \int_0^t \sigma_s^2 ds\right), \quad (4.1.1)$$

where r is the reference interest rate, f is the expense ratio and λ is the cost of borrowing. In practice, all the variables in (4.1.1) are observable except the cost of borrowing λ. For the bullish LETFs, $\lambda = 0$, we have an explicit formula. We consider three ways of pricing options on LETFs relative to the option on underlying ETF: 1) We calibrate a stochastic vol model (Heston) to the underlying ETF option market and simulate future market scenarios ("paths") using the calibrated model. Prices of options on LETF are computed by averaging discounted cashflows over the different paths. 2) We extend formulas 3.1.23 to the stochastic volatility environment and compute the value of the option on LETF as a function of options on the underlying ETF. 3) We present a non-parametric formula which links the volatility skew of an LETF with the volatility skew of the underlying ETF based on the observation that the realized variance of an LETF is always $|\beta|$ times the realized variance of the underlying ETF.

We will discuss the three approaches in later sections and compare our model prices with mid-market prices. The results suggest that the implied volatility curves we get from all the three approaches are consist with the market skew.

4.2 Heston Model Approach

The study has the following steps:

1. Clean the historical option price data obtained from Wharton Research Data Services (WRDS)
2. Given a particular date, calibrate the Heston model to the underlying ETF option market
3. Simulate sample paths from the calibrated Heston model and compute the LETF prices using (2.1.3)
4. Compute the model price for options on LETF by averaging the discounted cashflows over different paths and compare that with the mid-market price

4.2.1 Exact Solution of Heston Model

Heston proposed the following model (1993) (see [12]):

$$\begin{cases} dS &= S\sqrt{\nu_t}dW_t \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dZ_t \\ E[dW_t dZ_t] &= \rho dt \end{cases} \tag{4.2.1}$$

The closed form solution of a European call option for the model is (see [12] or [13])

$$C(S_t, \nu_t, t, T) = S_t P_1 - K e^{-(r-q)(T-t)} P_2 \tag{4.2.2}$$

where

$$\begin{aligned}
P_j(x, \nu_t, T, K) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln(K)} f_j(x, \nu_t, T, \phi)}{i\phi}\right) \\
x &= \ln(S_t) \\
f_j(x, \nu_t, T, \phi) &= \exp[C(T-t, \phi) + D(T-t, \phi)\nu_t + i\phi x] \\
C(\tau, \phi) &= (r-q)\phi i\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left(\frac{1 - ge^{d\tau}}{1-g}\right)\right] \\
D(\tau, \phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right) \\
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \\
d &= \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}
\end{aligned}$$

for $j = 1, 2$, and

$$u_1 = \frac{1}{2},\ u_2 = -\frac{1}{2},\ a = \kappa\theta,\ b_1 = \kappa + \lambda - \rho\sigma,\ b_2 = \kappa + \lambda$$

The formula looks a bit complicated, but actually quite explicit and can be easily evaluated in MATLAB. The only part that might give a slight problem is the limits of the integral. This integral cannot be computed exactly, but could be approximated with reasonable accuracy by using some numerical integration technique. The method used here is the MATLAB function `quad(@fun,a,b)`, which uses the *Adaptive Simpson's Rules* to numerically integrate `@fun` over `[a,b]`. It produces a result that has an error less than 10^{-6} or a user defined tolerance level which is prescribed by a fourth argument. The market price of volatility risk λ can be eliminated under the risk neutral measure. In fact, under the risk-neutral measure, we have

$$a = \kappa^* \theta^*, \ b_1 = \kappa^* - \rho\sigma, \ b_2 = \kappa^*, \ \kappa^* = \kappa + \lambda, \ \theta^* = \frac{\kappa\theta}{\kappa + \lambda}$$

As a result, given a set of parameters, we can compute the European call price as precise as we want in MATLAB. This gives us a way to calibrate the Heston model. We follow the same calibration procedure in [14]. We used the MATLAB code computing the exact solution of an European call option under Heston model provided in [14].

4.2.2 Numerical Results of the Heston Model Approach

All the data in this study is obtained from Wharton Research Data Services. For this particular sample study, the reference date is Oct 1st, 2009 and the underlying asset we choose is SPY, which is the ETF that tracks S&P 500 index. The leveraged bullish ETF, SSO, issued by ProShares, tracks twice the daily performance of the S&P 500 index. The reverse leveraged ETF, SDS, also issued by ProShares, tracks twice the inverse of the daily performance of the S&P 500 index. There are three different maturities in this study: 2009/10/17, 2009/11/21 and 2009/12/19. We plotted the market implied volatilities for SPY, SSO, SDS in figure 4.1, 4.2 and

4.3. We calibrate the Heston model to the SPY option market and the model parameters for (4.2.1) are

$$\kappa = 11.6028,\ \theta = 0.0754,\ c = 1.3209,\ \rho = -0.7698,\ \nu_0 = 0.0706 \tag{4.2.3}$$

In our calibration, we don't use any information on the LETF option markets (SSO and SDS).

To check the goodness of fit, we calculate the prices of options on SPY using the calibrated parameters (4.2.3) and compare with the market prices. The results are shown in figure 4.4, 4.5 and 4.6. We notice that, for 2009/10/17 contact, Heston model fails to capture the skew at both ends. It is well-known that stochastic volatility model is not good for calibrating short-dated options. For the other two maturities, 2009/11/21 and 2009/12/19, the calibration is excellent.

We simulate large number of paths for SPY using the calibrated Heston model and calculate the option prices for both SSO and SDS by averaging the discounted cashflows. We compare the model implied vol with the market vol. The results are shown from figure 4.7 to figure 4.12. The model implied vol agrees with the market implied vol well except for the front month contract (10/17/2009), due to the drawback of the Heston model in short-dated option calibration discussed above.

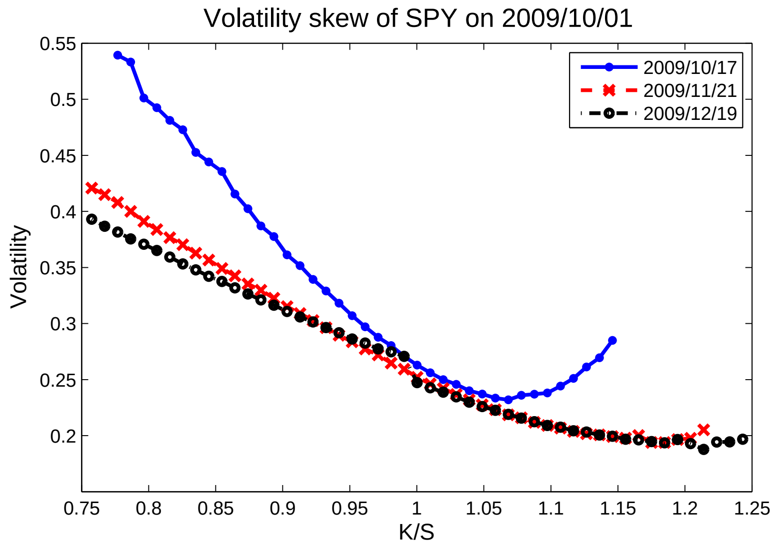

Figure 4.1: depicts the implied volatilities of SPY options on 2009/10/01 as a function of K/S for three different maturities.

