PI SPACES WITH ANALYTIC DIMENSION 1 AND ARBITRARY TOPOLOGICAL DIMENSION

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Abstract. For every $n$, we construct a metric measure space that is doubling, satisfies a Poincare inequality in the sense of Heinonen-Koskela, has topological dimension $n$, and has a measurable tangent bundle of dimension 1.

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1. Introduction

Since they were introduced in [HK98], PI spaces (metric measure spaces that are doubling and satisfy a Poincaré inequality) have been investigated extensively, leading to progress in many directions. In spite of this, there remains a gap between the structural constraints imposed by the existing theory and the properties exhibited by known examples. On the one hand existing examples come from a variety of different sources:

1. Sub-Riemannian manifolds.
2. Limits of sequences of Riemannian manifolds with a lower bound on the Ricci curvature [CC00].
3. Metric measure spaces satisfying synthetic Ricci curvature conditions [Raj12].
4. Certain Ahlfors-regular topological manifolds [Sem].
5. Examples with “small” singular sets [Laa02, Sem96].
6. Boundaries of hyperbolic groups [BP99].
7. Quotient constructions [Laa00].
8. Inverse limit constructions [CK15].
9. Spaces obtained from the above by taking products, gluing [HK98, Thm 6.15], or passing to nice subsets [MTW13].

On the other hand, in some respects this list is somewhat limited. For instance, the examples in (2), (4), (5) are rectifiable, and many in (3) are known to be rectifiable [MN, Gig, GMR], while those in (6), (7) and (8) admit a common description as limits of inverse systems and have very similar properties. Moreover, if \((X, \mu)\) denotes one of the above examples, then the infinitesimal structure of \((X, \mu)\) has a special form, in the sense that when one blows-up \((X, \mu)\) at \(\mu\)-a.e. point, one gets a metric measure space that is bilipschitz equivalent to the product of a Carnot group with an example as in (8).

Our purpose in this paper is to construct PI spaces that have different characteristics from previously known examples. Before stating our theorem, we recall that any PI space \((X, \mu)\) has a measurable (co)tangent bundle [Che99]; we will refer to its dimension as the analytic dimension of \((X, \mu)\).

**Theorem 1.1.** For every \(n\), there is a complete self-similar PI space \((X_\infty, \mu_\infty)\) with analytic dimension 1 and topological dimension \(n\). Furthermore, for some \(\alpha \in (0, 1)\):
There is a surjective David-Semmes regular map \( \hat{\pi}^\infty : (\mathbb{R}^n, d_\alpha) \to X_\infty \), where \( d_\alpha \) is the partial snowflake metric on \( \mathbb{R}^n \) given by
\[
d_\alpha(p, p') = |p_1 - p'_1| + \sum_{i=2}^n |p_i - p'_i|^\alpha.
\]
In particular, letting \( L^\alpha \) denote Lebesgue measure, for \( Q = 1 + (n-1)\alpha^{-1} \), the pushforward measure \( \mu_\infty = \hat{\pi}^\infty_\#(L^\alpha) \) is comparable to \( Q \)-dimensional Hausdorff measure \( H^Q \) on \( X_\infty \), and \( X_\infty \) is Ahlfors \( Q \)-regular.

(2) \( X_\infty \) has topological and Assouad-Nagata dimension \( n \) (see Section 6).

(3) \( (X_\infty, H^Q) \) satisfies a \((1,1)\)-Poincaré inequality (see Section 7).

(4) \( (X_\infty, H^Q) \) has analytic dimension 1: there is a single Lipschitz function \( x_\infty : X_\infty \to \mathbb{R} \) that is a chart on all of \( X_\infty \), i.e. it defines the measurable differentiable structure for \( (X_\infty, H^Q) \) (see Section 8).

(5) Let \( \hat{\Gamma} \) be the family of horizontal lines in \( \mathbb{R}^n \), equipped with the obvious measure. Then the pushforward of \( \hat{\Gamma} \) under the map \( \hat{\pi}^\infty : \mathbb{R}^n \to X_\infty \) gives a universal Alberti representation in the sense of [Bat15] for \( (X_\infty, \mu_\infty) \) (see Section 8).

(6) If \( \{p_k\} \subset X_\infty \), \( \lambda_k \subset (0, \infty) \) are arbitrary sequences, and \( (Z, z) \) is a pointed Gromov-Hausdorff limit of the sequence \( \{(\lambda_k X_\infty, p_k)\} \) of pointed rescalings of \( X_\infty \), then (2)-(4) hold for \( (Z, z) \). Moreover, there is a collection of at most \( N = N(n) \) David-Semmes regular maps \( (\mathbb{R}^n, d_\alpha) \to X_\infty \) whose images cover \( X_\infty \).

For comparison, we note that all the previously known examples with analytic dimension 1 have topological dimension 1 (see [BP99, Laa00, CK15]).

We refer the reader to Section 2 for an overview of the proof of Theorem 1.1 and to Section 11 for some generalizations.

We now pose some questions concerning the relation between the topological and the analytical structure of PI spaces.

The examples in Theorem 1.1 have small analytic dimension and arbitrarily large topological dimension. One may ask if the topological dimension can be small while the analytic dimension is large. This is not an interesting question, though: there are compact subsets \( X \subset [0, 1]^n \) with positive Lebesgue measure such that the metric measure space \( (X, \mathcal{L}^n) \) is a PI space with analytic dimension \( n \) and topological dimension 1 (see [MTW13] for the \( n = 2 \) case). Nonetheless, such
examples are rectifiable and look on the small scale like $\mathbb{R}^n$ itself, in
the sense that typical blow-ups are copies of $\mathbb{R}^n$. This motivates the
following revised version of the above question:

**Question 1.2.** Pick $n \geq 2$. Is there a PI space of analytic dimension
$n$ and Assouad-Nagata dimension 1?

The Assouad-Nagata dimension is a notion that is metric based, scale
invariant, well behaved with respect to Gromov-Hausdorff limits, and is
bounded below by the topological dimension (see Section 6); in particular,
the rectifiable examples mentioned above have Assouad-Nagata di-
mension $n$. Rather than using Assouad-Nagata dimension in Question
1.2, one could require instead that every blow-up (i.e. weak tangent) of $X$
has topological dimension 1.

The spaces in Theorem 1.1 have complicated local topology. One
may wonder if there are examples with similar properties that are topo-
logical manifolds:

**Question 1.3.** Pick $n \geq 2$. Suppose $(X, \mu)$ is a PI space homeo-
morphic to $\mathbb{R}^n$, such that any pointed Gromov-Hausdorff limit of any
sequence of pointed rescalings of $X$ is also homeomorphic to $\mathbb{R}^n$. What
are the possibilities for the analytic dimension of $(X, \mu)$? The Haus-
dorff dimension? For instance, is there such a PI space homeomorphic
to $\mathbb{R}^3$ (or even $\mathbb{R}^2$) with analytic dimension 1? Or one which is Ahlfors
$Q$-regular for large $Q$?

**Organization of the paper.** In Section 2 we define the metric space
$X_\infty$ in the $n = 2$ case and then discuss some of the key points in
the proof of Theorem 1.1. The $n = 2$ case of Theorem 1.1 is proven
in Sections 3-9 and the case of general $n$ is treated in Section 10.
In Section 11 we consider a more general class of examples of direct
systems that have many of the same features.

2. Overview of the proof of Theorem 1.1

In this section we define $X_\infty$ and related objects in the $n = 2$ case. We
also have an informal discussion of the proof of Theorem 1.1. The
proof itself appears in Sections 3-10.

**Standing assumptions:** The objects and notation introduced in
this section will be retained up through Section 9.
A combinatorial description of partial snowflake metrics on $\mathbb{R}^2$. Pick integers $m, m_v$ with $2 \leq m < m_v$.

For $j \in \mathbb{Z}$, let $Y_j$ be the cell complex associated with the tiling of $\mathbb{R}^2$ by the translates of the rectangle $[0, m^{-j}] \times [0, m_v^{-j}]$. Thus the translation group $(m^{-j}\mathbb{Z}) \times (m_v^{-j}\mathbb{Z})$ acts by cellular isomorphisms on $Y_j$. Given $k \geq 0$, we may view $Y_j + k$ as a $k$-fold iterated subdivision of $Y_j$, where at each iteration the 2-cells are subdivided $m$ times in the horizontal direction and $m_v$ times in the vertical direction. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\Phi((x, y)) = (m^{-1}x, m_v^{-1}y)$. Then $\Phi^k : \mathbb{R}^2 \to \mathbb{R}^2$ induces a cellular isomorphism $Y_j \to Y_j + k$ for all $j, k \in \mathbb{Z}$.

We now define a metric on $\mathbb{R}^2$ based on the combinatorial structure of the $Y_j$’s. Let $\hat{d}^\infty_Y$ be the largest pseudodistance on $\mathbb{R}^2$ with the property that for every $j$, each cell of $Y_j$ has $\hat{d}^\infty_Y$-diameter at most $m^{-j}$. One readily checks (see Lemma 5.1) that $\hat{d}^\infty_Y$ is comparable to the partial snowflake metric

$$d_\alpha((p_1, p_2), (p'_1, p'_2)) = |p_1 - p'_1| + |p_2 - p'_2|^\alpha,$$

where $\alpha = \log m / \log m_v$.

**The definition of $X_\infty$.** We will define the space $X_\infty$ as a quotient of $(\mathbb{R}^2, \hat{d}^\infty_Y)$, where the quotient is generated by certain identifications that respect the $x$-coordinate. One may compare this with Laakso’s construction of PI spaces as quotients of the product $[0, 1] \times C$, where $C$ is a Cantor set [Laa00]. Henceforth we will call a 1-cell of $Y_j$ horizontal (respectively vertical) if it is a translate of $[0, m^{-j}] \times \{0\}$ (respectively $\{0\} \times [0, m_v^{-j}]$).

Choose a large integer $L$ (e.g. $L = 100$), and set $m = 4$ and $m_v = 3L$.

For all $k, \ell \in \mathbb{Z}$, $i \in \{1, 2, 3\}$, we define the following pair of vertical 1-cells of $Y_1$ (see Figure 1):

$$a_{k,\ell,i} = \left\{ \frac{i}{4} + k \right\} \times [(3\ell + i - 1)m_v^{-1}, (3\ell + i)m_v^{-1}],$$

$$a'_{k,\ell,i} = \left\{ \frac{i}{4} + k \right\} \times [(3\ell + i)m_v^{-1}, (3\ell + i + 1)m_v^{-1}],$$

Note that $a'_{k,\ell,i}$ is the image of $a_{k,\ell,i}$ under the vertical translation $(x, y) \mapsto (x, y + m_v^{-1})$. The collections $\{a_{k,\ell,i} \mid k, \ell \in \mathbb{Z}, 1 \leq i \leq 3\}$, $\{a'_{k,\ell,i} \mid k, \ell \in \mathbb{Z}, 1 \leq i \leq 3\}$ are invariant under translation by $\mathbb{Z}^2$ and are contained in the union of vertical lines $\{(x, y) \in \mathbb{R}^2 \mid x \in \frac{1}{4}\mathbb{Z} \setminus \mathbb{Z}\}$. 


Next, we define an equivalence relation $\mathcal{R}$ on $\mathbb{R}^2$ by identifying $a_{k,\ell,i}$ with $a'_{k,\ell,i}$ by the vertical translation $(x, y) \mapsto (x, y + m_v^{-1})$ for all $k, \ell \in \mathbb{Z}$, $1 \leq i \leq 3$. Note that $\mathcal{R}$ is invariant under translation by $\mathbb{Z}^2$.

Let $\mathcal{R}_\infty$ be the equivalence relation on $\mathbb{R}^2$ generated by the collection of pushforwards $\Phi^j_\ast \mathcal{R}$ for all $j \in \mathbb{Z}$. We define $X_\infty$ to be the quotient $\mathbb{R}^2/\mathcal{R}_\infty$, and let $\hat{\pi}_\infty : \mathbb{R}^2 \to X_\infty$ be the quotient map. We metrize $X_\infty$ using the largest pseudodistance $\hat{d}_\infty^X$ on $X_\infty$ such that for every $j \in \mathbb{Z}$ and every 2-cell $\hat{\sigma}$ of $Y_j$, the projection $\hat{\pi}_\infty(\hat{\sigma}) \subset X_\infty$ has $\hat{d}_\infty^X$-diameter at most $m^{-j}$. It is not hard to see that $\hat{d}_\infty^X$ is the largest pseudodistance on $X_\infty$ such that $\hat{\pi}_\infty : (\mathbb{R}^2, \hat{d}_\infty^Y) \to (X_\infty, \hat{d}_\infty^X)$ is 1-Lipschitz. Henceforth we use $\hat{d}_\infty$ instead of $\hat{d}_\infty^X$ when there is no risk of confusion.
**PI SPACES WITH ANALYTIC DIMENSION 1**

**X\(_\infty\) as a direct limit of cell complexes.** While the definition of \(X\_\infty\) as a quotient \(\mathbb{R}^2/\mathcal{R}_\infty\) is transparent, it does not provide a convenient framework for understanding the structure of \(X\_\infty\). Instead, we will analyze \(X\_\infty\) by representing it as a direct limit.

For every \(j \in \mathbb{Z}\), let \(\mathcal{R}_j\) be the equivalence relation on \(\mathbb{R}^2\) generated by \(\Phi^i_\ast \mathcal{R}\) for all \(i \in \mathbb{Z}\) with \(i < j\), and let \(X_j\) be the quotient \(\mathbb{R}^2/\mathcal{R}_j\) equipped with quotient topology. Since \(\mathcal{R}_j^{-1} \subset \mathcal{R}_j\), the quotient maps induce a direct system of topological spaces

\[
\ldots \xrightarrow{\pi_{-1}} X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} \ldots
\]

For all \(i \in \mathbb{Z}, \ j \in \mathbb{Z} \cup \{\infty\}\) with \(i \leq j\), we denote the projection map \(X_i \to X_j\) by \(\pi_{i,j}\), the quotient map \(\mathbb{R}^2 \to X_j\) by \(\hat{\pi}_j\), and the composition \(\pi_{i,j} \circ \hat{\pi}_i : Y_i \to X_j\) by \(\hat{\pi}_{i,j}\).

We metrize \(X_j\) using the largest pseudodistance \(\hat{d}_j\) on \(X_j\) such that for every \(i \leq j\) and every cell \(\hat{\sigma}\) of \(Y_i\), the image \(\hat{\pi}_j(\hat{\sigma}) \subset X_j\) has \(\hat{d}_j\)-diameter at most \(m^{-i}\).

**Remark 2.1.** Our examples were partly inspired by [CK15, Sec. 11], which gives a construction of PI spaces of topological and analytic dimension \(n\). In fact, \(X\_\infty\) arose when we attempted to find an “anisotropic” variant of the cube complex examples of [CK15, Sec. 11]. However, this leads to a situation where the projection map \(\pi_0^\infty : X_\infty \to [0,1]^n\) is Lipschitz with respect to the partial snowflake metric on \([0,1]^n\), and is moreover a light map. This is incompatible with the existence of a Poincaré inequality in \(X_\infty\).

**Discussion of the proof.** We now give an indication of some of the key points in the proof of Theorem 1.1.

The first part of the proof, which appears in Sections 3-4, develops the combinatorial and metric structure of the direct system \(\{X_j\}\). Because the equivalence relation \(\mathcal{R}_j\) may be generated by identifying pairs of vertical 1-cells of the cell complex \(Y_j\) by vertical translations, the cell structure of \(Y_j\) descends to a cell structure on \(X_j\). This cell complex has controlled combinatorics; because the distance \(\hat{d}_j\) is defined combinatorially, this implies that \((X_j,\hat{d}_j)\) is doubling at the scale \(m^{-j}\). A key estimate (Proposition 4.12) compares \((X_j,\hat{d}_j)\) with \((X_\infty,\hat{d}_\infty)\) is that for every \(p,p' \in X_j\) one has

\[
\hat{d}_j(p,p') - 2m^{-j} \leq \hat{d}_\infty(\hat{\pi}_j^\infty(p),\hat{\pi}_j^\infty(p')) \leq \hat{d}_j(p,p').
\]

In particular, this implies that the sequence \(\{X_j\}\) Gromov-Hausdorff converges to \(X_\infty\).
With the foundation laid in Sections 3–4, several parts of Theorem 1.1 follow fairly easily:

- By a short argument (Lemma 5.2), one deduces the David-Semmes regularity of the projection \( \hat{\pi}^\infty : (\mathbb{R}^2, d^\infty_Y) \to X^\infty \), which gives part (1) of Theorem 1.1.
- The restriction of the projection \( \hat{\pi}^\infty : \mathbb{R}^2 \to X^\infty \) to the boundary of the unit square \( \partial[0,1]^2 \) is injective. By an elementary topological argument this implies that the topological dimension of \( X^\infty \) is at least 2 (see Lemma 5.2).
- Using (2.2), one shows that there is a "good cover" of \( X^\infty \) whose inverse image under \( \pi^\infty_j : X^j \to X^\infty \) approximates the decomposition of \( X^j \) into open cells. This proves that the Assouad-Nagata dimension of \( X^\infty \) is at most 2 (see Theorem 6.2).

