# CANNON'S CONJECTURE

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Cannon's Conjecture (CC) gives a group-theoretic generalization of the generic case of W. P. Thurston's famous Geometrization Conjecture (GC) for 3-dimensional manifolds recently proved by G. Perelman. CC is motivated by that part of the Geometrization Conjecture that concerns the problem of promoting a metric of variable negative curvature to a metric of constant negative curvature. CC claims that, if an infinite, finitely presented group acts roughly like the fundamental group of a hyperbolic 3-manifold near infinity, then the group can be realized as a group of $2 \times 2$ matrices with complex entries acting conformally on the 2-dimensional sphere $\mathbb{S}^2$ and by rigid motion on non Euclidean hyperbolic 3-dimensional space $\mathbb{H}^3$. Perelman's work for manifolds and its recent generalization to orbifolds establishes the result in the case that the group is in fact the fundamental group of a 3-dimensional manifold or orbifold but leaves the general case open.

Here is a precise statement of the conjecture followed by the supporting definitions.

**Cannon's Conjecture.** *Suppose that $G$ is an infinite, finitely presented group whose Cayley graph is Gromov-hyperbolic and whose space at infinity is the 2-sphere $\mathbb{S}^2$. Then $G$ is a Kleinian group.*

**Cayley graph.** Let $G$ denote a group having a finite generating set $C = \{c_1, \ldots, c_k\}$. Arthur Cayley associated with each such group $G$ a connected graph $\Gamma = \Gamma(G, C)$ that gives a geometric picture of the group $G$ and, in fact, supplies an abbreviated multiplication table for the group. The *Cayley graph* is defined as follows. The vertices of $\Gamma$ are the elements $g \in G$. Each edge of $\Gamma$ has the form $e = (g, c, g \cdot c)$, is directed from initial vertex $g \in G$ to terminal vertex $g \cdot c \in G$, and is labelled by element $c \in C$. The inverse of this edge has the form $e^{-1} = (g \cdot c, c^{-1}, g)$.

**Gromov-hyperbolic group.** Non Euclidean geometry has *thin triangles*, a simple property not satisfied in Euclidean space. Here is the appropriate definition as applied to Cayley graphs. The Cayley graph $\Gamma$ has a natural

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metric which assigns each edge the length of 1. Between each pair $p, q \in \Gamma$, there will be one or more shortest paths, which are called *geodesics*. A *geodesic triangle* $\Delta$ in $\Gamma$ then consists of three points $a$, $b$, and $c$ and geodesic paths $ab$, $bc$, and $ca$ between them. If $\delta > 0$ is a positive number, then we say that the triangle $\Delta$ is *$\delta$-thin* if the $\delta$-neighborhood of each point of $\Delta$ intersects at least two of the paths $ab$, $bc$, and $ca$. The Cayley graph $\Gamma$ and the group $G$ are said to be *Gromov-hyperbolic* if there is a positive number $\delta$ such that every geodesic triangle in $\Gamma$ is $\delta$-thin. This property, unusual as it may seem, is actually generic in the sense that it is satisfied by the Cayley graphs of most finitely presented groups.

**The space at infinity.** Every Gromov-hyperbolic group $G$ has a natural *space at infinity* defined as follows. Let 0 represent a fixed point of $\Gamma$. Each point at infinity is represented by a geodesic ray $R$ which begins at $0 = R(0)$ and is parametrized by distance $t \in [0, \infty)$ from the base point 0. Two rays $R$ and $S$ represent the same point at infinity if, for some positive bound $B$, $R(t)$ and $S(t)$ are within $B$ of one another for all values of $t$. Points represented by rays $R$ and $S$ are said to be *close* to each other if $R(t)$ and $S(t)$ are close to one another for all $t$ in a large initial interval $[0, T]$. The space at infinity is always finite dimensional, compact, and metrizable.

**Kleinian group.** A group $G$ is *Kleinian* if it can be realized as a group of $2 \times 2$ complex matrices acting conformally on the 2-sphere $\mathbb{S}^2$ and by rigid motions on hyperbolic 3-dimensional non Euclidean geometry. Here are the precise technical conditions that will not explained here: namely, the action on $\mathbb{H}^3$ should be isometric, properly discontinuous, and cocompact.