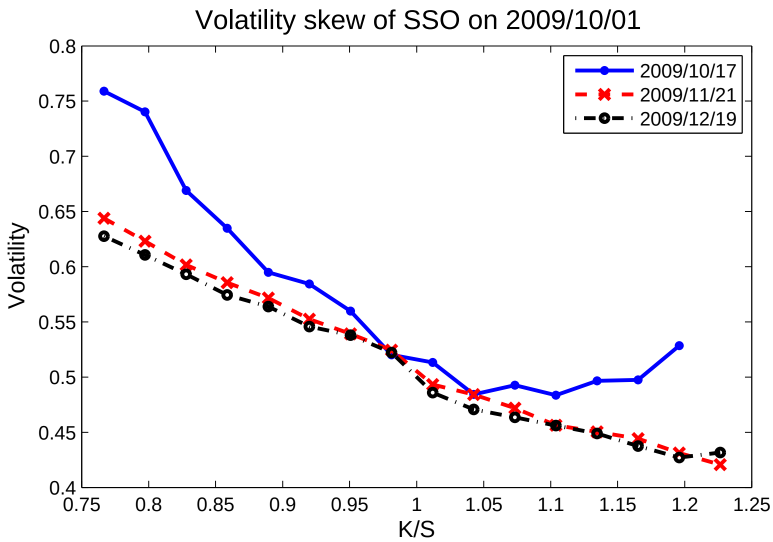

Figure 4.2: depicts the implied volatilities of SSO options on 2009/10/01 as a function of K/S for three different maturities.

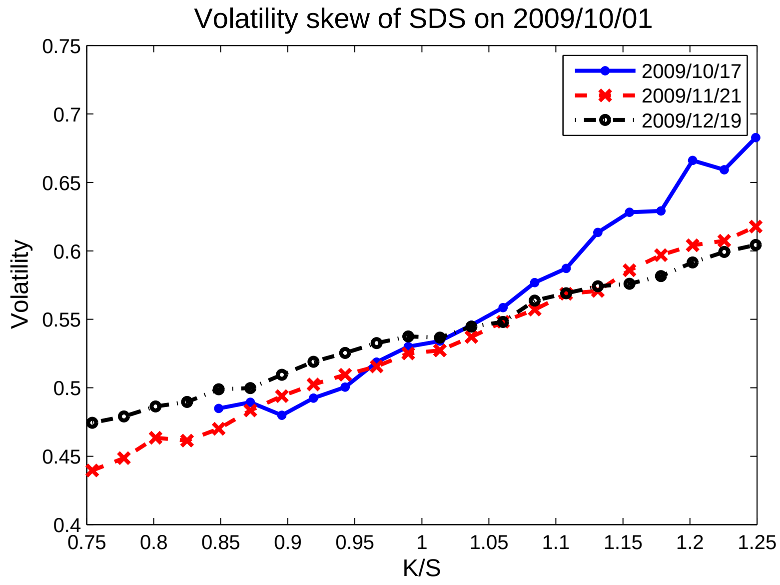

Figure 4.3: depicts the implied volatilities of SDS options on 2009/10/01 as a function of K/S for three different maturities. We see that the skew is inverted

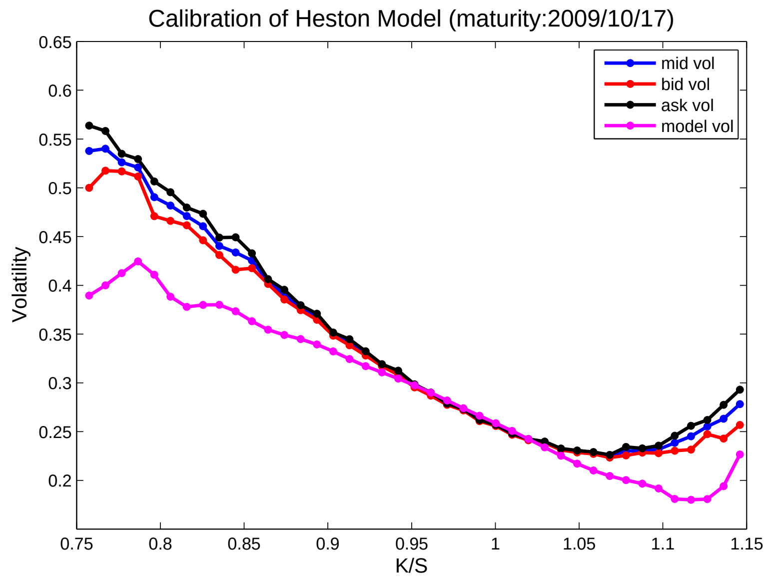

Figure 4.4: comparison of the implied volatilities from the market of SPY with Heston model volatilities using parameter values in 4.2.3, maturity:2009/10/17

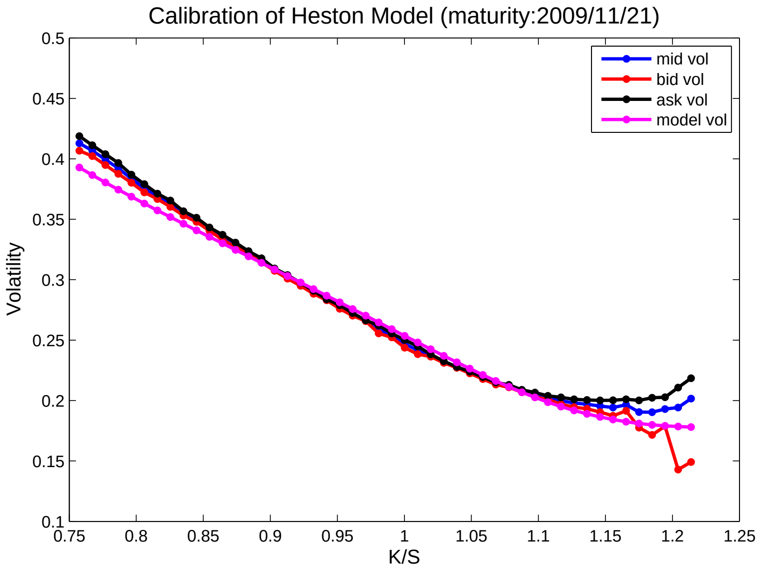

Figure 4.5: similar to figure 4.4, maturity: 2009/11/21

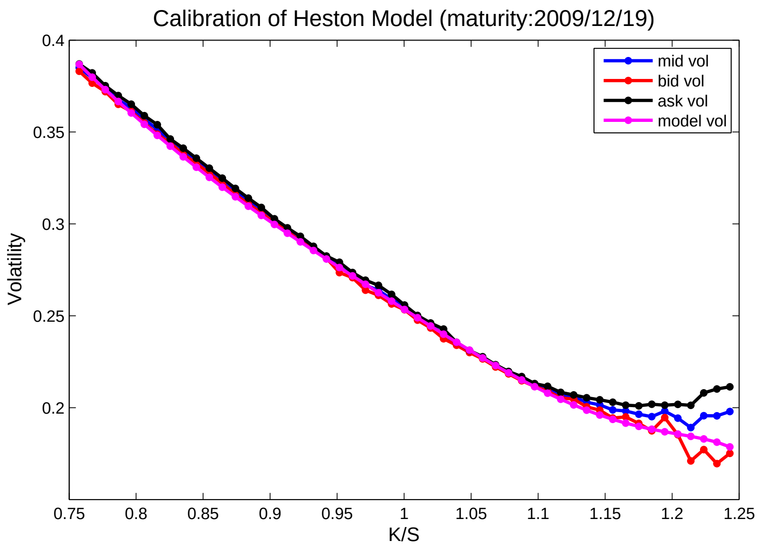

Figure 4.6: similar to figure 4.4, maturity: 2009/12/19

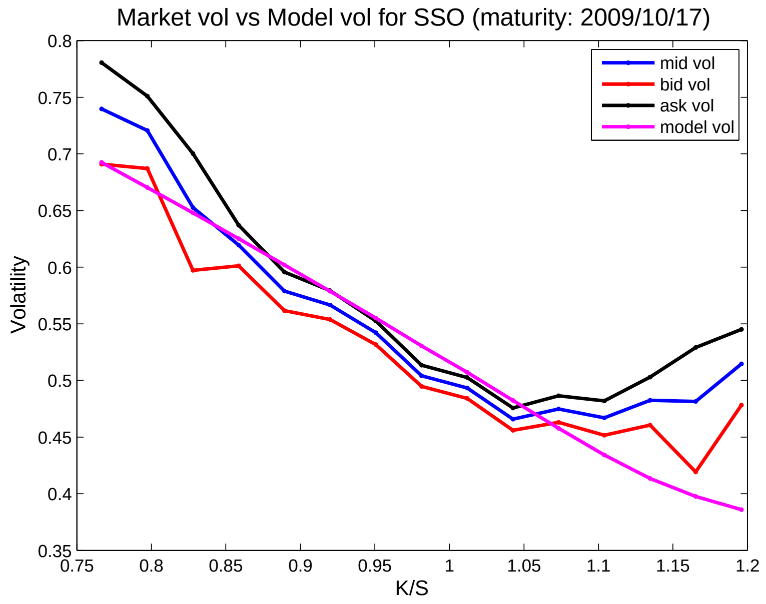

Figure 4.7: we compute the model price by averaging the payoff over simulated paths. Model vol is then computed by inverting the Black-Scholes formula. This is SSO 2009/10/17 contact.