The remaining assertions of Theorem 1.1 have to do with the analytical structure of \( X^\infty \), and are largely based on the existence of good families of curves, whose construction we now describe.

The starting point is the observation that the first coordinate \( x : \mathbb{R}^2 \to \mathbb{R} \) descends to a 1-Lipschitz function \( x^\infty : X^\infty \to \mathbb{R} \), and hence any horizontal geodesic segment in \( \mathbb{R}^2 \) projects to a geodesic in \( X^\infty \). Every 2-cell \( \hat{\sigma} \) of \( Y^j \) is a rectangle of width \( m^{-j} \) and height \( m_j^{-1} v \) so it yields a family \( \Gamma^\hat{\sigma} \) of geodesic segments in \( X^\infty \) of length \( m^{-j} \); since \( \Gamma^\sigma \) has a natural parametrization by an interval of length \( m_j^{-1} \), it carries a natural measure \( \nu^\sigma \).

If \( \hat{\sigma}, \hat{\sigma}' \) are 2-cells of \( Y^j \) such that the projections \( \hat{\pi}^j(\sigma), \hat{\pi}^j(\sigma') \) share a vertical 1-cell of \( X^j \), then the corresponding measured families of curves \( (\Gamma^\hat{\sigma}, \nu^\sigma), (\Gamma^\hat{\sigma}', \nu^\sigma') \) may be concatenated to form a new measured family of curves. More generally, if \( \sigma_1, \ldots, \sigma_\ell \) is a sequence of 2-cells of \( X^j \) that form a horizontal gallery\(^1\) in \( X^j \) (i.e. \( \sigma_{i-1} \) shares a vertical 1-cell with \( \sigma_i \) for all \( 1 < i \leq \ell \)) then one may concatenate the corresponding curve families. To produce an abundance of such curve families, a crucial property is the “horizontal gallery accessibility” (Lemma 7.1), which is an analog of the gallery diameter bound of [CK15, Sec. 11, condition (3)]. This says that if \( \sigma, \sigma' \) are 2-cells of \( X^j \) with controlled combinatorial distance in \( X^j \), then they may be joined by a horizontal gallery of controlled length. This accessibility property is due to the choice of the equivalence relation \( R \); notice that if \( \hat{\sigma}, \hat{\sigma}' \) are 2-cells of \( Y_1 \), then one may form a horizontal gallery between their projections.

\(^1\)We have borrowed the term “gallery” from the theory of Coxeter complexes and Tits buildings.
σ, σ′ ∈ X_1 by using the vertical identifications a_{k,ℓ,i} ↔ a'_{k,ℓ,i} that define R.

The (1, 1)-Poincaré inequality is proved using the standard approach via “pencils” of curves [Sem]. The pencils are built by combining the measured families of curves described above, for horizontal galleries of different scales, see Section 7.

To prove that the analytic dimension of X_∞ is 1, we define a “horizontal derivative” D_∞ u for any Lipschitz function u : X_∞ → R. By using horizontal galleries again, we show that at an approximate continuity point p of D_∞ u, we have

\[ u(q) - u(p) = (D_∞ u(p))(x_∞(q) - x_∞(p)) + o(d(q,p)), \]

see Section 8.

Remark 2.3. Rather than metrizing X_∞ using ˆd_∞, an alternate approach is to define a metric on X_∞ using a distinguished set of paths. For instance, following [Laa00] one could define, for all p, p′ ∈ X_∞, the distance d(p, p′) to be the infimal 1-dimensional Hausdorff measure of (ˆπ_∞)^{-1}(γ), where γ is a path from p to p′. This leads to an essentially equivalent analysis, with the details organized somewhat differently. We found the approach using ˆd_∞ more transparent.

Notational conventions. In the following a ≈ b will indicate that a and b are comparable up to a uniformly bounded multiplicative factor C, and sometimes we will also write a ≈ C b to highlight C. We will similarly use expressions like a ≲ b and a ≳ b.

3. The Cellular Structure of the Direct System \{X_j\}

In this section we examine different aspects of the combinatorial structure of the X_j’s. We remind the reader that we will retain the notation from Section 2 through Section 9. The following lemma lists the main properties of R and R_j that will be used later.

Lemma 3.1 (Properties of R).

(1) Nontrivial cosets of Φ^{-1}_∞ R belong to the union of vertical lines \{(x, y) ∈ R^2 \mid x ∈ m^{-j}Z \setminus m^{-j-1}Z\}, and any two points in the same coset lie in a vertical edge path of Y_j of combinatorial length at most 2.

(2) Cosets of R_j are contained in orbits of the action m^{-j}Z × m_v^{-j}Z ↷ R^2 by translations.
(3) For all $j \in \mathbb{Z}$, let $\mathcal{R}_{j+1}$ be the equivalence relation on $X_j$ generated by the pushforward $(\hat{\pi}_j \circ \Phi^j)_*(\mathcal{R})$. Then $\mathcal{R}_{j+1}$ is an equivalence relation on $X_j$, and the cosets of $\mathcal{R}_{j+1}$ are the fibers of $\pi_j : X_j \to X_{j+1}$.

(4) For $i = 0, 1$, let $\hat{\sigma}_i$ be a cell of $Y_j$, and $\hat{p}_i$ be an interior point of $\hat{\sigma}_i$. If $\hat{p}_0 \sim_{\mathcal{R}_j} \hat{p}_1$, then there is a translation $t \in m^{-j}\mathbb{Z} \times m_\circ^{-j}\mathbb{Z}$ such that $t(\hat{\sigma}_0) = \hat{\sigma}_1$, and $\hat{q}_0 \sim_{\mathcal{R}_j} t(\hat{q}_0)$ for all $\hat{q}_0 \in \hat{\sigma}_0$.

Proof. (1) and (2) follow immediately from the definition of $\mathcal{R}$ and $\Phi$.

(3) By (1) the nontrivial cosets of $\Phi_{j-1}^j \mathcal{R}$ intersect only trivial cosets of $\mathcal{R}_{j-1}$. This implies that $\mathcal{R}_{j+1}$ is an equivalence relation on $X_j$, and that its cosets are the fibers of $\pi_j : X_j \to X_{j+1}$.

(4) Since $\mathcal{R}_j$ is generated by $\Phi_i^j \mathcal{R}$ for $i < j$, it is generated by pairwise identifications of cells of $Y_j$ by translations. This implies (4). □

Lemma 3.2 (Cell structure of $X_j$).

(1) The collection of images of the open cells of $Y_j$ under the projection map $\hat{\pi}^j : Y_j \to X_j$ defines a CW complex structure on $X_j$.

(2) For every cell $\hat{\sigma}$ of $Y_j$, the restriction of $\hat{\pi}^j$ to $\hat{\sigma}$ is a characteristic map for the open cell $\pi^j(\text{Int}(\hat{\sigma})) \subset X_j$.

(3) If $\hat{\sigma}_0, \hat{\sigma}_1$ are cells of $Y_j$ then $\hat{\pi}^j(\text{Int}(\hat{\sigma}_0)) \cap \hat{\pi}^j(\text{Int}(\hat{\sigma}_1)) \neq \emptyset$ if and only if $\mathcal{R}_j$ identifies $\hat{\sigma}_0$ with $\hat{\sigma}_1$ by translation. In particular, if $\hat{\sigma}$ is a cell of $Y_j$, then $\hat{\pi}^j|_\sigma : \sigma \to X_j$ identifies certain pairs of faces of $\hat{\sigma}$ by translation.

(4) The cells of $Y_{j+k}$ project under $\hat{\pi}^j : \mathbb{R}^2 \to X_j$ to define a cell complex $X_j^{(k)}$, which is the $k$-fold iterated subdivision of $X_j$.

Proof. (1)-(3) We first show that $X_j$ is Hausdorff.

Pick $p \in X_j$. By Lemma 3.1(2) and the fact that $Y_j$ is invariant under $m^{-j}\mathbb{Z} \times m_\circ^{-j}\mathbb{Z}$, there is an $r > 0$ such that any cell $\hat{\sigma}$ of $Y_j$ that intersects $N_r((\hat{\pi}^j)^{-1}(p))$ must intersect $(\hat{\pi}^j)^{-1}(p)$. Here $N_r(S) = \{q \in \mathbb{R}^2 \mid d_{\mathbb{R}^2}(q, S) \leq r\}$ denotes the Euclidean metric $r$-neighborhood.

Claim. $N_r((\hat{\pi}^j)^{-1}(p))$ is a union of cosets of $\mathcal{R}_j$.

Suppose $\hat{q}' \in \mathbb{R}^2$ and $\hat{q}' \sim_{\mathcal{R}_j} \hat{q}$ for some $\hat{q} \in N_r((\hat{\pi}^j)^{-1}(p))$. By the choice of $r$, if $\text{Int}(\hat{\sigma})$ is the open cell of $Y_j$ containing $\hat{q}$, then $\hat{\sigma}$ contains some point $\hat{p} \in (\hat{\pi}^j)^{-1}(p)$. By Lemma 3.1(4) there is a translation...
Let $T_j^2$ denote the quotient space $\mathbb{R}^2/(m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z})$, and $\pi_{T_j^2} : \mathbb{R}^2 \to T_j^2$ be the quotient map. The open cells of $Y_j$ project under $\pi_{T_j^2}$ to open cells of $T_j^2$, inducing the standard cell structure on $T_j^2$. Note that by Lemma 3.1[2] there is a well-defined continuous map $X_j \to T_j^2$.

Now consider a cell $\hat{\sigma}$ of $Y_j$. The composition $Y_j \xrightarrow{\hat{\pi}^j} X_j \to T_j^2$ maps the interior of $\hat{\sigma}$ homeomorphically onto an open cell of $T_j^2$; it follows that $\hat{\pi}^j : Y_j \to X_j$ is a characteristic map for the open cell $\hat{\pi}^j((\text{Int}(\hat{\sigma})))$. If $\hat{\sigma}_0, \hat{\sigma}_1$ are cells of $Y_j$ such that $\hat{\pi}^j((\text{Int}(\hat{\sigma}_0))) \cap \hat{\pi}^j((\text{Int}(\hat{\sigma}_1))) \neq \emptyset$, then by Lemma 3.1[4] we have $\hat{\pi}^j((\text{Int}(\hat{\sigma}_0))) = \hat{\pi}^j((\text{Int}(\hat{\sigma}_1)))$. This shows that the collection of images of open cells of $Y_j$ is a decomposition of $X_j$ into disjoint open cells.

For any cell $\hat{\sigma}$ of $Y_j$, the closure of $\hat{\pi}^j((\text{Int}(\hat{\sigma})))$ is just $\hat{\pi}^j(\hat{\sigma})$, and is therefore contained in the union of the images of the open cells contained in $\hat{\sigma}$. The closure of any open cell of $X_j$ intersects only finitely many open cells.

Finally, if $C \subset X_j$, then $C$ is closed if and only if $(\hat{\pi}^j)^{-1}(C)$ is closed, which is equivalent to $(\hat{\pi}^j)^{-1}(C) \cap \hat{\sigma}$ being closed for every cell $\hat{\sigma}$ of $Y_j$, which happens if and only if $C \cap \hat{\pi}^j(\hat{\sigma})$ is closed.

Thus we have verified (1)-(3).

The proof of (4) is similar to the proof of (1).

Henceforth the notation $X_j$ and $X_j^{(k)}$ will refer to the cell complex structure established in Lemma 3.2. A 1-cell of $X_j^{(k)}$ is horizontal (vertical) if it is the image of a horizontal (vertical) cell of $Y_{j+k}$ under the projection map $\hat{\pi}^j : \mathbb{R}^2 \to X_j$.

We now analyze the behavior of the projection maps with respect to the combinatorial structure.
Lemma 3.3.  

(1) If \( p, p' \in X_j \) are distinct points and \( \pi_j(p) = \pi_j(p') \), then there is a vertical edge path in the 1-skeleton of \( X_j^{(1)} \) that contains \( p, p' \) and has combinatorial length at most 2.

(2) If \( \sigma, \sigma' \) are 2-cells of \( X_j \), and \( \pi_j(\sigma) \cap \pi_j(\sigma') \neq \emptyset \), then \( \sigma \cap \sigma' \neq \emptyset \).

(3) Suppose \( 0 \leq j \leq j' < \infty \) and \( \hat{\sigma}, \hat{\sigma}' \) are 2-cells of \( Y_j \) and \( Y_{j'} \), respectively. If \( \hat{\pi}^\infty(\hat{\sigma}) \cap \hat{\pi}^\infty(\hat{\sigma}') \neq \emptyset \), then \( \hat{\pi}^j(\hat{\sigma}) \cap \hat{\pi}^{j'}(\hat{\sigma}') \neq \emptyset \).

Proof. (1). This follows from (1) and (3) of Lemma 3.1.

(2). Suppose that \( p \in \sigma, p' \in \sigma' \), and \( \pi_j(p) = \pi_j(p') \). By (1) there is a vertical edge path in \( X_j^{(1)} \) of combinatorial length at most two joining \( p \) to \( p' \). Since the subdivision of \( \sigma \) in \( X_j^{(1)} \) has combinatorial “height” \( m_v \), it follows that \( \sigma \cap \sigma' \neq \emptyset \).

(3). By the definition of the direct limit, we have \( \hat{\pi}^\ell(\hat{\sigma}) \cap \hat{\pi}^\ell(\hat{\sigma}') \neq \emptyset \) for some \( \ell \). Letting \( k_0 + 1 \) be the minimal such \( \ell \), suppose we have \( k_0 \geq j' \). Since \( \sigma, \sigma' \) are finite unions of 2-cells of \( Y_{k_0} \), we may assume without loss of generality that \( j = j' = k_0 \). Then \( \hat{\pi}^{k_0}(\hat{\sigma}), \hat{\pi}^{k_0}(\hat{\sigma}') \) are disjoint 2-cells of \( X_{k_0} \) whose projections to \( X_{k_0+1} \) intersect. This contradicts (2). Thus \( k_0 + 1 \leq j' \), proving (3).

The next lemma shows that the \( X_j \)’s and the projection maps have bounded complexity.

Lemma 3.4.

(1) \( \pi_j : X_j \to X_{j+1} \) is injective on the 1-skeleton of \( X_j \).

(2) If \( p \in X_j \) is a 0-cell, then \( \pi_j^{-1}(\pi_j(p)) = \{p\} \), i.e. 0-cells experience no collapsing.

(3) For every \( p \in X_j \), the point inverse \( (\hat{\pi}^j)^{-1}(p) \) contains at most 3 points.

(4) For every \( j \), the link of any cell in \( X_j \) contains at most 24 cells.

(5) For every \( j \) and every 0-cell \( v \in X_j \), the link of \( v \) is connected: any two cells \( \sigma, \sigma' \) of \( X_j \) containing \( v \) may joined by a sequence \( \sigma = \tau_1, \ldots, \tau_{\ell} = \sigma' \) of cells of \( X_j \) containing \( v \), where \( \tau_{i-1} \) and \( \tau_i \) share a 1-cell.

(6) Every cell of \( X_j \) contains at most 9 open cells.

(7) There is an \( N = N(n) \) such that for every \( j \in \mathbb{Z} \), every combinatorial \( n \)-ball in \( X_j \) contains at most \( N \) cells.

(8) For every \( j \in \mathbb{Z} \) and every 2-cell \( \sigma \) of \( X_j \), the inverse image \( (\hat{\pi}^j)^{-1}(\sigma) \) may be covered by at most 27 2-cells of \( Y_j \).
\[ \text{Proof.} \] (1) and (2) follow from Lemma 3.1(3) and Lemma 3.3(1).

(3) By Lemma 3.1 a nontrivial coset of \( \mathcal{R}_j \) is a nontrivial coset of \( \Phi^i \mathcal{R} \) for some \( i < j \), and therefore contains at most 3 elements.

(4) It suffices to verify this for 0-cells. If \( v \) is a 0-cell of \( X_j \), then \( (\bar{\pi}_j)^{-1}(v) \) contains at most 3 vertices of \( Y_j \) by (3). Therefore \( v \) is contained in at most 12 2-cells and 12 1-cells.