**The difficulty in proving the conjecture is this.** The hypothesis gives a natural action of the group $G$ on two combinatorially defined objects, namely the Cayley graph and its topological 2-sphere space at infinity. The Cayley graph is in a precise sense analogous to non Euclidean hyperbolic 3-space $\mathbb{H}^3$, but the hypothesis gives no natural connection between the two. The space at infinity is, in a precise sense, analogous to the 2-sphere boundary $\mathbb{S}^2$ of $\mathbb{H}^3$, but the hypothesis gives no natural connection between the given topological space at infinity and the classical complex analytic structure on the round 2-sphere. There are uncountably many ways of introducing an analytic structure on the topological space at infinity, and at most one of these uncountably many structures is compatible with the group action and can satisfy the conjecture. That is, a search is required for the (possibly existing) one and only one needle in an uncountable haystack.

**Many approaches have been taken in attempts to prove the conjecture.** It is enough to show that it is possible to embed the Cayley graph into $\mathbb{H}^3$ in such a way that the metrics of the two spaces are comparable; but this is a difficult task. Other attacks are based on a combinatorial version of classical conformal modulus. Others try to minimize Hausdorff dimension among metrics on $\mathbb{S}^2$ compatible with the combinatorial group action. Another approach employs nonstandard Teichmüller theory. A fourth uses measures of regularity from geometric measure theory.

The modulus approach considers expanding balls in the metric on the Cayley graph. Each ball about the base point induces an open cover of the space at infinity. Each open cover of the space at infinity gives an approximate measure to the conformal shapes of sets at infinity. These approximate shapes can be individually optimized by combinatorial versions of the classical Riemann mapping theorem. Provided that these optimized shapes are not drastically distorted as the balls expand in radius so that the covers at infinity become finer and finer, then the desired analytic structure at infinity can be found. The unsurmounted difficulty is that one is required to compare the results of countably many optimizations of ever increasing complexity. [See the Combinatorial Riemann Mapping Theorem and Cannon-Swenson.] The necessary and sufficient conditions have been formulated as axioms to be checked.

The approach via Hausdorff dimension requires alteration of the metric at infinity in such a way as to be compatible with the group action yet reduce the Hausdorff dimension of the space at infinity to 2. As of yet, the dimension has been reduced to a value within the range $(2, 2+\epsilon)$, where $\epsilon > 0$ can be chosen arbitrarily.

Gromov hyperbolic groups all enjoy a linear recursive structure at infinity which gives a local combinatorial picture of the space at infinity by a planar subdivision rule. This subdivision rule can be defined by a self-map of a non-Hausdorff, compact surface. Analytic structures on this surface can presumably be classified by some nonstandard Teichmüller space. The self-map of the surface should induce a continuous self-map of the corresponding Teichmüller space, and the desired analytic structure at infinity should then arise from a fixed point of the Teichmüller map.

Another approach takes tools from geometric measure theory and seeks measures on the space at infinity such that the group action is sufficiently regular.

It may be that the conjecture will finally be resolved by modifications of Perelman's approach via flow of Ricci curvature.

**Related problems.** The general version of the problem is this: *Given an object defined combinatorially, optimize the shape of that object geometrically.*

For example, consider a knotted circle $K$ in the 3-dimensional sphere $\mathbb{S}^3$. Thurston proved that the complement $\mathbb{S}^3 \setminus K$ of $K$ in $\mathbb{S}^3$ has a unique geometric structure. Almost all of these unique structures are non Euclidean hyperbolic. In fact, according to the surveys of Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks, there are 1,701,936 prime knots having 16 or fewer crossings, and only 32 of these are not hyperbolic.

Similarly, according to a natural notion of density, almost all finitely presented groups are Gromov hyperbolic. For most of these, however, the space at infinity is not the 2-sphere.

Among the countably many compact 2-dimensional manifolds all but four admit a hyperbolic non-Euclidean structure.

Every rational function of one complex variable defines a mapping from the 2-sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ to intself. This map is *branched* in the sense that, in local coordinates, it looks like a power map $z \mapsto z^k$, with $k$ varying from point to point. A natural question asks, "If the local branching data is given for some self-map of the 2-sphere $\mathbb{S}^2$, when is there a rational map having that branching data?" For a large class of branched maps, Thurston has answered the question in terms of a certain obstruction which can sometimes be calculated. It is conjectured that the Thurston obstructions for branched maps are in some precise sense equivalent to the modulus axioms mentioned above relative to Gromov-hyperbolic groups.

The same subdivision rules that exist at infinity for Gromov-hyperbolic groups can often be defined for branched maps of the 2-sphere and for the complements of knotted circles in $\mathbb{S}^3$. For branched maps, the subdivision rules give a combinatorial analysis of the Julia set of the map. It is possible that subdivision rules may prove to be a unifying point of view in the study of Kleinian groups, knot groups, and rational maps.