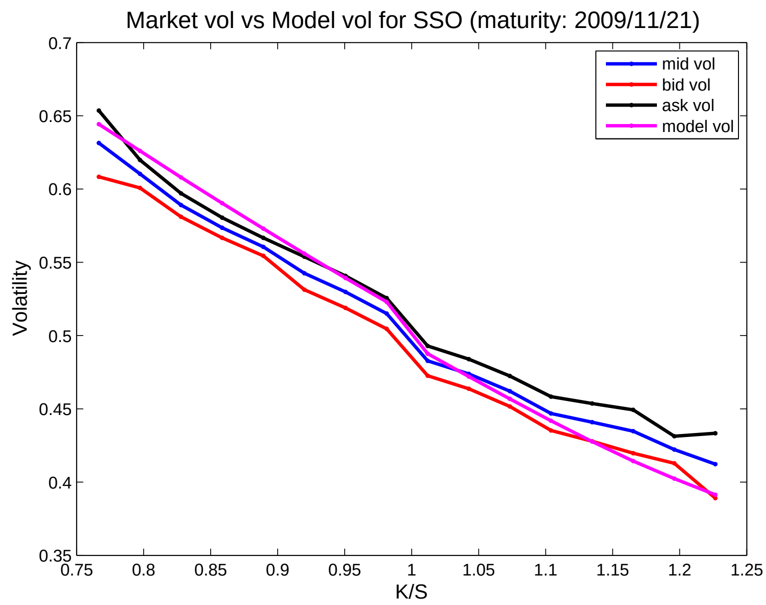

Figure 4.8: similar to 4.7, SSO 2009/11/21 contact.

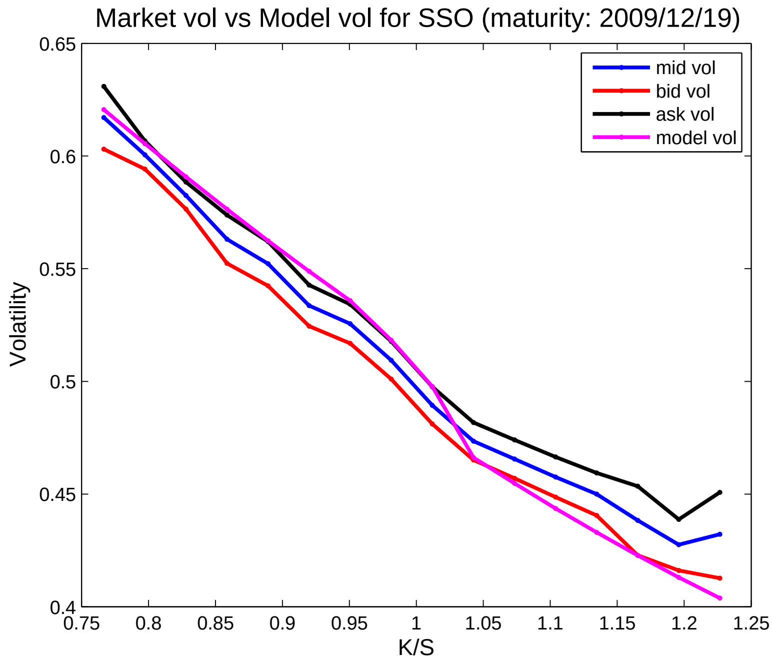

Figure 4.9: similar to 4.7, SSO 2009/12/19 contact.

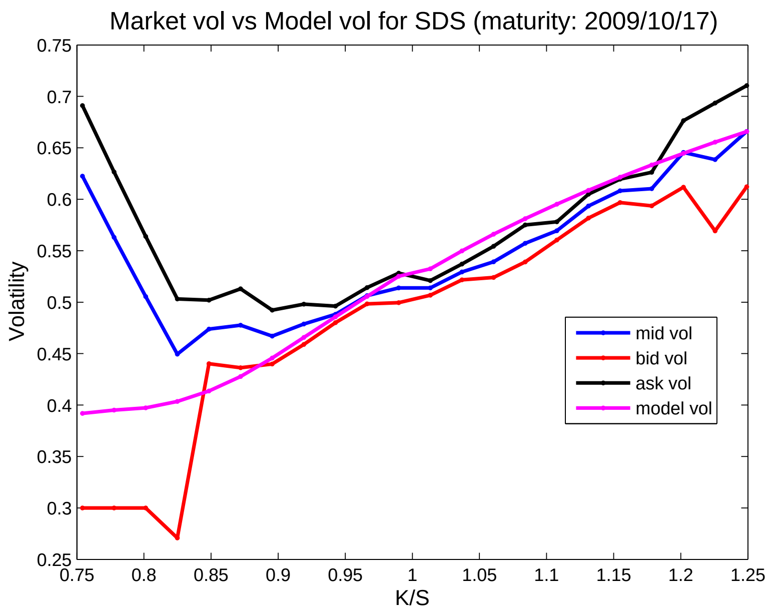

Figure 4.10: similar to 4.7, SDS 2009/10/17 contact.

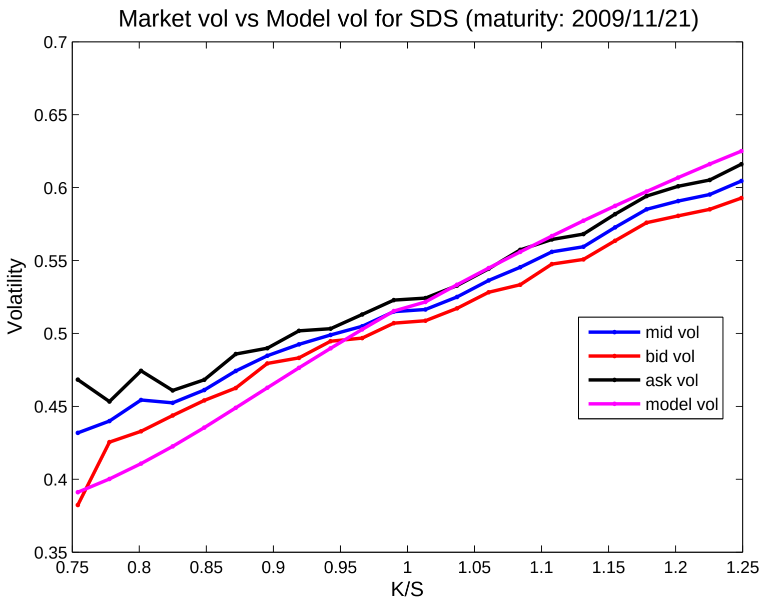

Figure 4.11: similar to 4.7, SDS 2009/11/21 contact.