(5) If \( (\bar{\pi}_j)^{-1}(v) \) contains only one point \( \hat{v} \in \mathbb{R}^2 \), then the link of \( v \) is isomorphic to the link of \( \hat{v} \). Otherwise for some \( i < j \) the set \( (\bar{\pi}_j)^{-1}(v) \) is a nontrivial coset of \( \Phi^i \mathcal{R} \), and has the form \( \{ \hat{v}, \hat{v} \pm m_v^{(i+1)} \} \) for some 0-cell \( \hat{v} \) of \( Y_{i+1} \), and the two vertical 1-cells emanating from \( \hat{v} \) are identified by \( \Phi^i \mathcal{R} \) with the vertical 1-cells emanating from \( \hat{v} \pm m_v^{(i+1)} \). Thus the link of \( v \) is homeomorphic to the quotient of three disjoint circles \( S^1, S^1_+, S^1_- \) by by identifying \( p_\pm \in S^1 \) with \( q_\pm \in S^1_\pm \), this is connected.

(6) This follows from Lemma 3.2(3).

(7) This follows by induction on \( n \), using (4) and (6).

(8) Suppose \( \tau \subset \sigma \) is an open cell of dimension \( d \). By Lemma 3.2(3), the inverse image of \( \tau \) under the projection \( \bar{\pi}_j : Y_j \to X_j \) is the disjoint union of the interiors of a collection \( \bar{\tau}_1, \ldots, \bar{\tau}_\ell \) of \( d \)-cells of \( Y_j \). By (3) we have \( \ell \leq 3 \). Combining this with (6) we get that \( (\bar{\pi}_j)^{-1}(\sigma) \) can be covered by at most 27 cells of \( Y_j \).

\[ \square \]

Let \( x, y \) denote the coordinate functions on \( \mathbb{R}^2 \). For every \( j \in \mathbb{Z} \), define \( y_j : \mathbb{R}^2 \to S^1(m_v^{-j}) = \mathbb{R}/m_v^{-j}\mathbb{Z} \) to be the composition of \( y : \mathbb{R}^2 \to \mathbb{R} \) with the quotient map \( \mathbb{R} \to \mathbb{R}/m_v^{-j}\mathbb{Z} \). We will metrize \( S^1(r) = \mathbb{R}/r\mathbb{Z} \) with the quotient metric \( d_{S^1(r)} \).

**Lemma 3.5** (The functions \( x_j \) and \( y_j \)).

1. For all \( j \in \mathbb{Z} \cup \{ \infty \} \) the function \( x \) descends to a function \( x_j : X_j \to \mathbb{R} \) and for all \( j \in \mathbb{Z} \) the function \( y_j \) descends to a function \( y_j : X_j \to S^1(m_v^{-j}) = \mathbb{R}/m_v^{-j}\mathbb{Z} \).
2. If \( \sigma \) is a 2-cell of \( X_j \), then the image of \( \sigma \) under \( x_j : X_j \to \mathbb{R} \) is an interval of length \( m_v^{-j} \) whose endpoints are the images of the vertical 1-cells of \( \sigma \).
3. If \( p, p' \in X_j \) and \( \pi_j(p) = \pi_j(p') \), then \( d(y_j(p), y_j(p')) \leq 2 m_v^{-j+1} \).
4. If \( \bar{\sigma} \) is a 2-cell of \( X_{j+1} \), then the inverse image \( \pi_j^{-1}(\bar{\sigma}) \subset X_j \) maps under \( y_j : X_j \to \mathbb{R}/m_v^{-j}\mathbb{Z} \) to a set of diameter at most \( 5m_v^{-j+1} \).
Proof. (1). Since the identifications generating \( R \) are given by vertical translations by multiples of \( m_v^{-1} \), the cosets of \( R \) are contained in the orbits of the translation action \( \{ e \} \times m_v^{-1}\mathbb{Z} \cong \mathbb{R}^2 \); likewise the cosets of \( R_j \) are contained in the orbits of \( \{ e \} \times m_v^{-j}\mathbb{Z} \). Therefore \( x \) and \( \hat{y}_j \) descend to \( X_j = X_0/R_j \).

(2). We have \( \sigma = \hat{\pi}^j(\hat{\sigma}) \) for some 2-cell \( \hat{\sigma} \) of \( Y_j \). Then \( y_j(\sigma) = y(\sigma) \), and the assertion is clear.

(3). If \( \bar{e} \) is a vertical 1-cell of \( X_j^{(1)} \), then \( \bar{e} = \hat{\pi}^j(e) \) for some vertical 1-cell \( e \) of \( Y_{j+1} \); therefore \( y_j(\bar{e}) \) has diameter \( m_v^{-(j+1)} \). Applying Lemma 3.3(1) we get \( d(y_j(p),y_j(p')) \leq 2 m_v^{-(j+1)} \).

(4). Pick \( p,p' \in X_\infty \). Choose \( \sigma \) a 2-cell of \( X_j^{(1)} \) such that \( \pi_j(\sigma) = \hat{\sigma} \). Then there exist \( q,q' \in \sigma \) such that \( \pi_j(p) = \pi_j(q) \) and \( \pi_j(p') = \pi_j(q') \). Since \( \text{diam}(y_j(\sigma)) = m_v^{-(j+1)} \), by (3) we get that \( \text{diam}(\pi_j)^{-1}(\hat{\sigma}) \leq 5 m_v^{-(j+1)} \).

4. Metric structure

We now analyze the metric \( \hat{d}_\infty \) by relating it to the geometry of the approximating cell complexes \( X_j \). The main result in this section is Proposition 4.12.

Let \( \hat{d}_j \) be the largest pseudodistance on \( X_j \) such that for every \( i \leq j \), and every 2-cell \( \hat{\sigma} \) of \( Y_i \), the diameter of \( \hat{\pi}^i(\hat{\sigma}) \) is \( \leq m^{-i} \).

Before proceeding, we introduce some additional terminology.

A chain in a set is a sequence \( S_1, \ldots, S_\ell \) of subsets such that \( S_{i-1} \cap S_i \neq \emptyset \) for all \( 1 < i \leq \ell \).

**Definition 4.1.** Let \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \) be a sequence, where \( \hat{\sigma}_i \) is a 2-cell of \( Y_{j_i} \). Then:

- For \( j \in \mathbb{Z} \cup \{ \infty \} \), the sequence \( \{ \hat{\sigma}_i \} \) is a \textbf{j-chain joining} \( p,p' \in X_j \) if the projections \( \hat{\pi}^0_j(\hat{\sigma}_0), \ldots, \hat{\pi}^j_j(\hat{\sigma}_\ell) \) form a chain in \( X_j \), and \( p \in \hat{\pi}^0_j(\hat{\sigma}_0), p' \in \hat{\pi}^j_j(\hat{\sigma}_\ell) \). By convention, the empty sequence is a chain joining every point to itself.
- The \textbf{length} and \textbf{generation} of the sequence \( \{ \hat{\sigma}_i \} \) are \( \sum_i m^{-j_i} \) and \( \max_i j_i \) respectively.

By Definition 4.1, for \( p,p' \in X_\infty \), we have

\[
\hat{d}_\infty(p,p') = \inf \{ \text{length}(\{ \hat{\sigma}_i \}) \mid \{ \hat{\sigma}_i \} \text{ is an } \infty \text{-chain joining } p,p' \},
\]
and for all $p, p' \in X_j$

$$\hat{d}_j(p, p') = \inf \left\{ \text{length}(\{\hat{\sigma}_i\}) \mid \{\hat{\sigma}_i\} \text{ is a } j\text{-chain of generation } \leq j \text{ joining } p, p' \right\}.$$ 

Note that it follows that $\hat{d}_j$ restricts to a discrete metric on each 2-cell $\sigma$ of $X_j$.

**Lemma 4.2.** If $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ is an $\infty$-chain of generation $g$, then it is a $j$-chain for all $j \geq g$.

**Proof.** This follows from Lemma 3.3(3). $\square$

**Lemma 4.3.** Let $C = C_1 \cup C_2$ be a collection of subsets of a set, and $S, S' \in C_1$. Assume that $C$ contains a chain from $S$ to $S'$, and that every element of $C_2$ is contained in some element of $C_1$. Then $C_1$ contains a chain from $S$ to $S'$.

**Proof.** Starting with a chain in $C$ from $S$ to $S'$, one may inductively reduce the number of elements from $C_2$ by replacing each one with an element of $C_1$ that contains it. $\square$

**Definition 4.4.** A **gallery** in $X_j^{(k)}$ is a sequence of 2-cells $\sigma_1, \ldots, \sigma_\ell$ of $X_j^{(k)}$ such that $\sigma_{i-1} \cap \sigma_i$ contains a 1-cell $e_i$ of $X_j^{(k)}$ for every $1 < i \leq \ell$; the gallery is **vertical** if for all $1 < i \leq \ell$ the 1-cell $e_i$ is horizontal, and **horizontal** if $e_i$ is vertical for all $1 < i \leq \ell$.

**Lemma 4.5.** There exist $N_1 = N_1(C), N_2 = N_2(C)$ such that for every $j \in \mathbb{Z}$, every $\hat{d}_j$-ball $B(p, Cm^{-j}) \subset X_j$ is contained in a combinatorial $N_1$-ball of $X_j$, and is contained in a union of at most $N_2$ 2-cells of $X_j$.

**Proof.** Fix $C < \infty$ and $p \in X_j$.

If $\hat{d}_j(p', p) < Cm^{-j}$, then there is a $j$-chain $\hat{\sigma}_0^0, \ldots, \hat{\sigma}_{\ell_0}^0$ from $p$ to $p'$ of length $< Cm^{-j}$. Letting $j_i$ be the generation of $\hat{\sigma}_0^0$, this yields

$$m^{-j_i} < Cm^{-j}, \quad \ell_0 \cdot m^{-j} < Cm^{-j}$$

so

$$j - j_i < \frac{\log C}{\log m}$$

and $\ell_0 < C$. Now (4.6) implies that $\hat{\sigma}_i^0$ contains at most $C_1 = C_1(C)$ 2-cells of $Y_j$. The collection $C$ of all 2-cells of $Y_j$ contained in $\cup_i \hat{\sigma}_i^0$ contains at most $C_1 \cdot \ell_0$ 2-cells. Also $C$ contains a $j$-chain $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$
from \( p \) to \( p' \), where \( \ell \leq C_1 \ell_0 < C_1 \cdot C \). Putting \( N_1 = C_1 \cdot C \), we have shown that \( p' \) lies in the combinatorial \( N_1 \)-ball of the open cell containing \( p \).

By Lemma \( 3.4[7] \) the \( N_1 \)-ball contains at most \( N_2 = N_2(C) \) cells.

\( \square \)

The next two lemmas provide lower bounds on the combinatorial length of certain types of chains of 2-cells in \( X_{j+1} \).

**Lemma 4.7.** Let \( \sigma \subset X_j \) be a 2-cell with vertical 1-cells \( e_1,e_2 \). If \( \bar{\sigma}_1,\ldots,\bar{\sigma}_\ell \subset \pi_j(\sigma) \) is a chain of 2-cells of \( X_{j+1} \) that joins \( \pi_j(e_1) \) to \( \pi_j(e_2) \), then \( \ell \geq m \).

**Proof.** Since \( \{\bar{\sigma}_j\} \) is a chain in \( X_{j+1} \), the images under \( x_{j+1} : X_{j+1} \to [0,1] \) form a chain in \( [0,1] \) joining \( x_{j+1}(\bar{\pi}_j(e_1)) \) to \( x_{j+1}(\bar{\pi}_j(e_2)) \). But by Lemma 3.5(2) the image \( x_{j+1}(\bar{\sigma}_j) \) is a segment of length \( m^{-j+1} \), while \( |x_{j+1}(\bar{\pi}_j(e_1)) - x_{j+1}(\bar{\pi}_j(e_2))| = m^{-j} \). Thus

\[
    m^{-j} = \text{diam}(\bar{\pi}_j(e_1) \cup \bar{\pi}_j(e_2)) \leq \sum_i \text{diam}(x_{j+1}(\bar{\sigma}_i)) = \ell \cdot m^{-(j+1)}
\]

so \( \ell \geq m \).

**Definition 4.8.** The open star (resp. closed star) of a cell \( \sigma \) in a cell complex is the union of the open (respectively closed) cells whose closure contains \( \sigma \).

**Lemma 4.9.** Let \( e \) be a vertical 1-cell of \( X_j \), and \( \text{St}(e,X_j) \) be the closed star of \( e \) in \( X_j \). Let \( \bar{\sigma}_1,\ldots,\bar{\sigma}_\ell \) be a chain of 2-cells in \( X_{j+1} \) such that:

1. \( \bar{\sigma}_1,\ldots,\bar{\sigma}_\ell \subset \pi_j(\text{St}(e,X_j)) \).
2. \( \bar{\sigma}_1 \cap \pi_j(e) \neq \emptyset \).
3. There is a 2-cell \( \tau \) of \( X_j \) that is not contained in \( \text{St}(e,X_j) \), such that \( \bar{\sigma}_\ell \) intersects \( \pi_j(\tau) \).

Then one of the following holds:

(a) For some \( \bar{\sigma}_i \), there is a \( p \in \pi_j^{-1}(\bar{\sigma}_i) \) such that \( d(y_j(p),0) \leq 2m_v^{-j+1} \).

(b) There is a 2-cell \( \sigma \) of \( X_j \) that is contained in \( \text{St}(e,X_j) \), with vertical 1-cells \( e,e_1 \) such that some subchain of \( \{\bar{\sigma}_j\} \) is contained in \( \pi_j(\sigma) \) and joins \( \pi_j(e) \) to \( \pi_j(e_1) \).

Moreover in case (a) we have

\[
    \ell \geq \frac{1}{5} \cdot \left[ m_v^{j+1} \text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_1)),0) - 2 \right].
\]
Proof. Suppose (a) holds. Since the sets \( \{ y_j(\pi_j^{-1}(\bar{s}_k)) \}_{1 \leq k \leq i} \) each have diameter \( \leq 5m_v^{-j+1} \) by Lemma 3.5.4, and they form a chain in \( S^1(m_v^{-j}) \) joining \( y_j(\pi_j^{-1}(\bar{s}_j)) \) to \( B(0, 2m_v^{-j+1}) \subset S^1(m_v^{-j}) \), the triangle inequality gives \( (4.10) \). Now suppose (a) does not hold, i.e., for every \( \bar{s}_i \) and every \( p \in \pi_j^{-1}(\bar{s}_i) \),

\[
d(\bar{y}_j(p), 0) > 2m_v^{-j+1}.
\]

We may assume without loss of generality that \( \bar{s}_1, \ldots, \bar{s}_\ell \) is a minimal chain satisfying the hypotheses of the Lemma, so that:

(i) For all \( i > 1 \), the 2-cell \( \bar{s}_i \) does not intersect \( \pi_j(e) \).

(ii) For all \( i < \ell \), the 2-cell \( \bar{s}_i \) does not intersect \( \pi_j(\tau') \) for any 2-cell \( \tau' \) of \( X_j \) not contained in \( \text{St}(e, X_j) \).

For every \( i \), there is a 2-cell \( \sigma_i \) of \( X_j^{(1)} \) such that \( \pi_j(\sigma_i) = \bar{s}_i \), and \( \sigma_i \subset \text{St}(e, X_j) \). The 2-cell \( \sigma_i \) is contained in a unique 2-cell \( \tau_i \) of \( X_j \) that contains \( e \).

Let \( \tau \) be as in (3), and pick \( \bar{p} \in \bar{s}_\ell \cap \pi_j(\tau) \). Choose \( p_\ell \in \sigma_\ell, p \in \tau \) such that \( \pi_j(p_\ell) = \pi_j(p) = \bar{p} \). By Lemma 3.3.1 there is a vertical edge path \( \gamma \) in \( X_j^{(1)} \) of combinatorial length at most 2 that contains \( p_\ell \) and \( p \). Because \( \tau \not\subset \text{St}(e, X_j) \), we know that \( e \not\subset \tau \), and so \( \tau \neq \tau_\ell \). Therefore \( \gamma \) must intersect \( \partial \tau_\ell \) and \( \partial \tau \). In view of \( (4.11) \) we conclude that \( p_\ell \) and \( p \) both lie in the interior of a vertical 1-cell \( e_1 \subset \tau_\ell \cap \tau \). Since \( e \not\subset \tau \), it follows that \( e_1 \neq e \), so \( e_1 \) is the second vertical 1-cell of \( \partial \tau_\ell \).