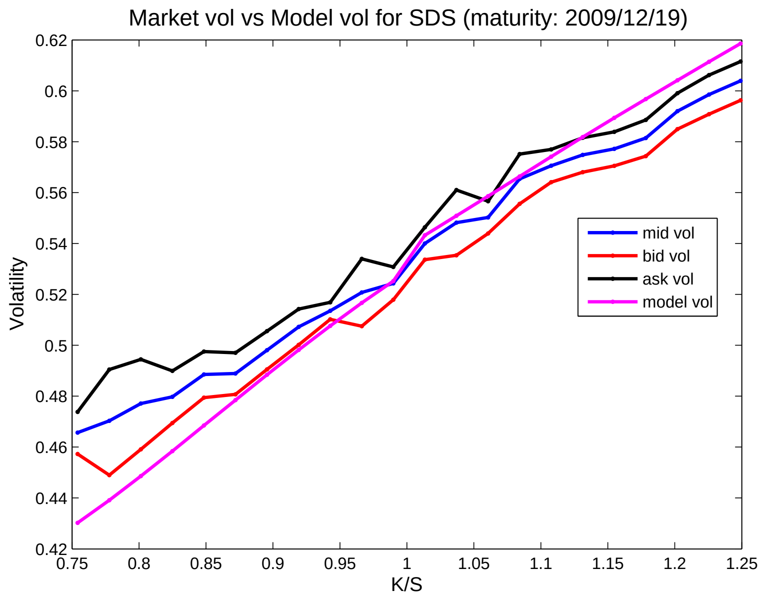

Figure 4.12: similar to 4.7, SDS 2009/12/19 contact.

4.3 Analytical Approximation Approach

Recall that under deterministic volatility assumption, we have (3.1.23) as follows:

$$
\begin{aligned}
C_{L^+}(k,t) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta \left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (\beta - 1) \int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t) dK \right) \\
C_{L^-}(k,t) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} |\beta| \left((k^*)^{\beta-1} P_S(S_0 k^*, t) + (1 - \beta) \int_{0}^{k^*} K^{\beta-2} P_S(S_0 K, t) dK \right) \\
P_{L^+}(k,t) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta \left((k^*)^{\beta-1} P_S(S_0 k^*, t) + (\beta - 1) \int_{0}^{k^*} K^{\beta-2} P_S(S_0 K, t) dK \right) \\
P_{L^-}(k,t) &= \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} |\beta| \left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (1 - \beta) \int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t) dK \right)
\end{aligned}
$$

where $k^* = \left(\frac{k}{L_0} e^{\frac{\beta^2-\beta}{2}V_t} \right)^{\frac{1}{\beta}}$. A natural way to extend them to the stochastic volatility environment is to replace $V_t = \int_0^t \sigma_s^2 ds$ by $V_t = E[\int_0^t \sigma_s^2 ds] = (\text{implied vol})^2 \times t$. That is replacing a random variable by its mean value. The implied vol should be the implied vol of SPY, but that is strike dependent. Which implied vol should we use? The idea is, if SSO goes to the strike k at maturity, the "most likely" vol for SPY should be (implied vol of SSO at strike k)/2. Thus, if we have the skew of the underlying option market, it is straightforward to obtain the skew of the LETF option market using (3.1.23). All the information is available in the market and we tested it with our SPY example. The results are shown from figure 4.13 to figure 4.18. This method actually gives a better fit than the Heston approach, especially for the short-dated options. This shows that using $V_t = (\text{most likely vol})^2 \times t$ is a good way of extending to the stochastic volatility environment.

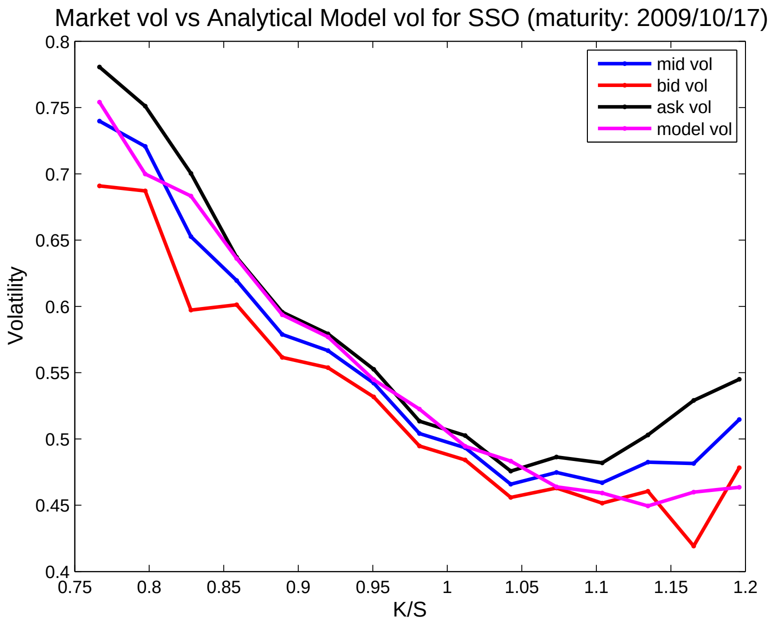

Figure 4.13: We derive the volatility skew of the LETF from formula 3.1.23 with $V_t = \int_0^t \sigma_s^2 ds$ replaced by $V_t = E[\int_0^t \sigma_s^2 ds] = (\text{implied vol})^2 \times t$. This is SSO 2009/10/17 contact.

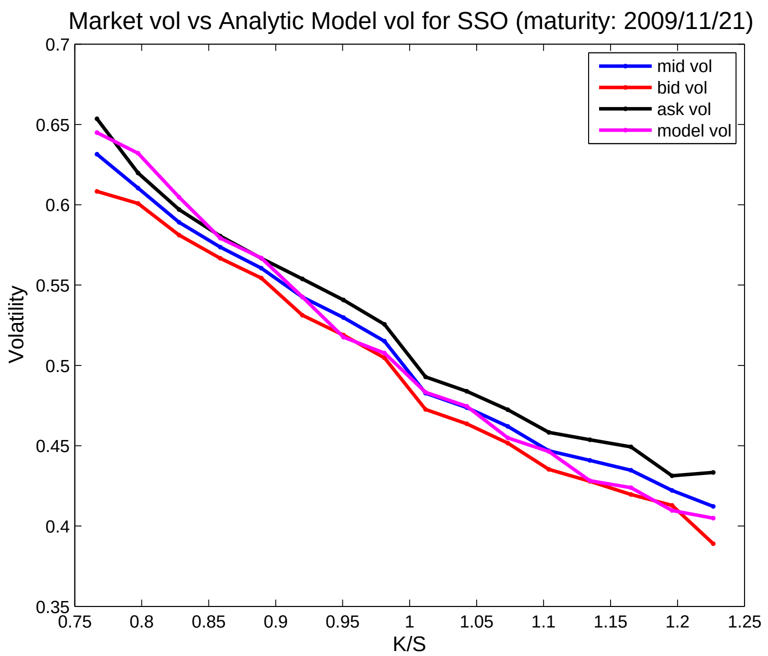

Figure 4.14: similar to 4.13, SSO 2009/11/21 contact.

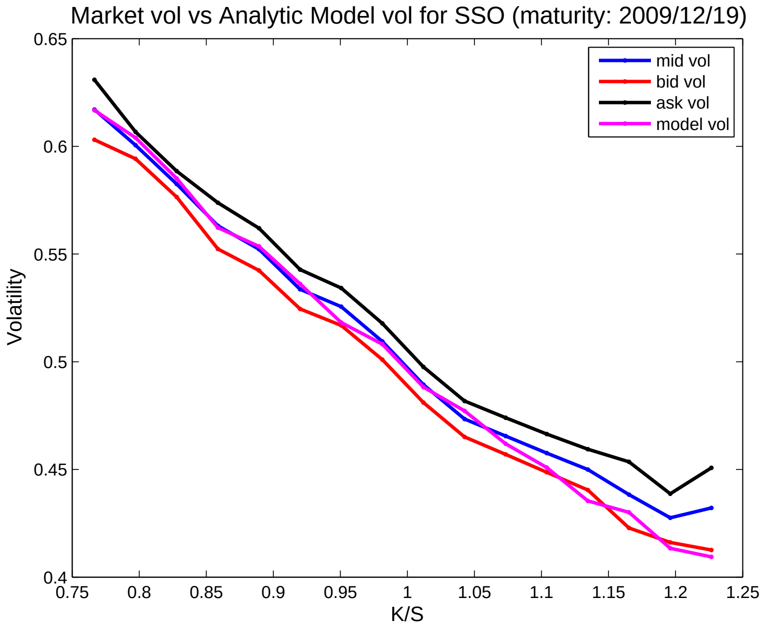

Figure 4.15: similar to 4.13, SSO 2009/12/19 contact.

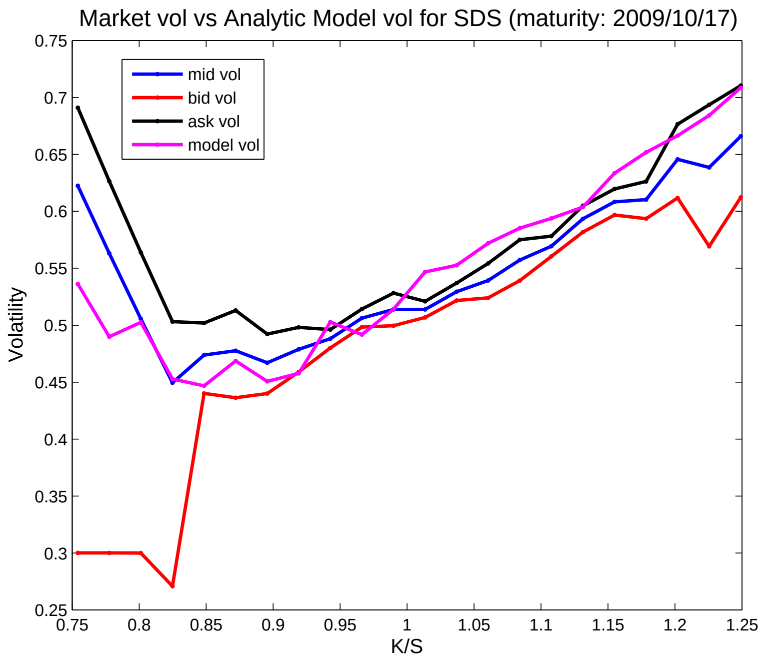

Figure 4.16: similar to 4.13, SDS 2009/10/17 contact.

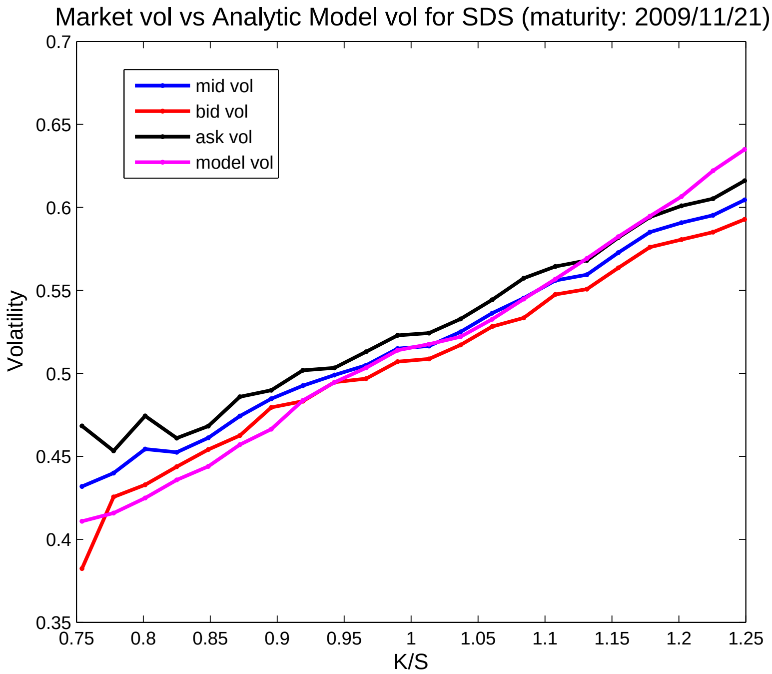

Figure 4.17: similar to 4.13, SDS 2009/11/21 contact.

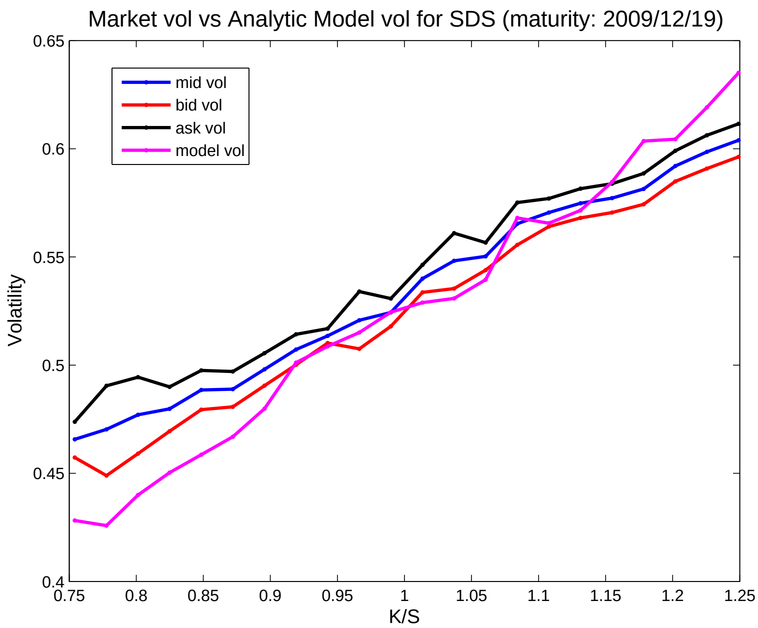

Figure 4.18: similar to 4.13, SDS 2009/12/19 contact.

4.4 Non-parametric Approximation Approach

In the Black-Scholes framework, what really matters is the volatility of the option. In another word, if we want to price an option on LETF, all we have to do is to get the correct volatility relative to the underlying ETF option market. For example, suppose we have an at-the-money (ATM) call option on SSO, given the information on SPY options market, what should be the volatility of this option? We know that the realized volatility of SSO will be exactly twice of the realized volatility of SPY because the daily return of SSO is double of the return of SPY. According to (2.2.4), if $L_t = L_0$, then $S_t = S_0 \left(e^{\frac{\beta^2 - \beta}{2} V_t} \right)^{\frac{1}{\beta}} \neq S_0$, which is the "most likely" strike. Thus, in this example, the ATM option on SSO should have *twice* of the volatility of the SPY option with strike $S_0 k^* = S_0 \left(e^{\frac{\beta^2 - \beta}{2} V_t} \right)^{\frac{1}{\beta}}$. That is

$$\sigma_L(k) = 2\sigma_S(S_0 k^*), \tag{4.4.1}$$

where $\sigma_L(k)$ is the volatility of the option on L with strike k and $\sigma_S(S_0 k^*)$ is the volatility of the option on S with strike $S_0 k^*$. This simple formula reveals the hidden link between the volatility skew of the LETF and the volatility skew of the underlying ETF. Because of its importance, we write it again in a more general form.

$$\sigma_L(k) = |\beta| \sigma_S(S_0 k^*), \tag{4.4.2}$$

where β is the leverage ratio between L and S. So if we want to get the skew of SSO, we compute the "most likely" strikes for all the available strikes of SSO in the market and multiply the implied volatility of the corresponding "most likely" strike options by two. We validate this idea with our SPY example and the results are shown from figure 4.19 to figure 4.24. We see that this non-parametric approach matches the market incredibly well despite its simplicity!