Let \( i_0 \) be the minimal \( i \) such that the subchain \( \bar{s}_1, \ldots, \bar{s}_\ell \) is contained in \( \tau_\ell \), i.e., such that \( \tau_1 = \ldots = \tau_\ell \). Suppose \( i_0 > 1 \). Choose \( p_{i_0-1} \in \pi_{i_0-1} \), \( p_{i_0} \in \sigma_{i_0} \) such that \( \pi_j(p_{i_0-1}) = \pi_j(p_{i_0}) \in \bar{s}_{i_0} \cap \bar{s}_{i_0} \subset \pi_j(\tau_{i_0-1}) \cap \pi_j(\tau_\ell) \). Reasoning as above, we get that \( p_{i_0-1} \) and \( p_{i_0} \) belong to the interior of a vertical 1-cell \( e' \) of \( X_j \), where \( e' \subset \partial \tau_{i_0-1} \cap \partial \tau_\ell \). We cannot have \( e' = e \) or \( e' = e_1 \) by the minimality of the chain \( \bar{s}_1, \ldots, \bar{s}_\ell \). This is a contradiction. Hence \( i_0 = 1 \) and \( \bar{s}_1, \ldots, \bar{s}_\ell \subset \tau_\ell \). Putting \( \sigma = \tau_\ell \) we have shown that (b) holds.

**Proposition 4.12.** For all \( k \in \mathbb{Z} \) and \( p, p' \in X_k \),

\[
(4.13) \quad d_\infty(\pi_k(p), \pi_k(p')) \geq d_k(p, p') - 2m^{-k}.
\]

**Proof.** Fix \( k \in \mathbb{Z} \), \( p, p' \in X_k \), and let \( \bar{p} = \pi_k(p) \), \( \bar{p}' = \pi_k(p') \). Choose 2-cells \( \bar{s}, \bar{s}' \) of \( Y_k \) such that \( p \in \pi_k(\bar{s}), p' \in \pi_k(\bar{s}') \).

Let \( \bar{s}_0^{0}, \ldots, \bar{s}_0^{0} \) be an \( \infty \)-chain joining \( \bar{p} \) to \( \bar{p}' \). We enlarge this to a new \( \infty \)-chain \( \bar{s}, \bar{s}_0^{0}, \ldots, \bar{s}_0^{0}, \bar{s}' \), which we relabel as \( \bar{s}_1, \ldots, \bar{s}_\ell \), where
\(\ell = \ell_0 + 2\). Then \(\{\sigma_i\}\) is an \(\infty\)-chain joining \(\bar{p}\) to \(\bar{p}'\) of length
\[
\leq \text{length}(\{\sigma_0^0\}) + 2m^{-k}.
\]

We will show that we can reduce the generation of the chain \(\{\sigma_i\}\) to \(k\) without increasing its length, so as to obtain a \(k\)-chain from \(\bar{p}\) to \(\bar{p}'\) of length at most \(\text{length}(\{\sigma_i\})\). This implies that \(\hat{d}_k(p, p') \leq \text{length}(\{\sigma_0^0\}) + 2m^{-k}\). Taking the infimum over all \(\infty\)-chains from \(p\) to \(p'\) yields (4.13).

Let \(j + 1\) be the generation of \(\{\sigma_i\}\). If \(j + 1 \leq k\) we are done, so we assume that \(j \geq k\). Moreover, we may assume that there is no other \(\infty\)-chain \(\sigma'_0, \ldots, \sigma'_{\ell'}\) such that:

- \(\sigma'_0 = \sigma_0, \sigma'_{\ell'} = \sigma_{\ell}\).
- \(\text{length}(\{\sigma'_i\}) \leq \text{length}(\{\sigma_i\})\).
- The generation of \(\{\sigma'_i\}\) is at most \(j + 1\).
- \(\{\sigma'_i\}\) has fewer cells of generation \(j + 1\) than \(\{\sigma_i\}\).

Since \(\{\sigma_i\}\) is an \(\infty\)-chain, by Lemma 3.3 the sequence \(\sigma_1, \ldots, \sigma_\ell\) is a \((j + 1)\)-chain.

Before proceeding further, we first indicate the rough idea of the argument. Although it takes values in \(S^1(m_v^{-j})\), we view \(y_j : X_j \to S^1(m_v^{-j})\) as a “height function”. Since \(m_v\) is much larger than \(m\), a string of cells in \(\sigma_1, \ldots, \sigma_{\ell}\) of maximal generation \(j + 1 > k\) cannot move efficiently in the “vertical” direction, i.e. it cannot change \(y_j\) efficiently. Thus any such string is forced to move roughly horizontally, and it may then be replaced by cells of lower generation without increasing \(\text{length}(\{\sigma_i\})\). This contradicts the choice of \(\{\sigma_i\}\).

We now resume the proof of the proposition.

Let \(\bar{\sigma}_{i_1}, \ldots, \bar{\sigma}_{i_2}\) be a maximal string of consecutive cells from \(\{\sigma_i\}\) of generation \(j + 1\), so that both \(\bar{\sigma}_{i_1 - 1}\) and \(\bar{\sigma}_{i_2 + 1}\) have generation \(\leq j\). For \(1 \leq i \leq \ell\), let \(\sigma_i = \tilde{\pi}^j(\tilde{\sigma}_i)\), \(\bar{\sigma}_i = \tilde{\pi}^{j+1}(\tilde{\sigma}_i)\). For \(i_1 \leq i \leq i_2\) let \(\tau_i = \tilde{\pi}^j(\tilde{\tau}_i)\).

**Step 1.** If \(e\) is a vertical 1-cell of \(X_j\), and \(\bar{\sigma}_{i_3} \cap \pi_j(e) \neq \emptyset\) for some \(i_1 \leq i_3 \leq i_2\), then \(\text{dist}(y_j((\pi_j)^{-1}(\bar{\sigma}_{i_3})), 0) < 11m \cdot m_v^{-j+1}\).

Let \(\bar{\sigma}_{i_4}, \ldots, \bar{\sigma}_{i_5}\) be a maximal subsequence of \(\bar{\sigma}_{i_1}, \ldots, \bar{\sigma}_{i_2}\) such that:

- \(i_4 \leq i_3 \leq i_5\), i.e. \(\bar{\sigma}_{i_3}\) belongs to the subsequence.
- The cells \(\bar{\sigma}_{i_4}, \ldots, \bar{\sigma}_{i_5}\) are all contained in \(\pi_j(\text{St}(e, X_j))\).
Claim. At least one of the two cells $\bar{\sigma}_{i4}, \bar{\sigma}_{i5}$ must intersect $\pi_j(\tau)$, for some 2-cell $\tau$ of $X_j$ that is not contained in $St(e,X_j)$.

Proof of claim. Suppose the claim were false.

Assume first that $\sigma_{i4-1}$ has generation $j + 1$. By the maximality of $\bar{\sigma}_{i3}, \ldots, \bar{\sigma}_{i5}$, we have $\bar{\sigma}_{i4-1} \not\subset \pi_j(St(e,X_j))$, so $\tau_{i4-1} \not\subset St(e,X_j)$. Then $\emptyset \neq \bar{\sigma}_{i4-1} \cap \bar{\sigma}_{i4} \subset \pi_j(\tau_{i4-1}) \cap \bar{\sigma}_{i4}$, proving the claim. Thus we may assume that $\sigma_{i4-1}$ has generation $\leq j$, and likewise for $\sigma_{i5+1}$.

Since $\sigma_{i4-1}$ is a union of 2-cells of $X_j$, for some 2-cell $\tau$ of $X_j$, we have $\tau \subset \sigma_{i4-1}$ and $\bar{\sigma}_{i4} \cap \pi_j(\tau) \neq \emptyset$. If $\tau \not\subset St(e,X_j)$, then the claim follows, so we assume that $\tau \subset St(e,X_j)$, and in particular $e \subset \tau$. Likewise we may assume that $\sigma_{i5+1}$ contains a 2-cell $\tau'$ of $X_j$ that contains $e$. Now $\sigma_{i4-1} \cap \sigma_{i5+1} \neq \emptyset$, so we may shorten the $j + 1$-chain $\{\hat{\sigma}_i\}$ by deleting $\hat{\sigma}_{i4}, \ldots, \hat{\sigma}_{i5}$, which is a contradiction. Thus the claim holds.

We now assume without loss of generality that the claim holds for $\bar{\sigma}_{i5}$. Hence $\bar{\sigma}_{i3}, \ldots, \bar{\sigma}_{i5}$ satisfies the hypotheses of Lemma 4.9.

First suppose that conclusion (a) of Lemma 4.9 holds. Then (4.10) gives

$$i_5 - i_3 + 1 \geq \frac{1}{5} \cdot [m_j^{(j+1)} \cdot \text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i3})), 0) - 2]$$

Because $\bar{\sigma}_{i3}, \bar{\sigma}_{i5} \subset St(e,X_j)$, we have $\tau_{i3}, \tau_{i5} \subset St(e,X_j)$, so $e \subset \tau_{i3} \cap \tau_{i5}$. It follows that we obtain a $(j + 1)$-chain with fewer cells of generation $j + 1$ by replacing $\bar{\sigma}_{i3}, \ldots, \bar{\sigma}_{i5}$ with the two cells $\tau_{i3}, \tau_{i5}$. This new chain has length

$$\text{length}(\{\hat{\sigma}_i\}) = \text{length}(\{\hat{\sigma}_i\}) + \text{length}(\tau_{i3}, \tau_{i5})$$

$$= \text{length}(\{\hat{\sigma}_i\}) - m^{-j}(i_5 - i_3 + 1) - 2m^{-j}$$

Using (4.14) we get

$$\text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i3})), 0) \leq (10m + 2)m^{-(j+1)} < 11m \cdot m^{-(j+1)}.$$
This completes Step 1.

**Step 2.** We have

\[(4.15) \quad \text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_3})),0) < 16m \cdot m_v^{-(j+1)}\]

for all \(i_1 \leq i_3 \leq i_2\).

Suppose \(i_3\) violates \((4.15)\). Choose \(i_4\) maximal such that \(i_3 \leq i_4 \leq i_2\) and \(\tau_i = \tau_{i_3}\) for all \(i_3 \leq i \leq i_4\). Since we obtain a comparison chain by replacing \(\sigma_{i_3}, \ldots, \hat{\sigma}_{i_4}\) with \(\bar{\tau}_{i_3}\), we get that

\[m^{-(j+1)}(i_4 - i_3 + 1) = \text{length}(\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_4}) < \text{length}(\bar{\tau}_{i_3}) = m^{-j},\]

i.e. \(i_5 - i_3 + 1 < m\). Applying Lemma \((3.5)\) and the fact that the sets \(y_j(\pi_j^{-1}(\bar{\sigma}_i))\) form a chain in \(S^1(m_v^{-j})\), we get

\[(4.16) \quad \text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_4})),0) \geq 11m \cdot m_v^{-(j+1)} .\]

First suppose \(i_4 < i_2\), i.e. that \(\text{gen}(\hat{\sigma}_{i_4+1}) = j + 1\). Then \(\tau_{i_4+1} = \tau_{i_4}\) and \(\pi_j(\tau_{i_4+1}) \cap \pi_j(\tau_{i_4}) \neq \emptyset\). By \((4.16)\) it follows that \(\bar{\sigma}_{i_4}\) must intersect \(\pi_j(e)\) for some vertical 1-cell \(e\) of \(\tau_{i_4}\). But then \((4.16)\) contradicts Step 1.

Now suppose \(i_4 = i_2\). Then \(\text{gen}(\hat{\sigma}_{i_4+1}) \leq j\), and so \(\sigma_{i_4+1}\) contains a 2-cell \(\tau\) of \(X_j\) such that \(\hat{\sigma}_{i_4} \cap \pi_j(\tau) \neq \emptyset\). If \(\tau = \tau_{i_3} = \tau_{i_4}\), then we may discard \(\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_4}\), contradicting the minimality of the chain \(\hat{\sigma}_1, \ldots, \hat{\sigma}_f\). Therefore \(\tau \neq \tau_{i_3}\), and reasoning as above we find that \(\hat{\sigma}_{i_4} \cap \pi_j(e) \neq \emptyset\) for some vertical 1-cell of \(\tau_{i_3}\). Using \((4.16)\), we again get a contradiction to Step 1.

**Step 3.** Shifting \(\bar{\sigma}_{i_1}, \ldots, \bar{\sigma}_{i_2}\) toward the 1-skeleton of \(X_j\).

Pick \(i_1 \leq i \leq i_2\). Since \(\sigma_i \subset \pi_j^{-1}(\bar{\sigma}_i)\), by Step 2 and Lemma \((3.5)\) we have

\[
\text{dist}(y_j(\sigma_i),0) \leq \text{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_i)),0) + \text{diam}(y_j(\pi_j^{-1}(\bar{\sigma}_i)))
\]

\[< (16m + 5) \cdot m_v^{-(j+1)} .\]

Hence there is vertical gallery (see Definition \((4.4)\)) \(\mu_{i_1}, \ldots, \mu_{i_{\ell_i}}\) in \(X_j^{(1)}\) with \(\ell_i < 16m + 5\) that starts from \(\sigma_i = \mu_{i_1}\), such that \(\mu_{i_{\ell_i}}\) contains a horizontal 1-cell \(e_i^j\) of \(X_j^{(1)}\) that is contained in a horizontal 1-cell \(e_i\) of \(X_j\), see Figure \((??)\). Since the gallery is vertical, we have \(x_j(\sigma_{i-1}) = x_j(e_i^{j-1}), x_j(\sigma_i) = x_j(e_i^j) \subset \mathbb{R}\).

**Claim.** \(e_i^{j-1} \cap e_i^j \neq \emptyset\) for all \(i_1 < i \leq i_2\).
Proof of claim. The images $x_j(\sigma_{i-1})$, $x_j(\sigma_i) \subset \mathbb{R}$ are intervals of length $m^{-i+1}$ that have an endpoint in common because $x_j(\sigma_{i-1}) = x_{j+1}(\bar{\sigma}_{i-1})$, $x_j(\sigma_i) = x_{j+1}(\bar{\sigma}_i)$, and $\bar{\sigma}_{i-1} \cap \bar{\sigma}_i \neq \emptyset$. We may therefore choose 0-cells $v_{i-1}, v_i$ of $X_j^{(1)}$ such that $v_{i-1} \in e_{i-1}', v_i \in e_i'$, and $x_j(v_{i-1}) = x_{j+1}(\bar{\sigma}_{i-1})$.

Choose 0-cells $p_{i-1}, p_i$ of $X_j^{(1)}$ such that $p_{i-1} \in \sigma_{i-1}$, $p_i \in \sigma_i$, $x_j(p_{i-1}) = \pi_j(p_{i-1})$, $x_j(p_i) = x_j(v_{i-1}) = x_{j+1}(\bar{\sigma}_{i-1})$. We may join $p_{i-1}$ to $p_i$ with a vertical edge path in $X_j^{(1)}$ of combinatorial length at most 2 by Lemma 3.3(1). Using the vertical cells in the vertical galleries $\{\mu_{i-1n}\}, \{\mu_{in}\}$, we get vertical edge paths $\gamma_{i-1}, \gamma_i$ in $X_j^{(1)}$ joining $v_{i-1}$ to $p_{i-1}$ and $v_i$ to $p_i$, respectively, where $\gamma_{i-1}, \gamma_i$ have combinatorial length $< 16m + 5$. Concatenating $\gamma_{i-1}, \gamma, \gamma_i$ we get a vertical edge path joining $v_{i-1}$ to $v_i$ of combinatorial length $< 32m + 12 = 140$. Since $m_\nu = 3L \geq 300$, this forces $v_{i-1} = v_i$, proving the claim. \hfill \Box

Claim. $\sigma_{i-1} \cap e_{i-1}' \neq \emptyset$ and $e_{i-1}' \cap \sigma_{i+1} \neq \emptyset$.

Proof of claim. There is a 2-cell $\tau \subset \sigma_{i-1}$ of $X_j$ such that $\pi_j(\tau) \cap \bar{\sigma}_1 \neq \emptyset$. We may find 0-cells $p, p_i$ of $X_j^{(1)}$ such that $p \in \tau$, $p_i \in \sigma_i$, and $\pi_j(p) = \pi_j(p_i)$. There is a vertical edge path $\gamma$ in $X_j^{(1)}$ from $p$ to $p_i$ of combinatorial length at most 2, and a vertical edge path $\gamma_i$ in $X_j^{(1)}$ of combinatorial length $< 16m$ joining $p_i$ to an endpoint $v_{i-1}'$ of $e_{i-1}'$. Combining $\gamma$ and $\gamma_i$ we get a vertical edge path of length $< 16m + 2$ in $X_j^{(1)}$ starting at $p \in \tau$ and ending in the 1-skeleton of $X_j$. This implies that $v_{i-1} \in \partial \tau$. Hence $\sigma_{i-1} \cap e_{i-1}' \neq \emptyset$ and the claim holds. \hfill \Box

Combining the two claims, we get that

$$\sigma_1, \ldots, \sigma_{i-1}, e_{i-1}', \ldots, e_{i_2}', \sigma_{i_2} \cup \sigma_{i_2+1}, \ldots, \sigma_{\ell}$$

forms a chain in $X_j$. Therefore, after modifying $\bar{\sigma}_i$ for $i_1 \leq i \leq i_2$ if necessary, we may assume without loss of generality that $\sigma_i = \mu_\ell$, and in particular $\sigma_i$ contains $e_{i}'$.