4.5 Conclusion

In this chapter, we studied three different ways of pricing options on LETF relative to options on underlying ETF. Our model option prices on LETF are in line with market prices except minor problem with Heston approach in short-dated contracts. The second approach utilizes the value decomposition formulas (3.1.23) we developed in Chapter 3. The non-parametric approach is based on the observation that the realized volatility of LETF is $|\beta|$ times of that of the underlying ETF. By finding the right corresponding strike, which is the "most likely" strike, we have a simple way of relating the implied volatility skew of the LETF with the implied volatility skew of the underlying ETF.

We will further use this idea of the "most likely" strike in the next chapter to discuss how to hedge an LETF option with the underlying options.

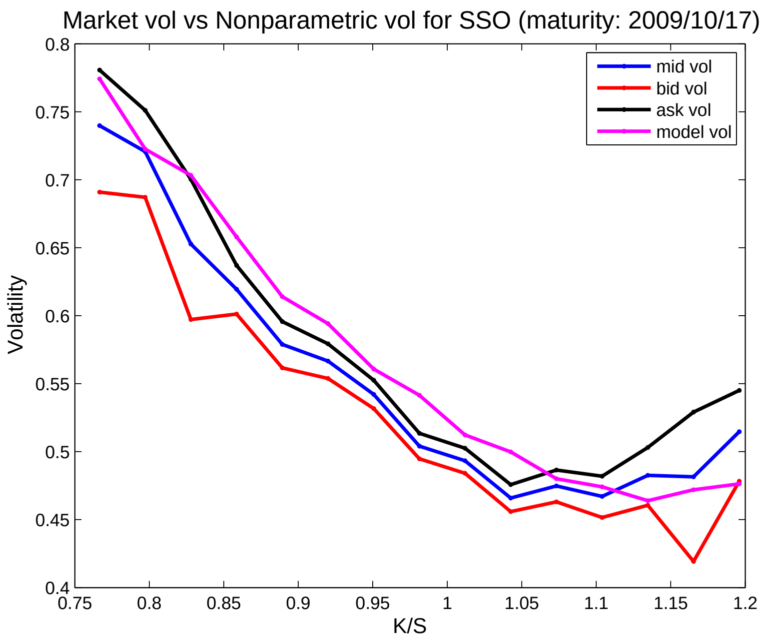

Figure 4.19: Based on the non-parametric approach, we get the implied volatility skew of SSO by the "most likely" strike construction. This is SSO 2009/10/17 contact.

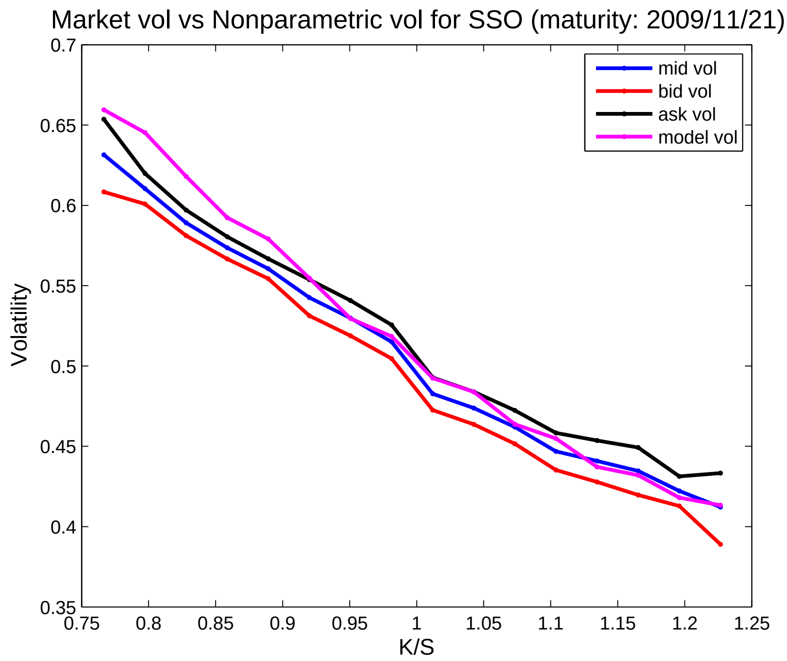

Figure 4.20: similar to 4.19, SSO 2009/11/21 contact.

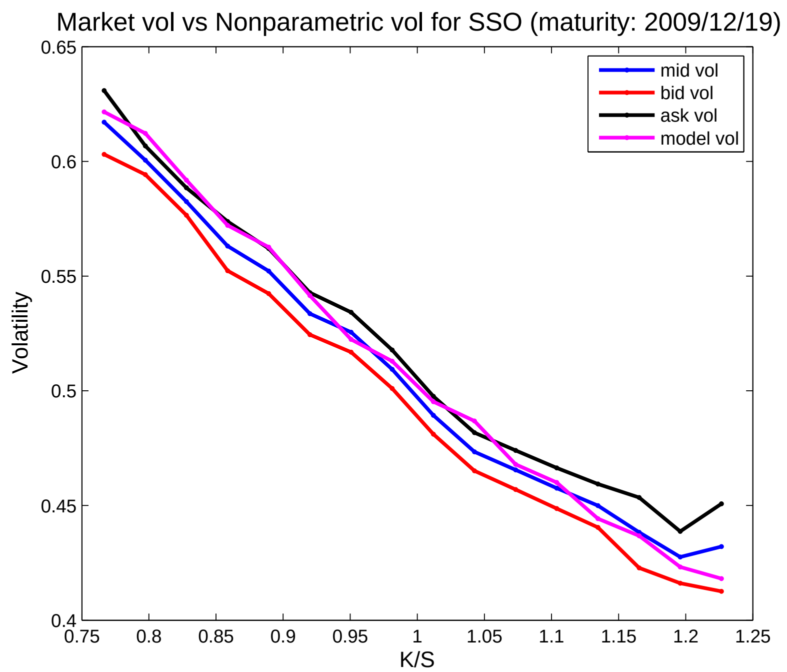

Figure 4.21: similar to 4.19, SSO 2009/12/19 contact.

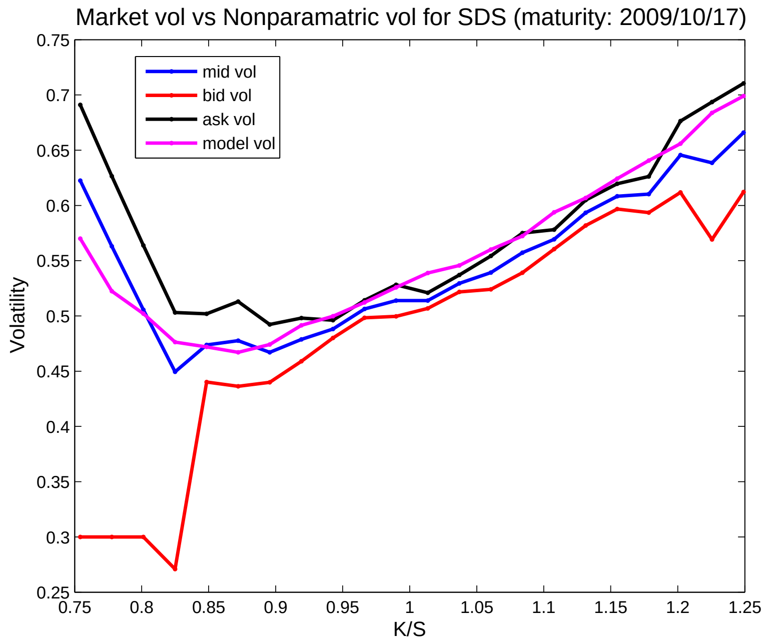

Figure 4.22: similar to 4.19, SDS 2009/10/17 contact.