Step 4. The final contradiction.

Pick $i_3$ such that $i_1 \leq i_3 \leq i_2$.

Suppose some interior point of $e_{i_3}$ is contained in $\sigma_{i_1} \cup \sigma_{i_2+1}$. Since $\sigma_{i_1} \cup \sigma_{i_2+1}$ is a subcomplex of $X_j$, this implies that $e_{i_3}$ is contained in $\sigma_{i_1} \cup \sigma_{i_2+1}$. Therefore applying Lemma 4.3 we may delete the edge $e_{i_3}'$ from the collection (4.17), and it will still contain a chain, and likewise
we may delete \( \hat{\sigma}_{i_3} \) from the collection \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \) and it will still contain a chain; this contradicts the minimality of \( \{ \hat{\sigma}_i \} \).

Now suppose \( e_{i_3} \) is contained in \( e'_{i_1} \cup \ldots \cup e'_{i_2} \). Taking the union of the collection (4.17) with \( e_{i_3} \) and applying Lemma 4.3, it follows that we may remove from the collection (4.17) each \( e'_{i} \) that is contained in \( e_{i_3} \), and add \( e_{i_3} \), and the resulting collection will contain a chain from \( \sigma_1 \) to \( \sigma_\ell \). Therefore we may remove from \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \) each \( \hat{\sigma}_i \) with \( i_1 \leq i \leq i_2 \) such that \( e'_{i} \subset e_{i_3} \), and add the 2-cell \( \hat{\tau}_{i_3} \), and the resulting collection will contain a \( j+1 \)-chain that has length at most \( \hat{d}_\infty(\{ \hat{\sigma}_i \}) \) and fewer cells of generation \( j+1 \), contradicting the definition of \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \).

Therefore \( e_{i_3} \) is not contained in \( \sigma_{i_1-1} \cup \sigma_{i_1} \cup \ldots \cup \sigma_{i_2} \cup \sigma_{i_2+1} \). But then we may remove from the \( j+1 \)-chain \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \) some 2-cell \( \hat{\sigma}_i \), with \( i_1 \leq i \leq i_2 \) and \( e'_{i} \subset e_{i_3} \) and still have a chain. This contradicts the minimality of \( \hat{\sigma}_1, \ldots, \hat{\sigma}_\ell \).

We conclude that \( j+1 \leq k \), completing the proof of Proposition 4.12.

\[ \square \]

The following two corollaries of Proposition 4.12 relate the distance in \( X_\infty \) or \( X_j \) with the combinatorial distance.

**Corollary 4.18.** For every \( C \) there is an \( N = N(C) \) such that if \( p, p' \in X_j \) and \( \hat{d}_\infty(\pi_j^\infty(p), \pi_j^\infty(p')) < Cm^{-j} \), then there is a chain of at most \( N \) cells of \( X_j \) joining \( p \) to \( p' \).

*Proof.* This follows from Proposition 4.12 and Lemma 4.5. \( \square \)

**Corollary 4.19.** For \( p_0, p_1 \in X_\infty \) let

\[ J(p_0, p_1) = \left\{ j \mid (\pi_j^\infty)^{-1}(p_0), (\pi_j^\infty)^{-1}(p_1) \text{ do not intersect adjacent cells of } X_j \right\}, \]

and let \( \underline{J}(p_0, p_1) = \inf J(p_0, p_1) \), where as usual the infimum of the empty set is \( \infty \). Then

\[ m^{-\underline{J}(p_0, p_1)} \leq \hat{d}_\infty(p_0, p_1) \leq 2m \cdot m^{-\underline{J}(p_0, p_1)}, \]

where by convention we let \( m^{-\underline{J}(p_0, p_1)} = 0 \) when \( \underline{J}(p_0, p_1) = \infty \).

*Proof.* Note that \( \underline{J}(p_0, p_1) = \inf J(p_0, p_1) = 1 + \sup(\mathbb{Z} \setminus J(p_0, p_1)) \).
If \( k \not\in J(p_0, p_1) \), then \((\pi_k^\infty)^{-1}(p_0)\) and \((\pi_k^\infty)^{-1}(p_1)\) intersect adjacent cells, and this gives \( \hat{d}_\infty(p_0, p_1) \leq 2m^{-k} \) by the definition of \( \hat{d}_\infty \). Thus

\[
\hat{d}_\infty(p_0, p_1) \leq 2 \inf \{ m^{-k} \mid k \not\in J(p_0, p_1) \} \leq 2m \cdot \inf \{ m^{-(k+1)} \mid k \not\in J(p_0, p_1) \} = 2m \cdot \sup \{ m^{-k} \mid k \in J(p_0, p_1) \} = 2m \cdot m^{-1(p_0,p_1)}. 
\]

If \( k \in J(p_0, p_1) \) then then any \( k \)-chain connecting \((\pi_k^\infty)^{-1}(p_0)\) with \((\pi_k^\infty)^{-1}(p_1)\) must contain at least 3 cells, and hence

\[
(4.22) \quad \hat{d}_k((\pi_k^\infty)^{-1}(p_0), (\pi_k^\infty)^{-1}(p_1)) \geq 3m^{-k}. 
\]

The lower bound in (4.21) then follows applying Proposition 4.12. \( \square \)

We remark that the following lemma is not really essential to the discussion. Even without knowing that \( \hat{d}_\infty \) is a distance, we could quotient out the sets of zero diameter and work in the resulting metric space, cf. Section II

**Lemma 4.23.** Keeping the notation from Corollary 4.19, the set \( J(p_0, p_1) \) is nonempty iff \( p_0 \neq p_1 \); in particular \( \hat{d}_\infty \) is a distance function on \( X_\infty \).

**Proof.** Clearly \( J(p_0, p_1) \neq \emptyset \implies p_0 \neq p_1 \). We will show that if \( J(p_0, p_1) \) is empty then \( p_0 = p_1 \).

For all \( j \in \mathbb{Z} \), since \( j \not\in J(p_0, p_1) \), the sets \((\pi_j^\infty)^{-1}(p_0)\), \((\pi_j^\infty)^{-1}(p_1)\) intersect adjacent cells of \( X_j \), and therefore the values of \( x_j \) on these sets agree to within error \( 2m^{-j} \). Since \( x_\infty(p_i) = x_j((\pi_j^\infty)^{-1}(p_i)) \), we get that \( |x_\infty(p_0) - x_\infty(p_1)| \leq 2m^{-j} \). As this holds for all \( j \) we conclude that \( x_\infty(p_0) = x_\infty(p_1) \).

Let \( t = x_\infty(p_0) = x_\infty(p_1) \).

If \( t \in \cup_{j \in \mathbb{Z}} (m^{-j}\mathbb{Z}) \) select \( j_0 \in \mathbb{Z} \) such that \( t \in m^{-j_0}\mathbb{Z} \), and otherwise let \( j_0 \in \mathbb{Z} \) be arbitrary. Then \( t \not\in m^{-(j+1)}\mathbb{Z} \setminus m^{-j}\mathbb{Z} \) for every \( j \geq j_0 \), and by the definition of \( R_j \), it follows that \((\pi_j^\infty)^{-1}(p_i)\) contains a unique element \( \hat{p}_i \in X_{j_i} \) for \( i = 0, 1 \).

Let \( j_1 \geq 0 \) be arbitrary.

**Claim.** \( \hat{p}_0 \) may be joined to \( \hat{p}_1 \) by a chain of at most four 2-cells of \( X_{j_0} \).

Pick \( j \geq j_0 \), to be determined later.
If \( t \in m^{-j}\mathbb{Z} \), we define \( t_{\pm} = t \pm m^{-j} \); if \( t \not\in m^{-j}\mathbb{Z} \) then for some \( k_j \in \mathbb{Z} \) we have \( t \in (k_j m^{-j}, (k_j + 1)m^{-j}) \), and then we let \( t_{\pm} = k_j m^{-j} \), \( t_{+} = (k_j + 1)m^{-j} \). Let \( g_j = \min(\{\ell \mid t_{-} \in m^{-\ell}\mathbb{Z}\}, \{\ell \mid t_{+} \in m^{-\ell}\mathbb{Z}\}) \). Since \( |t_{\pm} - t| \leq m^{-j} \), we will have
\[
(4.24) \quad \min(g_j - j_0, j - j_0) > j_1
\]
for all but finitely many \( j \), so we assume [4.24] holds.

Choose adjacent cells \( \tau_0, \tau_1 \) of \( X_j \) such that \( \tau_i \cap (\pi_j^{\infty})^{-1}(p_i) \neq \emptyset \), and cells \( \hat{\tau}_0, \hat{\tau}_1 \) of \( X_{j_0} \) such that \( \pi_{j_0}^{i_j}(\hat{\tau}_i) = \tau_i \). Let \( v \in \tau_0 \cap \tau_1 \) be a 0-cell of \( X_j \), and choose \( \hat{v}_i \in \hat{\tau}_i \) such that \( \pi_{j_0}^{i_j}(\hat{v}_i) = v \).

Case 1: \( x_j(v) = t \). As \( \pi_j^{i_j} \) is injective on \( x_j^{-1}(t) \), we have \( \hat{v}_0 = \hat{v}_1 \), so \( \hat{\tau}_0 \cap \hat{\tau}_1 \neq \emptyset \). Thus \( \hat{p}_0, \hat{p}_1 \) may be joined by a chain of at most two 2-cells of \( X_{j_0}^{(j_0 - j_0)} \), and hence also by a chain of at most two 2-cells of \( X_{j_0}^{(i_j)} \), proving the claim.

Case 2: \( x_j(v) \neq t \). Then \( x_j(v) \in \{t_{-}, t_{+}\} \). Since \( \pi_{j_0}^{i_j}(\hat{v}_i) = v \) it follows that \( \hat{v}_0 \) may be joined to \( \hat{v}_0 \) by a vertical edge path in \( X_{j_0}^{(j_0 - j_0)} \) of combinatorial length at most 2. Hence \( \hat{p}_0 \) may be joined to \( \hat{p}_1 \) may be joined by a chain of at most four 2-cells of \( X_{j_0}^{(i_j)} \), and the claim holds in this case.

Since \( j_1 \) is arbitrary, the claim forces \( \hat{p}_0 = \hat{p}_1 \).

\[ \square \]

5. David-Semmes regularity of the projection \( \mathbb{R}^2 \to X_\infty \) and the lower bound on the topological dimension

In this section we prove part (1) of Theorem 1.1, and the lower bound in part (2), in the \( n = 2 \) case.

Let \( d_\alpha^\infty \), \( \alpha \), and \( d_\alpha \) be as in Section 2. We recall that \( \mathcal{L}^2 \) denotes Lebesgue measure on \( \mathbb{R}^2 \), and for \( j \in \mathbb{Z} \cup \{\infty\} \) we let \( \mu_j = (\hat{\tau}_j)^\# \mathcal{L}^2 \).

**Lemma 5.1.** There is a constant \( C \) such that \( C^{-1} d_\alpha \leq d_\alpha^\infty \leq C d_\alpha \). Moreover, the \( Q \)-dimensional Hausdorff measure is uniformly comparable to a Lebesgue measure.

**Proof.** The metric on \( Y = \mathbb{R}^2 \) is uniformly comparable to a product metric \( d_{hor} \times d_{vert} \); \( d_{hor} \) is just the standard metric on \( \mathbb{R} \); on the other hand, \( d_{vert} \) is the largest pseudodistance which makes each cell of generation \( j \in \mathbb{Z} \) of the standard \( m_\alpha \)-adic subdivision of \( \mathbb{R} \) have diameter at most \( m^{-j} \). Specifically, for any \( x, x' \in \mathbb{R} \), the distance \( d_{vert}(x, x') \)
is the infimum of the quantities $\sum_i \text{length}(\sigma_i)$, where $\sigma_1, \ldots, \sigma_t$ is a chain of $m_v$-adic intervals joining $x$ to $x'$, and length($\sigma_i$) = $m^{-j_i}$ when $\sigma_i$ has generation $j_i$ (cf. Section 4); since $|x - x'| \leq \sum_i m_v^{-j_i} = \sum_i (\text{length}(\sigma_i))^{\frac{1}{2}}$ one readily checks that $d_{\text{vert}}$ is uniformly comparable to the snowflake of the standard metric on $\mathbb{R}$ by the power $\alpha = \frac{\log m}{\log m_v}$.

Finally, as $\hat{d}_{\infty}$-balls of radius $r$ have Lebesgue measure $\approx r^Q$, we conclude that $\mathcal{H}^Q \approx L^2$.

**Lemma 5.2** (David-Semmes regularity of $\hat{\pi}^\infty$).

1. The projection map $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}_{\infty}^\infty) \to (X_{\infty}, \hat{d}_{\infty})$ is a David-Semmes regular map.
2. $(X_{\infty}, \hat{d}_{\infty})$ is Ahlfors $Q$-regular, where $Q = 1 + \alpha^{-1}$.
3. $(X_{\infty}, \hat{d}_{\infty})$ has topological dimension at least 2.

**Proof.**

1. Suppose $\sigma$ is a 2-cell of $Y_j$. Then by the definition of $\hat{d}_{\infty}$, the pullback of $(\hat{\pi}^\infty)^* \hat{d}_{\infty}$ is a pseudodistance on $\mathbb{R}^2$ with respect to which the diameter of $\sigma$ is $\leq m^{-j}$. From the definition of $\hat{d}_{\infty}$ we therefore have $\hat{d}_{\infty} \leq (\hat{\pi}^\infty)^* \hat{d}_{\infty}$, so $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}_{\infty}^\infty) \to (X_{\infty}, \hat{d}_{\infty})$ is $1$-Lipschitz.

Pick $\bar{p} \in X_{\infty}$, $r > 0$.

We want to show that $(\hat{\pi}^\infty)^{-1}(B(\bar{p}, r))$ may be covered by a controlled number of balls of radius comparable to $r$ in $(\mathbb{R}^2, \hat{d}_{\infty}^\infty)$. To that end, we choose $j \in \mathbb{Z}$ such that $m^{-(j+1)} < r \leq m^{-j}$, and pick $p \in (\pi_j^\infty)^{-1}(\bar{p})$.

Since $(\pi_j^\infty)^{-1}(B(\bar{p}, m^{-j})) \subset B(p, 3m^{-j})$ by Proposition 4.12 we have

$$((\hat{\pi}^\infty)^{-1}(B(\bar{p}, r)) \subset (\hat{\pi}^\infty)^{-1}(B(\bar{p}, m^{-j}))$$

$$= (\hat{\pi}_j)^{-1}(\pi_j^\infty)^{-1}(B(\bar{p}, m^{-j})) \subset (\hat{\pi}_j)^{-1}(B(p, 3m^{-j}))$$.

Note that $B(p, 3m^{-j})$ is contained in a controlled number of 2-cells of $X_j$ by Lemma 3.4[7], and the inverse image of each of these may be covered by at most 27 cells of $Y_j$ by Lemma 3.4[8]. Since a cell of $Y_j$ has $\hat{d}_{\infty}^\infty$-diameter $\leq m^{-j} < m \cdot r$, we are done.

2. This follows immediately from 1 since regular maps preserve Ahlfors-regularity.

3. By 1 the point inverses $(\hat{\pi}^\infty)^{-1}(\bar{p})$ have controlled cardinality; in particular, they are totally disconnected. Therefore $\hat{\pi}^\infty$ cannot decrease the topological dimension (see for example [Eng78, Thm. 1.24.4]); therefore, the topological dimension of $X_{\infty}$ is at least that of $\mathbb{R}^2$. 


One may give a more concrete proof that $X_{\infty}$ has topological dimension $\geq 2$ along the following lines. Let $f$ be the restriction of $\hat{\pi}_{\infty} : \mathbb{R}^2 \to X_{\infty}$ to the boundary of $[0,1]^2$; note also that $f$ is injective. To see this, suppose that $p, p' \in \partial[0,1]^2$ are distinct points with $f(p) = f(p')$. Then $x(p) = x_{\infty}(f(p)) = x_{\infty}(f(p')) = x(p')$, and by the definition of the equivalence relation we must have $x(p) = x(p') \in \mathbb{Z} \setminus \mathbb{Z}^{(j-1)}$ for some $j \in \mathbb{Z}$. In view of the definition of $R_{\infty}$, by inspection we get a contradiction to the fact that $p, p' \in \partial[0,1]^2$.

Letting $\gamma = f(\partial[0,1]^2) \subset X_{\infty}$ be the image, we apply the Tietze extension theorem to extend $g = f^{-1} : \gamma \to \partial[0,1]^2$ to a continuous map $\hat{g} : X_{\infty} \to \mathbb{R}^2$. By degree theory, the origin is a stable value of the composition $[0,1]^2 \xrightarrow{\hat{\pi}_{\infty}} X_{\infty} \xrightarrow{\hat{g}} \mathbb{R}^2$, so $\hat{g}$ cannot be approximated by a map to $\mathbb{R}^2 \setminus \{0\}$. Therefore $X_{\infty}$ has topological dimension at least 2.

$\square$

6. THE UPPER BOUND ON THE ASSOUAD-NAGATA DIMENSION

The proof of the upper bound on the topological dimension is more subtle than the proof of the lower bound. Note that the existence of a map $\mathbb{R}^2 \to X_{\infty}$ with finite point inverses is not by itself enough to imply that $X_{\infty}$ has topological dimension 2: recall that the Peano curve $[0,1] \to [0,1]^2$ is a finite-to-one surjective map, showing that such maps can increase the topological dimension.

We now recall the definition of the Assouad-Nagata dimension. Let $\mathcal{C}$ be a cover of a metric space $X$. Then $\mathcal{C}$ is $r$-bounded if $\text{diam}(C) \leq r$ for all $C \in \mathcal{C}$, and $\mathcal{C}$ has $r$-multiplicity at most $k$ if every ball of radius $r$ intersects at most $k$ elements of $\mathcal{C}$.

**Definition 6.1.** [Ass82, LS05] The Assouad-Nagata dimension of a metric space $X$ is the infimum of the integers $n \geq 0$ such that for some $c > 0$ and every $r > 0$, there is a cover $\mathcal{C}$ of $X$ that is $cr$-bounded and has $r$-multiplicity at most $n + 1$.

Note that the Assouad-Nagata dimension is bounded below by the topological dimension [LS05, Prop. 2.2].

**Theorem 6.2.** The Assouad-Nagata dimension of $X_{\infty}$ is at most 2.

**Proof.** We exhibit “good coverings” in the sense of [LS05, Prop. 2.5](4): we will show that for every $k \in \mathbb{Z}$, there is a cover $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ of $X_{\infty}$
such that $C_i$ is $m^{-k}$-bounded and has $m^{-(k+1)}$-multiplicity 1 for all $i \in \{0, 1, 2\}$. This implies that the Assouad-Nagata dimension of $X_\infty$ is at most 2.

Consider the family $\Omega_2$ of 2-cells of $X_k$; from each $\sigma \in \Omega_2$ produce a subset $c_\sigma$ by taking the closure of the subset of $\sigma^{(1)}$ obtained by removing those points $p \in \sigma^{(1)}$ which satisfy one of the following:

$$
\begin{align*}
    d_R (x(p), \min x(\sigma^{(1)})) &\leq m^{-(k+1)}/2, \\
    d_R (x(p), \max x(\sigma^{(1)})) &\leq m^{-(k+1)}/2, \\
    d_{S^1} (y_k(p), 0) &\leq 3m_v^{-(k+1)}.
\end{align*}
$$

(6.3)

Note that $\text{diam} \hat{\pi}_k^\infty (c_\sigma) \leq m^{-k}$. Consider now different cells $\sigma_0, \sigma_1 \in \Omega_2$; for $i = 0, 1$ let $p_i \in \hat{\pi}_k^\infty (c_\sigma_i)$; as $c_\sigma_i$ does not meet the 1-skeleton of $X_k$, any cell $\tau$ of $X_{k+1}$ intersecting $(\hat{\pi}_k^\infty)^{-1}(p_i)$ must intersect a given cell $\tau_i$ which belongs to the set of those subcells of $\hat{\pi}_k^{k+1}(\sigma_i)$ which meet $\hat{\pi}_k^{k+1}(c_\sigma_i)$. Let $\tau'_i$ be a cell adjacent to $\tau_i$; recall that $X_{k+1}$ is obtained from $X_k$ by quotenting by $(\hat{\pi}_k \circ \Phi^k)_* \mathcal{R}_0$; and thus (6.3) guarantees that either $\tau'_0$ and $\tau'_1$ are not adjacent or the sets $x(\tau'_0)$ and $x(\tau'_1)$ are at distance at least $m^{-(k+1)}$. Thus, applying Corollary 4.19 and the fact that $x$ is 1-Lipschitz, we obtain

$$
\hat{d}_\infty (\hat{\pi}_k^\infty (c_{\sigma_0}), \hat{\pi}_k^\infty (c_{\sigma_1})) \geq m^{-(k+1)}.
$$

(6.4)

Let $\Omega_1$ be the collection of 1-cells of $X_k$; from $e \in \Omega_1$ produce a subset $c_e$ as follows: let $e^{(1)}$ be the subdivision of $e$ in $X_k^{(1)}$ and let $C_e$ be the collection of those points $p$ which belong to a 2-cell of $X_{k+1}^{(1)}$ which intersects $e$; then, if $e$ is vertical, $c_e$ is obtained from $C_e$ by taking the points $p \in C_e$ satisfying:

$$
\begin{align*}
    d_R (x(p), x(e)) &\leq m^{-(k+1)}/2 \\
    d_{S^1} (y_k(\tau), 0) &\geq 5m_v^{-(k+1)};
\end{align*}
$$

(6.5)

if $e$ is horizontal, $c_e$ is obtained from $C_e$ by taking the points $p \in C_e$ satisfying:

$$
\begin{align*}
    d_R (x(p), \{\max x(e), \min x(e)\}) &\geq m^{-(k+1)} \\
    d_{S^1} (y_k(p), 0) &\leq 3m_v^{-(k+1)}.
\end{align*}
$$

(6.6)

Let $e_0, e_1$ be distinct cells in $\Omega_1$. Then either $x(e_0)$ and $x(e_1)$ are at distance $m^{-(k+1)}$ apart, or there are no adjacent cells $\tau_i$ of $X_{k+1}$ such that $\tau_i \cap \hat{\pi}_k^{k+1}(c_{e_i}) \neq \emptyset$. Thus by Corollary 4.19 we conclude that:

$$
\hat{d}_\infty (\hat{\pi}_k^{k+1}(e_0), \hat{\pi}_k^{k+1}(e_1)) \geq m^{-(k+1)}.
$$

(6.7)
Let $\Omega_0$ be the collection of vertices of $X_k$. For $v \in \Omega_0$ consider the set $C_v$ consisting of those 2-cells of $X_k^{(1)}$ within combinatorial distance 5 from a cell containing $v$; then $c_v$ consists of those points $p \in C_v$ satisfying:

\begin{align}
 d_R(x(p), x(v)) &\leq m^{-(k+1)} \\
 d_S(y_k(p), 0) &\leq 5m^{-(k+1)}. 
\end{align}

Then $\text{diam } \hat{\pi}_k^\infty(c_v) \leq m^{-(k+1)}$. Let $v_0$ and $v_1$ distinct vertices. If $x(v_0) \neq x(v_1)$ then $\hat{d}_\infty(\hat{\pi}_k^\infty(c_v_0), \hat{\pi}_k^\infty(c_v_1)) \geq m^{-(k+1)}$. Otherwise, there are no adjacent cells $\tau_i$ of $X_{k+1}$ such that $\tau_i \cap \hat{\pi}_k^{k+1}(c_{v_i}) \neq \emptyset$. Thus Corollary 4.19 gives:

\begin{align}
 \hat{d}_\infty(\hat{\pi}_k^\infty(c_{v_0}), \hat{\pi}_k^\infty(c_{v_1})) &\geq m^{-(k+1)}. 
\end{align}

The families $\{\hat{\pi}_k^\infty(c_\sigma)\}_{\sigma \in \Omega_2}$, $\{\hat{\pi}_k^\infty(c_e)\}_{e \in \Omega_1}$ and $\{\hat{\pi}_k^\infty(c_v)\}_{v \in \Omega_0}$ provide a good covering of $X_\infty$. □

### 7. The Poincaré Inequality

In this section we prove that $(X_\infty, \mu)$ satisfies a Poincaré inequality. Following Semmes [Sem], we do this by showing that any two points $p, q \in X_\infty$ can be joined by a “pencil” — a good measured family of curves. To obtain such a family, we first construct a string of horizontal galleries that connects $p$ to $q$; we then convert this to a measured family of curves by replacing each 2-cell with the corresponding (appropriately normalized) measured family of horizontal geodesics, and then concatenating.


**Lemma 7.1.** For each $C > 0$ there is an $L = L(C)$ such that the following holds. For every $j \in \mathbb{Z}$, and every pair $\sigma_0, \sigma_1$ of 2-cells of $X_j$ such that

\begin{align}
 \hat{d}_\infty(\hat{\pi}_j^\infty(\sigma_0), \hat{\pi}_j^\infty(\sigma_1)) &\leq Cm^{-j}; 
\end{align}

there is a horizontal gallery $T$ in $X_j$ whose combinatorial length is at most $L$ and which starts at $\sigma_0$ and ends at $\sigma_1$.

**Proof.** We first show that if $\sigma_0 \cap \sigma_1 \neq \emptyset$ there is a horizontal gallery $S$ of length at most $L_0$ starting at $\sigma_0$ and ending at $\sigma_1$. If $\sigma_0$ and $\sigma_1$ share a vertical 1-cell we can just take $S = \{\sigma_0, \sigma_1\}$. If $\sigma_0$ and $\sigma_1$ share a horizontal 1-cell, we know by the definition of $R_{j-1}$ that there is a horizontal gallery $S$ of length at most $C_0 = C_0(m, m_v)$ which starts
at $\sigma_0$ and ends at $\sigma_1$. If $\sigma_0$ and $\sigma_1$ share only a 0-cell, we know by Lemma 3.4 that we can find a gallery $S$ joining them and of length at most $C_1 = C_1(m, m_0)$. This can be turned into a horizontal gallery of length at most $C_0 C_1$ by replacing consecutive cells which share a horizontal edge by a horizontal gallery of length at most $C_0$ connecting them. Thus one can take $L_0 = C_0 C_1$.

We now turn to the general case. By Corollary 4.18 there is a chain $T_0$ in $X_j$ consisting of at most $N(C)$ 2-cells, which starts at $\sigma_0$ and ends at $\sigma_1$. For any pair of consecutive 2-cells in $T_0$ we can use the discussion in the previous paragraph to construct a horizontal gallery connecting them. We thus obtain a horizontal gallery $T$ of length at most $N(C)L_0$ starting at $\sigma_0$ and ending at $\sigma_1$. $\square$

Next, using Lemma 7.1 we will show that one can connect any pair of points $p, q \in X_\infty$ by stringing together geometrically shrinking sequences of horizontal galleries. This is made precise in the following definition.

**Definition 7.3.** Let $p, q \in X_\infty$. A string of galleries connecting $p$ to $q$ consists of four sequences $\{\sigma_j\}_{j \geq j_0}$, $\{\tau_j\}_{j \geq j_0}$, $\{G_j\}_{j > j_0}$, $\{T_j\}_{j > j_0}$ which satisfy the following additional conditions for some constant $C_0$:

1. $(G1)$: $\hat{d}_\infty(p, q) \approx_{C_0} m^{-j_0}$;
2. $(G2)$: $\sigma_j$ and $\tau_j$ are cells of $X_j$ and $\sigma_{j_0} = \tau_{j_0}$;
3. $(G3)$: We have the following control on the distances from $p$ and $q$:

\begin{align}
\hat{d}_\infty(p, \pi_j^\infty(\sigma_j)) &\leq C_0 m^{-j}; \\
\hat{d}_\infty(q, \pi_j^\infty(\tau_j)) &\leq C_0 m^{-j};
\end{align}

(7.4)

4. $(G4)$: For each $j \geq j_0$ the vertical faces of $\sigma_j$ and $\tau_j$ are ordered:
   - we denote those of $\sigma_j$ by $e_j^{(0)}$, $e_j^{(1)}$, and those of $\tau_j$ by $f_j^{(0)}$ and $f_j^{(1)}$. We also require $e_j^{(0)} = f_j^{(0)}$;
   - $(G5)$: $G_j$ (resp. $T_j$) is a collection of $m$ horizontal galleries $\{G_j^{(1)}, \ldots, G_j^{(m)}\}$ (resp. $\{T_j^{(1)}, \ldots, T_j^{(m)}\}$); let $\{e_{j-1,1}^{(0)}, \ldots, e_{j-1,m}^{(0)}\}$ (resp. $\{f_{j-1,1}^{(0)}, \ldots, f_{j-1,m}^{(0)}\}$) denote the 1-cells in the first subdivision of $e_{j-1}^{(0)}$ (resp. $f_{j-1}^{(0)}$); then $G_j^{(i)}$ (resp. $T_j^{(i)}$) is a horizontal gallery of at most $C_0$ cells which connects $e_j^{(1)}$ (resp. $\pi_{j-1}(f_j^{(1)}))$ to $\pi_{j-1}(e_{j-1,1}^{(0)})$ (resp. $f_j^{(1)}$).

**Lemma 7.5.** Suppose $p, q \in X_\infty$ and $\hat{d}_\infty(p, q) \in [m^{-j_0 - 1}, m^{-j_0})$. Then there is a string of galleries connecting $p$ to $q$ where the constant $C_0$ does not depend on the pair $p, q$. 

Proof. As \( \hat{d}_\infty(p, q) < m^{-j_0} \) by Proposition 4.12 we conclude that:

\[
\begin{align*}
\hat{d}_{j_0} \left( (\pi_{j_0}^\infty)^{-1}(p), (\pi_{j_0}^\infty)^{-1}(q) \right) &< 3m^{-j_0}, \\
\end{align*}
\]

and thus \( (\pi_{j_0-1}^\infty)^{-1}(p) \) and \( (\pi_{j_0-1}^\infty)^{-1}(q) \) intersect adjacent cells of \( X_{j_0-1} \); we can therefore find a \( C_0 \) independent of \( p, q \) and a single cell \( \sigma_{j_0} = \tau_{j_0} \) of \( X_{j_0} \) such that \((7.4)\) holds with \( j = j_0 \). For each \( j \) choose \( \sigma_j \) and \( \tau_j \) such that \( (\pi_j^\infty)^{-1}(p) \cap \sigma_j \neq \emptyset \) and \( (\pi_j^\infty)^{-1}(q) \cap \tau_j \neq \emptyset \). We now choose a first and a last vertical edge for each \( \sigma_j \) and \( \tau_j \) as in \((G4)\). We now construct the collection of galleries \( \{G_j\}_{j>j_0} \) and \( \{T_j\}_{j>j_0} \). By symmetry we just focus on the construction of \( G_j \). By the choice of the cells \( \{\sigma_j\}_{j\geq j_0} \) we have that:

\[
\hat{d}_{j} \left( \sigma_j, \pi_{j-1}(\sigma_{j-1}) \right) \leq C_0 m^{-j};
\]

therefore, by possibly enlarging \( C_0 \), for each \( i \in \{1, \cdots, m\} \) we can choose by Lemma \[7.1\] a horizontal gallery consisting of at most \( C_0 \) cells

\[
(7.8) \quad G_j^{(i)} = \{\theta_1, \cdots, \theta_L\} \subset X_j
\]

which joins \( e_j^{(1)} \) to \( \pi_{j-1}(e_{j-1}^{(1)}) \).

\[\square\]

**Corollary 7.9.** Let \( p, q \in X_\infty \) be such that \( \hat{d}_\infty(p, q) \in [m^{-j_0-1}, m^{-j_0}) \); then there is a universal constant \( C_1 \) such that, for each \( M > j_0 \), there is a horizontal gallery \( G_M = \{\sigma_0, \cdots, \sigma_L\} \) consisting of 2-cells of \( X_M \) and whose length is at most \( C_0 m^{-j_0} \), where one has:

\[
\begin{align*}
\hat{d}_\infty(\pi_M^\infty(\sigma_0), p) &\leq C_0 m^{-M}, \\
\hat{d}_\infty(\pi_M^\infty(\sigma_L), q) &\leq C_0 m^{-M}.
\end{align*}
\]

**Proof.** We use Lemma \[7.5\] to take a string of galleries connecting \( p \) to \( q \) of length at most \( C(C_0) m^{-j_0} \), where \( C_0 \) is the constant in Lemma \[7.5\].