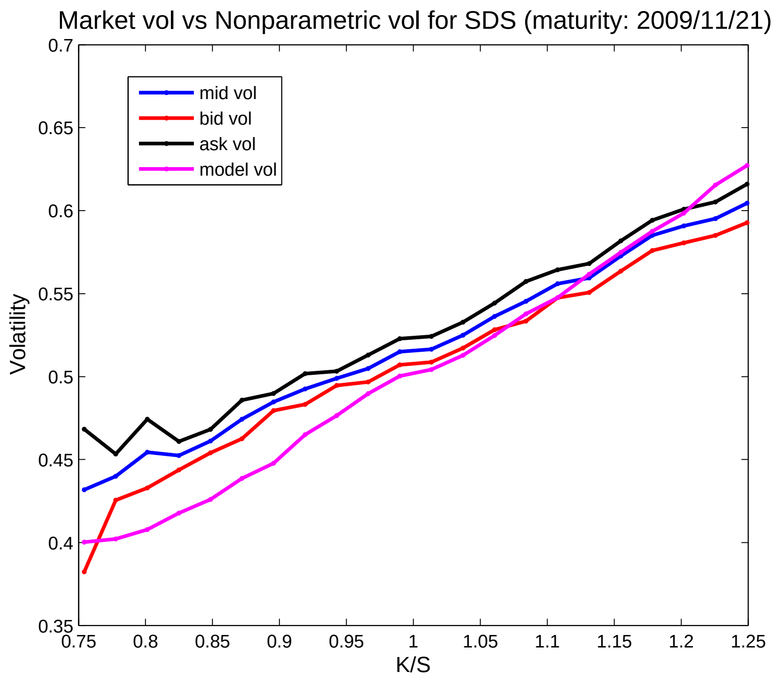

Figure 4.23: similar to 4.19, SDS 2009/11/21 contact.

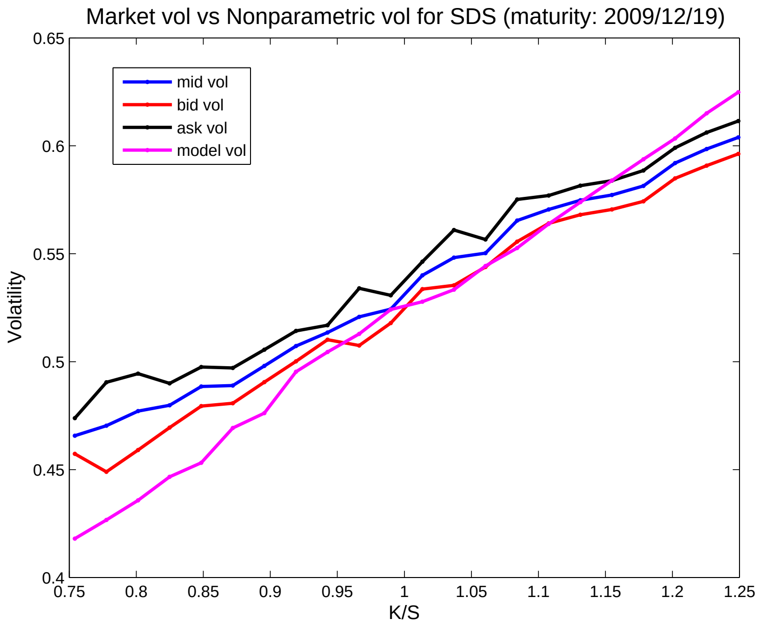

Figure 4.24: similar to 4.19, SDS 2009/12/19 contact.

Chapter 5

Hedging Options on Leveraged ETFs

The idea of relative-value pricing in the last chapter motivates us to think if there exists a way of trading one option market against the other two option markets. One possible way to do this is by regression. We could regress the payoff of the option that we are interested in against the payoffs of options which are available for hedging. For example, we can simulate a large number of paths for SPY and regress the payoff of an option on SSO against the payoffs of options on SPY. Another way is to hedge with the "most likely" contract. We present two examples in detail and compared the two approaches. The results demonstrate that the hedging with the "most likely" strike contact is the best hedge in terms of vega risk.

For a particular example, we study how to hedge call options on SSD or SDS with options on SPY. We regressed the payoff of an LETF option on payoffs of the underlying ETF options. We compare the hedging coefficients obtained by regression with hedging ratio from formula (3.1.23). The result shows that the two approaches give the same hedging coefficients and the "most likely" strike option

Table 5.1: The call option on SSO under consideration

date	maturity	spot	strike	call/put	bid	ask	implied vol
2009/10/01	2009/12/19	32.61	33	call	2.7	2.8	0.5243

is the best hedge. In fact, the integration term in (3.1.23) is insignificant in terms of vega. Most of the vega risk is hedged by the "most likely" strike option.

5.1 Example 1: Hedging a SSO Call Option with SPY Call Options

The SSO call option that we are interested in is shown in table 5.1. The analytical formula which will be tested is the first one in (3.1.23). Since $\beta = 2$ here, we have

$$C_{L^+}(k,t) = 2\frac{L_0}{S_0}e^{V_t}\left(k^*C_S(S_0k^*,t) + \int_{k^*}^{\infty} C_S(S_0K,t)dK\right) \tag{5.1.1}$$

To verify that, we regress the payoff of this option against the options on SPY with strike from 80 to 130. We further impose a condition that all the regression coefficients have to be nonnegative. The result is shown in figure 5.1. It agrees with (3.1.23) quite well. We have two peaks at strike 104 and 105, which corresponds to the first term in (5.1.1). The reason for the existence of the two peaks (sometimes there could be more than 2) is that we are using discrete strikes which are available in the market rather than continuous strikes. The theoretical "most likely" strike is $S_0k^* = S_0\left(\frac{k}{L_0}e^{\frac{\beta^2-\beta}{2}V_t}\right)^{\frac{1}{\beta}}$, where $V_t = E[\int_0^t \sigma_s^2 ds] = (\text{implied vol})^2 \times t$. As we discussed in the last chapter, the correct implied vol to use is (implied vol of SSO at strike k)/2. Thus, we have $S_0k^* = 102.97\left(\frac{33}{32.61}e^{\frac{4-2}{2}\frac{0.5243^2}{4}\frac{55}{252}}\right)^{\frac{1}{2}} = 104.3637$. The theoretical hedging ratio is $\frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta(k^*)^{\beta-1} = 0.6517$ according to (5.1.1). From the numerical result, the hedging coefficients for strike 104 and 105 are 0.3327

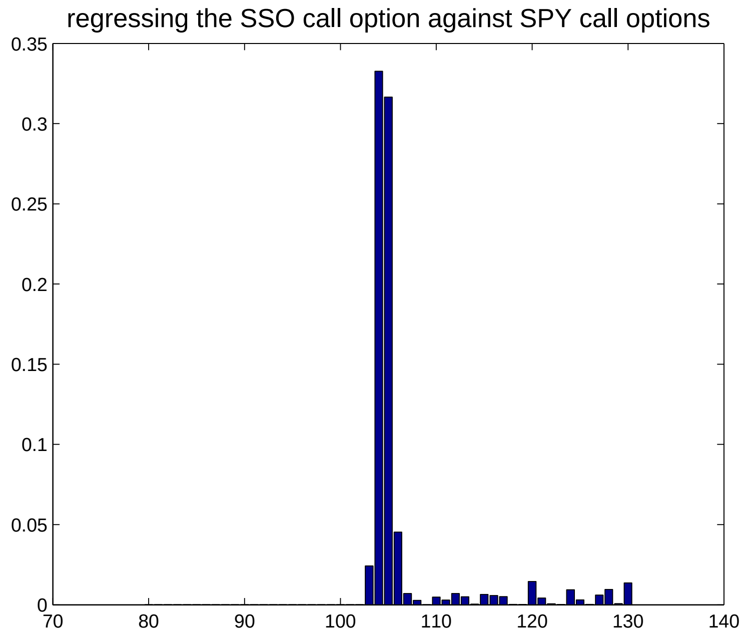

Figure 5.1: depicts the hedging coefficients of SPY options. Notice the coefficients peak at 104 and 105, which have values 0.3327 and 0.3165 respectively. The "most likely strike" is $k^* = 104.3637$. The R^2 is 0.9989

and 0.3165 respectively. The sum of them is 0.6492, which is very close to the theoretical hedging ratio.