We now truncate this string to obtain a horizontal gallery in \( X_M \):

\[
T = \left\{ \sigma_M, G_M^{(1)}, \pi_{M-1}(\sigma_{M-1}), \pi_{M-1}(G_{M-1}^{(1)}), \cdots, \pi_{j_0}^M(\sigma_{j_0}), \pi_{j_0}^M(T_{j_0+1}^{(1)}), \cdots, \pi_M \right\};
\]

we now inductively modify \( T \) to increase the minimal generation of cells in \( T \) to end up with a gallery consisting of only 2-cells of \( X_M \). At the first step, we take the end cell \( \sigma \) of \( G_{j_0+1}^{(1)} \) in \( X_{j_0+1} \), and the first cell \( \tau \) of \( T_{j_0+1}^{(1)} \) in \( X_{j_0+1} \); as both cells intersect some vertical faces of a pair of 2-cells in \( \pi_{j_0-1}^M(\sigma_{j_0}) \), we can find a horizontal gallery \( S \) in \( X_{j_0+1} \) connecting \( \sigma \) to \( \tau \) whose length is at most \( C(H)m^{-j_0} \). We then replace
π_j_0^M(σ_j_0) by π_j_0+1^M(S). A similar process is applied now to replace the cells of T of generation j_0 + 1 by cells of generation j_0 + 2. The increase in length is of a factor C(H)m_j_0 for each cell of generation j_0 + 1. Continuing inductively, one obtains a gallery T consisting of cells of generation M and whose length is ≤ C(C_0, H, m)m_j_0.

7.2. Constructing Semmes pencils.

Definition 7.12. A measured family of curves (Γ, I_Γ, Ω_Γ, ν_Γ) is a measurable function Γ : I_Γ × Ω_Γ → X, where I_Γ is an interval of ℝ, such that ν_Γ is a probability measure on Ω_Γ, and there is an L such that for each ω ∈ Ω_Γ the map t ↦ Γ(t, ω) is a Lipschitz curve in X of Lipschitz constant at most L.

We say that a measured family of curves (Γ, I_Γ, Ω_Γ, ν_Γ) joins a set S_0 to a set S_1 if for ν_Γ-a.e. ω ∈ Ω_Γ one has Γ(min I_Γ, ω) ∈ S_0 and Γ(max I_Γ, ω) ∈ S_1. Given a subinterval I ⊂ I_Γ we denote by Γ_I the restriction Γ : I × Ω_Γ → X.

To a measured family of curves (Γ, I_Γ, Ω_Γ, ν_Γ) we associate a measure µ_Γ by:

\[ \mu_\Gamma = |\Gamma|_\#(L^1 \times \nu_\Gamma). \]

In the following we will often simply write Γ instead of (Γ, I_Γ, Ω_Γ, ν_Γ). We finally define the support of Γ by:

\[ \text{spt } \Gamma = \{ p \in X_\infty : \forall r > 0 \text{ } \nu_\Gamma(\{ \omega : \Gamma(I_\Gamma, \omega) \cap B(p, r) \neq \emptyset \}) > 0 \}. \]

Definition 7.15. For a metric measure space (X, µ) recall the definition of the Riesz potential µ_p centered at p:

\[ \mu_p = \frac{d(p, \cdot)}{\mu(B(p, d(p, \cdot)))} \mu. \]

We will denote the Riesz potential of (X, µ_∞) centered on p ∈ X_∞ by µ_∞,p. To prove the Poincaré inequality and to establish a bound on the analytic dimension we rely on the following theorem.

Theorem 7.17. There is a universal constant C depending only on (m, m_v) such that the following holds.

(C1): For p, q ∈ X_∞, there is a measured family of curves Γ joining p to q;

(C2): The Lipschitz constant of Γ is at most C and L^1(I_Γ) ≤ Cd_∞(p, q);
\( (C3): \) The measure \( \mu_\Gamma \) is controlled by the Riesz potentials centered at \( p \) and \( q \):

\[
\mu_\Gamma \leq C(\mu_\infty, p \cup B(p, Cd(p, q)) + \mu_\infty, q \cup B(q, Cd(p, q))).
\]

**Proof.** Using Lemma 7.5, we take a string of galleries \( S = \{ \{ \sigma_j \}_{j \geq j_0}, \{ \tau_j \}_{j \geq j_0}, \{ \mathcal{G}_j \}_{j > j_0}, \{ \mathcal{T}_j \}_{j > j_0} \} \) connecting \( p \) to \( q \). The idea of the proof is based on the following observations: first, one can use \( S \) to string together horizontal segments to obtain Lipschitz paths joining \( p \) to \( q \). Secondly, the Fubini representations of the measures on the cells of \( S \) give rise to a “natural” transverse measure on the family of quasi-geodesics connecting \( p \) to \( q \). To ease the exposition, we have divided the proof in two steps. In the first we discuss how to build a measured family of curves joining \( \pi_\infty j_0 (\sigma_j) \) to \( p \). In the second we join together two measured families of curves connecting the points \( p \) and \( q \) to \( \pi_\infty j_0 (\sigma_j) = \pi_\infty j_0 (\tau_j) \).

**Step 1: Construction of a measured family of curves joining \( \pi_\infty j_0 (\sigma_j) \) to \( p \).**

We now construct a measured family of curves \( (\Gamma_p, I_p, \Omega_p, \nu_p) \) joining \( \pi_\infty j_0 (\sigma_j) \) to \( p \). We let:

\[
I_p = \left[ 0, \sum_{j_0 \leq s} m^{-s} \right],
\]

and introduce the intervals:

\[
A_i = \left[ \sum_{j_0 \leq s < i} m^{-s}, \sum_{j_0 \leq s < i} m^{-s} + \frac{m^{-i}}{2} \right] \quad (i \geq j_0)
\]

\[
B_i = \left[ \sum_{j_0 \leq s < i-1} m^{-s} + \frac{m^{-i+1}}{2}, \sum_{j_0 \leq s < i} m^{-s} \right] \quad (i > j_0).
\]

Note that by convention a sum over an empty set of indices is taken to be 0; for example, \( A_{j_0} = [0, m^{-j_0}/2] \) and \( B_{j_0+1} = [m^{-j_0}/2, m^{-j_0}] \). For later convenience let \( A_i = [a_i^{(0)}, a_i^{(1)}] \) and \( B_i = [b_i^{(0)}, b_i^{(1)}] \). We choose a cell \( \tilde{\sigma}_{j_0} \subset Y_{j_0} \) such that \( \hat{\pi}_{j_0}(\tilde{\sigma}_{j_0}) = \sigma_{j_0} \) and let \( \tilde{e}_{j_0}^{(0)} \) and \( \tilde{e}_{j_0}^{(1)} \) denote its vertical faces. Let \( \Omega_p \) denote the subset of \( \tilde{e}_{j_0}^{(1)} \) obtained by removing points \( p' \) such that \( y(p') = km^{-j}_q \) for \( (k, j) \in \mathbb{Z}^2 \). Note that \( \Omega_p \) has full \( \mathcal{H}^1 \)-measure in \( \tilde{e}_{j_0}^{(1)} \). Let \( \nu_p = \mathcal{H}^1 \cup \Omega_p \).

We define a map \( \hat{H}_{j_0} : A_{j_0} \times \Omega_p \to \tilde{\sigma}_{j_0} \) by letting \( \hat{H}_{j_0}(\cdot, p') \) denote the unique constant velocity horizontal curve in \( \tilde{\sigma}_{j_0} \) starting at \( p' \) and
ending in \( e_{j_0}^{(0)} \). We let \( \Gamma_p|A_{j_0} \times \Omega_p = \hat{\pi}_{j_0} \circ \hat{H}_{j_0} \). Then Fubini’s Theorem implies that:

\[
(7.21) \quad (\hat{H}_{j_0})_* (\mathcal{L}^1 \times \nu_p \ll A_{j_0} \times \Omega_p) = m_{v_{j_0}}^j \mathcal{L}^2 \ll \hat{\sigma}_{j_0}.
\]

Note that the Lipschitz constant of \( \hat{H}_{j_0} \) is at most 2.

As there is a uniform bound on the number of cells of \( Y_{j_0} \) that project to \( \sigma_{j_0} \), we conclude from \( (7.21) \) that:

\[
(7.22) \quad (\Gamma_p)|_*(\mathcal{L}^1 \times \nu_p \ll A_{j_0} \times \Omega_p) \approx m_{v_{j_0}}^j \mu \ll \text{spt}(\Gamma_p|A_{j_0} \times \Omega_p).
\]

Let \( \mathcal{G}_{j_0+1} = \{ \mathcal{G}_{j_0+1}^{(i)} \}_{1 \leq i \leq m} \) and note that for \( p' \in \Omega_p \) there is a unique \( i = i(p') \) such that the point \( \hat{\pi}_{j_0+1} \circ \hat{H}_{j_0}(a_{j_0}^{(1)}, p') \) belongs to the last cell of \( \mathcal{G}_{j_0+1}^{(i)} \). We now write \( \mathcal{G}_{j_0+1}^{(i)} = \{ \theta_1, \ldots, \theta_L \} \) and for each \( 1 \leq k \leq L \) choose a cell \( \hat{\theta}_k \subset Y_{j_0+1} \) such that \( \hat{\pi}_{j_0+1}^{-1}(\hat{\theta}_k) = \theta_k \). Intuitively, the collection of cells \( \mathcal{C}_{j_0+1} = \{ \theta_k \}_{1 \leq k \leq L} \) induces a broken horizontal path \( \hat{J}(\cdot, p') : B_{j_0+1} \to Y_{j_0+1} \). More precisely, observe that \( \hat{\theta}_L \cap (\hat{\pi}_{j_0+1})^{-1}(\hat{\pi}_{j_0}^{-1}(\hat{H}_{j_0}(a_{j_0}^{(1)}, p'))) \) consists of a single point \( p'_{\theta_L} \). Then there is a unique horizontal constant velocity path \( \xi_{\theta_L} : [b_{j_0+1}^{(0)}, b_{j_0+1}^{(0)} + L^{-1}m^{-j_0}/2] \to \hat{\theta}_L \) which starts at \( p'_{\theta_L} \) and ends on the vertical 1-cell of \( \hat{\theta}_L \) other than the one containing \( p'_{\theta_L} \). Let \( q'_{\theta_L} = \xi_{\theta_L}(b_{j_0+1}^{(0)} + L^{-1}m^{-j_0}/2) \). Then again \( \hat{\theta}_{L-1} \cap (\hat{\pi}_{j_0+1})^{-1}(\hat{\pi}_{j_0+1}^{-1}(q'_{\theta_L})) \) consists of a single point \( p'_{\theta_{L-1}} \). This happens because \( p' \in \Omega_p \) forces the points \( p'_{\theta_L} \) and \( q'_{\theta_L} \) to lie outside the 0-skeleton of the cells containing them. We then observe that there is a unique horizontal constant velocity path \( \xi_{\theta_{L-1}} : [b_{j_0+1}^{(0)} + L^{-1}m^{-j_0}/2, b_{j_0+1}^{(0)} + 2L^{-1}m^{-j_0}/2] \to \hat{\theta}_{L-1} \) which starts at \( p'_{\theta_{L-1}} \) and ends on the 1-cell opposite to the one containing \( p'_{\theta_{L-1}} \). If \( \xi_{\theta_{L}} \) continues by backward induction, constructing \( \xi_{\theta_k} \) knowing \( q'_{\theta_{k+1}} \) as we did for \( \theta_{L-1} \), and we omit the details. We then let for \( 1 \leq k \leq L \):

\[
(7.23) \quad \hat{J}(\cdot, p')(b_{j_0+1}^{(0)} + kL^{-1}m^{-j_0}/2, b_{j_0+1}^{(0)} + (k+1)L^{-1}m^{-j_0}/2) = \xi_{k-L},
\]

and extend \( \hat{J}(\cdot, p') \) to \( B_{j_0+1} \) so that it is continuous from the left and \( \hat{J}(b_{j_0+1}^{(0)}, p') = p'_{\theta_L} \). We then let

\[
(7.24) \quad \Gamma_p|B_{j_0+1} \times \Omega_p = \hat{\pi}_{j_0+1} \circ \hat{J}.
\]

Note that \( \Gamma_p(\cdot, p')|B_{j_0+1} \) is a Lipschitz curve joining \( \hat{\pi}_{j_0}^{-1}(\hat{H}_{j_0}(a_{j_0}^{(1)}, p')) \) to a point on the last face \( e_{j_0+1}^{(1)} \) of \( \sigma_{j_0+1} \), and whose Lipschitz constant is
at most $2L \leq 2C$. Taking into account that for $i_1 \neq i_2$ the cardinality of the set:

$$(7.25) \quad \left\{ (\theta, \theta') \in G_{j_0+1}^{(i_1)} \times G_{j_0+1}^{(i_2)} : \mu \left( \hat{\pi}_{j_0+1}^\infty (\theta) \cap \hat{\pi}_{j_0+1}^\infty (\theta') \right) > 0 \right\}$$

is uniformly bounded, an argument similar to the one which yielded (7.22) gives:

$$(7.26) \quad (\Gamma_p)|_\#(\mathcal{L}^1 \times \nu \ll B_{j_0+1} \times \Omega_p) \approx m_{i_0}^{j_0+1} \mu \ll \text{spt}(\Gamma_p|B_{j_0+1} \times \Omega_p).$$

The construction then continues by induction on the intervals $A_i$, $B_i$. Note that the choice of $\Omega_p$ guarantees that, knowing $\Gamma_p(\cdot, p')|\bigcup_{s \leq i} A_s \cap \bigcup_{s=j_0+1} B_s$, there is a unique extension $\Gamma_p|B_i \times \Omega_p$ which is obtained using steps similar to those we used to construct $\Gamma_p|B_{j_0+1} \times \Omega_p$. The point is again that $(\hat{\pi}_{j_0+1}^\infty)^{-1}(\Gamma(a_i^{(1)}, p'))$ does not intersect the 0-skeleton of $Y_{i+1}$ and so the extension to $B_{i+1}$ is uniquely determined. In particular, one has also the following analogue of (7.26):

$$(7.27) \quad (\Gamma_p)|_\#(\mathcal{L}^1 \times \nu \ll B_{i+1} \times \Omega_p) \approx m_{i_0}^i \mu \ll \text{spt}(\Gamma_p|B_{i+1} \times \Omega_p).$$

Similarly, knowing $\Gamma_p(\cdot, p')|\bigcup_{s \leq i} A_s \cap \bigcup_{s=j_0+1} B_s$, there is a unique extension $\Gamma_p|A_{i+1} \times \Omega_p$ which is constructed similarly to how we constructed $\Gamma_p|A_{j_0} \times \Omega_p$. Moreover, one has the following analogue of (7.22):

$$(7.28) \quad (\Gamma_p)|_\#(\mathcal{L}^1 \times \nu \ll A_{i+1} \times \Omega_p) \approx m_{i_0}^{i+1} \mu \ll \text{spt}(\Gamma_p|A_{i+1} \times \Omega_p).$$

We finally let $\hat{\Gamma}_p(\sum_{s=j_0}^\infty m^{-s}, \cdot) = p$ because $\Gamma_p(a_i^{(1)}, p') \rightarrow p$ and $\Gamma_p(b_i^{(1)}, p') \rightarrow p$ uniformly in $p'$ as $i \rightarrow \infty$. Thus $\hat{\Gamma}_p(\cdot, p')$ is a Lipschitz curve of Lipschitz constant at most $2C$ which joins $\hat{\pi}_{j_0}^\infty(p')$ to $p$. Note that because of the choice of the string of galleries $S$ (compare (7.4)), there is a universal constant $C_0$ such that:

$$(7.29) \quad \text{spt} \hat{\Gamma}_p|(A_i \cup B_i) \times \Omega_p \subset B(p, C_0 m^{-i}).$$

Let $\mu_{\hat{\Gamma}_p}$ denote the measure $(\hat{\Gamma}_p)|_\#(\mathcal{L}^1 \times \nu \ll [0, \sum_{s=j_0}^\infty m^{-s}] \times \Omega_p)$. Thus (7.28) and (7.27) imply that:

$$(7.30) \quad \frac{d\mu_{\hat{\Gamma}_p}}{d\mu} | (X_\infty \setminus B(p, C_0 m^{-j})) \lesssim \sum_{i=-\infty}^j m_i^j \lesssim m_i^j,$$

which gives

$$(7.31) \quad \mu_{\hat{\Gamma}_p} \lesssim \mu_{\infty, p} \ll B(p, C m^{-j_0}),$$

where $\mu_{\infty, p}$ denotes the Riesz potential centered on $p$.

**Step 2: Joining two measured families of curves.** We use Step 1 to build measured families of curves $\Gamma_p$ and $\Gamma_q$ joining $\pi_{j_0}^\infty(\sigma_{j_0}) = \pi_{j_0}^\infty(\tau_{j_0})$
to \( p \) and \( q \) respectively. Note that by condition (G4) in the properties of the string \( S \), both \( \Gamma_p \) and \( \Gamma_q \) start at \( \pi_{j_0}(e_{j_0}^{(1)}) = \pi_{j_0}(f_{j_0}^{(0)}) \), where \( e_{j_0}^{(1)} \) denotes the last face of \( \sigma_{j_0} \) and \( f_{j_0}^{(0)} \) the first face of \( \tau_{j_0} \).