If we weight the regression coefficients by the vega of the option, we find that the vega exposure is concentrated on the "most likely" strike as shown in figure 5.2. In fact, the two "most likely" strikes covers over 95% of the vega exposure. The vega of the target option is 12.6 with respect to underlying ETF volatility. The sum of the vega of the two "most likely" options is 12.1. If one would think of hedging the SSO call option with SPY call option of the same moneyness, he would choose the strike to be $102.97 * [1 + 0.5 * (33/32.61 - 1)] = 102.35$. This option only covers 83.2% of the total vega exposure. In another word, the idea of using the same moneyness hedge is not as good as the "most likely" strike hedging.

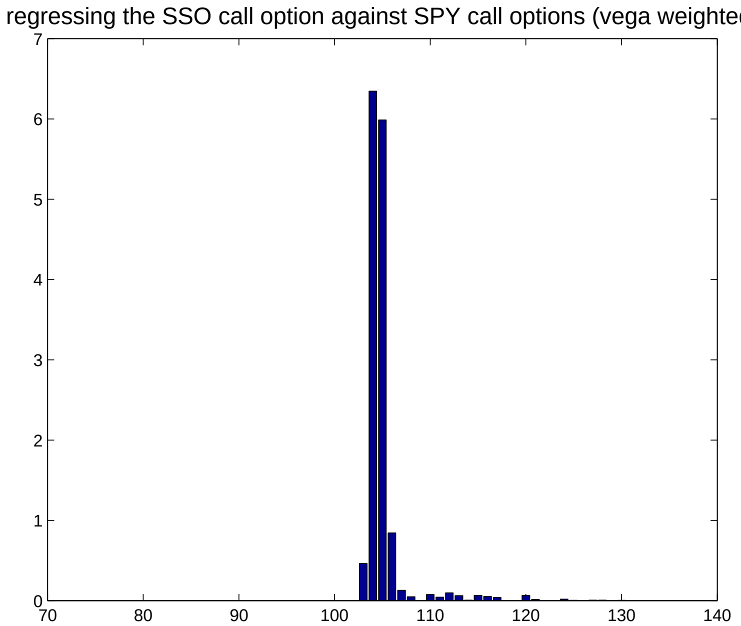

Figure 5.2: depicts the hedging coefficients weighted by the option vegas. As we can see, the "most likely" option covers most of the vega risk.

Table 5.2: The call option on SDS under consideration

date	maturity	spot	strike	call/put	bid	ask	implied vol
2009/10/01	2009/12/19	42.43	43	call	3.9	4	0.5367

5.2 Example 2: Hedging a SDS Call Option with SPY Put Options

Now let's consider hedging a bearish LETF call option with the underlying ETF put options. This corresponds to the second equation in (3.1.23). We have $\beta = -2$ in this case,

$$C_{L^-}(k,t) = 2\frac{L_0}{S_0}e^{-3V_t}\left((k^*)^{-3}P_S(S_0k^*,t) + 3\int_0^{k^*} K^{-4}P_S(S_0K,t)dK\right) \quad (5.2.1)$$

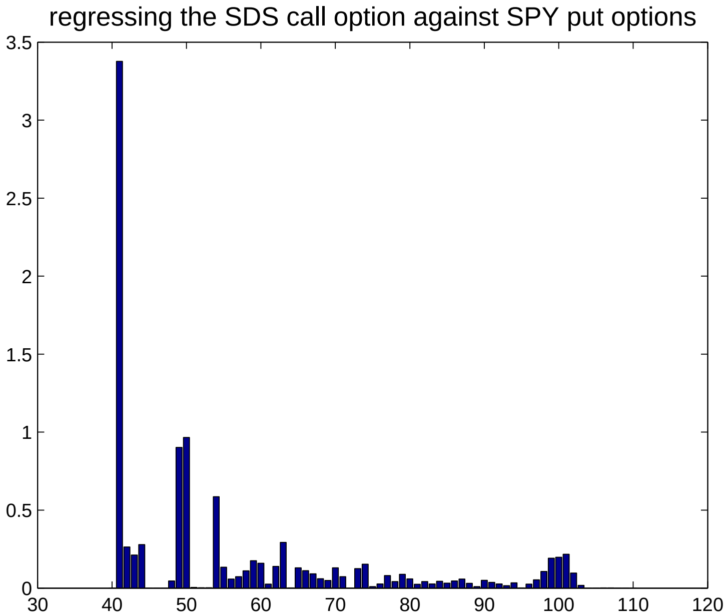

Figure 5.3: depicts the hedging coefficients of SPY options. The coefficient increases as the strike goes to 0, which agrees with equation 5.2.1. The R^2 is 0.9914

Notice, in the second term, K^{-4} goes to infinity as $K \to 0$. That means the hedging coefficient explodes as the strike goes to 0. This is consistent with the result shown in figure 5.3. The "most likely" strike is $S_0 k^* = S_0 \left(\frac{k}{L_0} e^{3V_t}\right)^{-\frac{1}{2}} = 99.97$. To see the picture in a better way, we weight the hedging coefficients by the vega of the options. The result is shown in 5.4. As we can see, the vega risk is concentrated near the "most likely" strike. The top three bars have strikes 99 ,100 and 101. Again, the "most likely" contracts cover most of the vega risk.

Finally, we apply this hedging methodology to all the options of SSO and SDS in the study (214 contacts in total). We summarize the result in Table 5.3. The method provides a very good vega hedging.

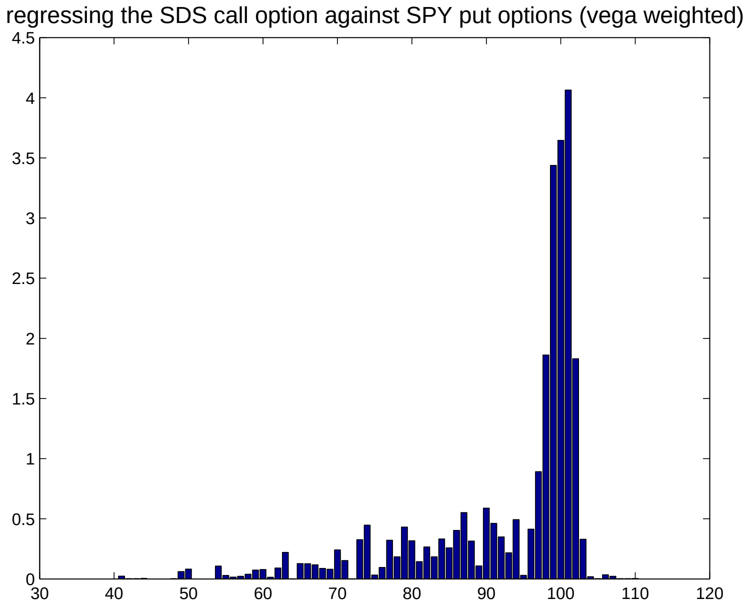

Figure 5.4: we weight the hedging coefficients by the vega of the options. Again, the vega exposure is concentrated around the "most likely" strike, The "most likely strike" is 99.97.

Table 5.3: Summary statistics of hedging with "most likely" strike

ticker	maturity	mean R^2	average % of vega hedged	standard dev
SSO	10/17/2009	0.9980	100.21%	1.95%
SSO	11/21/2009	0.9982	98.47%	1.40%
SSO	12/19/2009	0.9980	95.98%	5.76%
SDS	10/17/2009	0.9963	101.84%	1.04%
SDS	11/21/2009	0.9925	93.56%	3.59%
SDS	12/19/2009	0.9899	89.12%	3.77%

5.3 Conclusion

In this chapter, we explored the idea of hedging an option on LETF with the option on underlying ETF with the "most likely" strike. The result shows that the "most likely" contracts are the right hedge in terms of vega risk and this is confirmed by numerical regression. This is a consequence from the relationship in (4.4.2).

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