We now want to concatenate the reverse of \( \Gamma_p \) and \( \Gamma_q \). In fact, \( \Gamma_p \) is defined on \( I_p \times \Omega_p \) and \( \Gamma_q \) on \( I_q \times \Omega_p \) where \( \Omega_p \) is a full-measure subset of \( e_{j_0}^{(1)} \) and for \( p' \in \Omega_p \) we have \( \Gamma_p(\min I_p, p') = \Gamma_q(\min I_q, p') = p' \). Note also that the transverse measures \( \nu_p \) and \( \nu_q \) agree; we can thus concatenate the reverse of \( \Gamma_p \) and \( \Gamma_q \) obtaining a measured family of curves \( \Gamma_{pq} \). More precisely let:

\[
I_{\Gamma_{pq}} = I_p \cup (I_q + \max I_p - \min I_q),
\]

and define:

\[
\Gamma_{pq}(t, \omega) = \begin{cases} 
\Gamma_p(\max I_p - t) & \text{if } t \in [0, \mathcal{L}^1(I_{\Gamma_p})] \\
\Gamma_q(t - \min I_q + \max I_{\Gamma_p}) & \text{if } t \in [\mathcal{L}^1(I_{\Gamma_p}), \mathcal{L}^1(I_{\Gamma_p}) + \mathcal{L}^1(I_{\Gamma_q})].
\end{cases}
\]

For the measure \( \nu_{pq} \) we take \( \nu_p \). Now (C1) follows from the choice of the string \( S \), and (C2) follows since \( \Gamma_p \) and \( \Gamma_q \) are \( 2C \)-Lipschitz and because \( I_{\Gamma_{pq}} \) is given by (7.32). Note that:

\[
\mu_{\Gamma} = (\Gamma_{pq})|_\#(\mathcal{L}^1 \times \nu_{pq} \ll I_{\Gamma_{pq}} \times \Omega_p) = \mu_{\Gamma_p} + \mu_{\Gamma_q},
\]

and so we get (7.18) in (C3) by using (7.30).

Theorem 7.35. The metric measure space \( (X_\infty, \mu) \) admits a \((1,1)\)-Poincaré inequality.

\[
|u(p) - u(q)| \leq C \left( \int_{B(p,Cd(p,q))} g \, d\mu_{\infty,p} + \int_{B(q,Cd(p,q))} g \, d\mu_{\infty,q} \right).
\]

Let \( \Gamma \) be a measured family of curves joining \( p \) to \( q \) as in Theorem 7.17. Then for each \( \omega \in \Omega_{\Gamma} \) one has:

\[
|u(p) - u(q)| \leq \int_{I_{\Gamma}} g(\Gamma(t, \omega)) \, dt;
\]
integrating (7.37) in \( \nu \), one obtains:

\[
|u(p) - u(q)| \leq \int g \, d\nu,
\]

and (7.36) follows from (7.18). \( \square \)

8. The analytic dimension

In this section we prove parts (4) and (5) of Theorem 1.1, see Theorem 8.5.

We let \( \text{Lip}_b(X) \) denote the set of bounded real-valued Lipschitz functions defined on \( X \), which is a Banach algebra with norm given by:

\[
(8.1) \max\left( \|f\|_\infty, \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \right).
\]

**Definition 8.2.** Consider a bounded linear operator \( D : \text{Lip}_b(X) \to L^\infty(\mu) \) where \( \mu \) is a Radon measure on the metric space \( X \). Suppose that \( F : X \to Y \) is Lipschitz; then one obtains the pushforward \( F_#D : \text{Lip}_b(Y) \to L^\infty(F_#\mu) \) of \( D \) as follows; given \( u \in \text{Lip}_b(Y) \), \( F_#Du \) is determined by the requirement:

\[
(8.3) \int_Y g \, F_#Du \, dF_#u = \int_X g \circ F \, D(u \circ F) \, d\mu \quad (\forall g \in L^1(F_#\mu)).
\]

The collection of metric measure spaces \( \{ (X_j, \mu_j) \}_{j \in \mathbb{Z}} \) and maps \( \{ \pi_j \}_{j \in \mathbb{Z} \cup \{ \infty \}} \) gives rise to a direct system of operators \( \{ D_j \}_{j \in \mathbb{Z}} \) as follows. Let \( \hat{D} \) be the horizontal derivative operator:

\[
(8.4) \hat{D} : \text{Lip}_b(\mathbb{R}^2) \to L^\infty(L^2);
\]

we then let \( D_j = \hat{\pi}_j^# \hat{D} \) for \( j \in \mathbb{Z} \cup \{ \infty \} \). Note that if \( k \geq j \), \( \hat{\pi}^k = \hat{\pi}^j \circ \hat{\pi}^j \) and so \( \pi^k_# D_j = D_k \). In fact, the operators \( D_j \) are derivations in the sense of Weaver [Wea00], i.e. satisfy a product rule and a weak* continuity axiom. These properties are easy to verify, but will not be used in the following, except in the alternative argument given in Remark 8.24.

**Theorem 8.5.** \((X_\infty, \mu)\) has analytic dimension 1 and \((X_\infty, x_\infty)\) is a differentiability chart. Moreover, (5) in Theorem 1.1 holds.

**Proof.** Let \( f \) be a Lipschitz function; as the analytic dimension is a local property, we will assume \( f \) to be bounded. Consider an approximate continuity point \( p \) of \( D_\infty f \) and fix \( M \in \mathbb{N} \); using that \( p \) is an approximate continuity point of \( D_\infty f \), for each \( \varepsilon > 0 \) one can choose
$r_0(\varepsilon)$ such that, for any $r \leq r_0$, any $i \leq M + \lfloor \log_m(1/r) \rfloor$ and any cell $\hat{\sigma}_i$ of $Y_i$ satisfying

\begin{equation}
\hat{\pi}_i^\infty(\hat{\sigma}_i) \subset B(p, 2Cr),
\end{equation}

one has:

\begin{equation}
\int_{\hat{\pi}_i^\infty(\hat{\sigma}_i)} |D_\infty f(p) - D_\infty f| \, d\mu \leq \varepsilon.
\end{equation}

Let $q \in X_\infty$ satisfy $d(p, q) \leq r$. We let:

\begin{equation}
 u = f - f(p) - D_\infty f(p)(x_\infty - x_\infty(p));
\end{equation}

we will show that:

\begin{equation}
|u(p) - u(q)| \lesssim (m^{-M} + \varepsilon)r,
\end{equation}

from which the Theorem follows as $M$ and $\varepsilon$ are arbitrary.

Let $M_r = \lfloor \log_m(1/r) \rfloor + M$; we use Corollary 7.9 to obtain a horizontal gallery $G_{M_r} = \{\sigma_1, \ldots, \sigma_{L_r}\}$ of 2-cells of $X_{M_r}$ such that $\pi_{M_r}^\infty(G_{M_r})$ “almost connects” $p$ to $q$. More precisely, one has:

\begin{equation}
\begin{aligned}
d(\pi_{M_r}^\infty(\sigma_1), p) &\leq Crm^{-M} \\
d(\pi_{M_r}^\infty(\sigma_{L_r}), q) &\leq Crm^{-M};
\end{aligned}
\end{equation}

and

\begin{equation}
L_r m^{-M} r \leq Cr,
\end{equation}

where $C$ is a universal constant.

For $j \in \mathbb{Z}$ let $S_j$ denote the 1-skeleton of $Y_j$; then $S_j$ is $\hat{\pi}^j$-saturated, i.e. $(\hat{\pi}^j)^{-1}(\hat{\pi}^j(S_j)) = S_j$; as $S_j$ is $L^2$-null, we conclude that $\hat{\pi}^j(S_j)$ is $\mu_j$-null. Note that $\hat{\pi}^j$ restricts to a homeomorphism mapping $Y_j \setminus S_j$ onto $X_j \setminus \hat{\pi}^j(S_j)$; in particular, $\hat{\pi}^j$ maps open cells of $Y_j$ homeomorphically onto open cells of $X_j$. Let $v \in \text{Lip}_b(X_j)$ and $\sigma$ be an open cell of $X_j$; using the definition of pushforward, we conclude that $D_j v$ coincides $\mu_j$-a.e. with the horizontal derivative $\hat{D}(v \circ \hat{\pi}^j)$, i.e.:

\begin{equation}
D_j v = \hat{D}(v \circ \hat{\pi}^j) \circ (\hat{\pi}^j)^{-1} \quad \text{(on } \sigma\text{)}.
\end{equation}

Consider two adjacent cells $\sigma_i, \sigma_{i+1}$ of $G_{M_r}$; as $\sigma_i$ and $\sigma_{i+1}$ share a vertical face, the Fubini representation of $\mu_{M_r} \ll (\sigma_i \cup \sigma_{i+1})$ and the fundamental Theorem of Calculus imply
(8.13) \[
\left| \int_{\sigma_i} (u \circ \pi_{\infty}^M) \, d\mu_M - \int_{\sigma_{i+1}} (u \circ \pi_{\infty}^M) \, d\mu_M \right| \lesssim m^{-M} \int_{\sigma_i \cup \sigma_{i+1}} |D_M(u \circ \pi_{\infty}^M)| \, d\mu_M.
\]

The goal is now to replace the derivative \(D_M(u \circ \pi_{\infty}^M)\) with \(D_\infty u\) in the right hand side of (8.13). Fix a 2-cell \(\sigma\) of \(X_M\) and let \(\hat{\sigma}\) be the 2-cell of \(Y_M\) such that \(\pi^M(\hat{\sigma}) = \sigma\). The map \(\hat{\pi}^M\) restricts to a homeomorphism between the interiors of \(\hat{\sigma}\) and \(\sigma\); we now define \(g \in L^1(\mu_M)\) by letting:

(8.14) \[
\begin{aligned}
g &= \begin{cases} 
\chi_\sigma \mu_M(\sigma) \left( \text{sgn}(D_M(u \circ \pi_{\infty}^M)) \right) & \text{on the interior of } \sigma \\
0 & \text{elsewhere};
\end{cases}
\end{aligned}
\]

then we have:

(8.15) \[
g \circ \hat{\pi}^M = \frac{\chi_\sigma}{\mathcal{L}^2(\hat{\sigma})} \text{sgn}(D_M(u \circ \pi_{\infty}^M)) \circ \hat{\pi}^M,
\]

and using the definition of pushforward (8.3) we get:

(8.16) \[
\left| \int_{\sigma} |D_M(u \circ \pi_{\infty}^M)| \, d\mu_M \right| \leq \left| \int_{\hat{\sigma}} \hat{D}(u \circ \hat{\pi}^\infty) \right| \, d\mathcal{L}^2.
\]

Let \(\hat{S}\) be the union of the 1-skeleta of \(Y_j\) for \(j \in \mathbb{Z}\); then \(\hat{S}\) is \(\mathcal{L}^2\)-null and \(\hat{\pi}^\infty\)-saturated so that \(\mu_\infty(\hat{\pi}^\infty(\hat{S})) = 0\). Now \(\hat{\pi}^\infty\) restricts to a homeomorphism between \(\mathbb{R}^2 \setminus \hat{S}\) and \(X_\infty \setminus \hat{\pi}^\infty(\hat{S})\); in particular, we can define \(g \in L^1(\mu_\infty)\) as:

(8.17) \[
\begin{aligned}
g &= \begin{cases} 
\chi_{\hat{\pi}^\infty(\hat{S})} \mu_\infty(\hat{\pi}^\infty(\hat{S})) \left( \text{sgn}(\hat{D}(u \circ \hat{\pi}^\infty)) \right) \circ (\hat{\pi}^\infty)^{-1} & \text{on } X_\infty \setminus \hat{\pi}^\infty(\hat{\sigma}) \\
0 & \text{elsewhere};
\end{cases}
\end{aligned}
\]

then:

(8.18) \[
g \circ \hat{\pi}^\infty = \frac{\chi_\sigma}{\mathcal{L}^2(\hat{\sigma})} \text{sgn}\hat{D}(u \circ \hat{\pi}^\infty)
\]

as an element of \(L^1(\mathcal{L}^2)\); therefore, using the definition of pushforward (8.3), we get:

(8.19) \[
\left| \int_{\hat{\sigma}} \hat{D}(u \circ \hat{\pi}^\infty) \right| \, d\mathcal{L}^2 \leq \left| \int_{\hat{\pi}^\infty(\hat{\sigma})} D_\infty u \right| \, d\mu_\infty.
\]
Combining (8.19), (8.16), (8.13) we get:
\[ \bigg| \int_{\sigma_i} (u \circ \pi_{\infty}^M) \, d\mu_M - \int_{\sigma_{i+1}} (u \circ \pi_{\infty}^M) \, d\mu_M \bigg| \leq m^{-M} \varepsilon r \bigg| D_{\infty} u \bigg| \, d\mu_\infty, \]
and using (8.7) we conclude that:
\[ \bigg| \int_{\sigma_i} (u \circ \pi_{\infty}^M) \, d\mu_M - \int_{\sigma_{i+1}} (u \circ \pi_{\infty}^M) \, d\mu_M \bigg| \leq m^{-M} r \varepsilon. \]

Summing over \( i \) and using (8.11) we obtain:
\[ \bigg| \int_{\pi_{\infty}^{\hat{\Gamma}}(\sigma)} u \, d\mu_\infty - \int_{\pi_{\infty}^{\hat{\Gamma}}(\sigma')} u \, d\mu_\infty \bigg| = \bigg| \int_{\sigma} u \circ \pi_{\infty}^M \, d\mu_M - \int_{\sigma'} u \circ \pi_{\infty}^M \, d\mu_M \bigg| \leq C \varepsilon r; \]
using (8.10) and that \( u \) is Lipschitz, we obtain:
\[ |u(p) - u(q)| \leq 2 \text{LIP}(u) rm^{-M} + \varepsilon Cr, \]
which gives (8.9).

Finally, the horizontal derivative operator \( D_{\infty} \) is associated with the pushforward under \( \hat{\pi}_{\infty} \) of the family \( \hat{\Gamma} \) of horizontal lines of \( \mathbb{R}^n \) equipped with the obvious measure; now (8.16) can be interpreted as saying that this family of lines gives a universal Alberti representation in the sense of Bate. Note also that the horizontal geodesics are really gradient curves for \( f \), in the sense that \( f \) decreases/increases along them with optimal speed in the sense of [Sch13], see also Remark 8.28.

Remark 8.24. There is an alternative approach to the proof that the analytic dimension is 1, which uses directly the measured families of curves constructed in Theorem 7.17. This approach is based on Weaver derivations. By [Sch14, Thm. 5.9] it suffices to show that for each Lipschitz function \( f \) the inequality:
\[ \text{Lip}\ f \lesssim |D_{\infty} f| \]
holds \( \mu_\infty \)-a.e. Let \( p \) be a Lebesgue point for \( |D_{\infty} f| \). Given a point \( q \), we consider the measured family of curves \( \Gamma \) joining \( p \) to \( q \) which was constructed in Theorem 7.17 by inspecting the construction of \( \Gamma \) and using estimates like (8.16), (8.19) one concludes that there is a universal constant \( C \) such that:
\[ |f(p) - f(q)| \leq C \int_{B(p,Cd(p,q)) \cup B(q,Cd(p,q))} |D_{\infty} f| \, (d\mu_{\infty,p} + d\mu_{\infty,q}); \]
recall that \( \mu_{\infty,p}(B(p,Cd(p,q))) \approx \mu_{\infty,q}(B(p,Cd(p,q))) \approx d(p,q) \). As \( p \) is a Lebesgue point for \( |D_{\infty} f| \), the fact that \( \mu_\infty \) is doubling implies
that:

\[
\lim_{q \to p} \int_{B(p,Cd(p,q))} |D_\infty f| \, d\mu_\infty,p = |D_\infty f|(p)
\]

(8.27)

\[
\lim_{q \to p} \int_{B(q,Cd(p,q))} |D_\infty f| \, d\mu_\infty,q = |D_\infty f|(q).
\]

Thus, dividing (8.26) by \(d(p,q)\) and taking the lim sup as \(q \to p\), (8.25) follows.

**Remark 8.28.** Let \(f\) be Lipschitz and \(g_f\) denote its generalized minimal upper gradient. By [Che99, Secs. 5,6] (see [CKS] for a simplified proof) we know that \(g_f = \text{Lip}_f \mu_\infty\)-a.e. However, in these examples one can obtain a direct argument under the following lines. As we are dealing with a PI space we already know that \(\text{Lip}_f \leq Cg_f\); moreover, as the Sobolev space \(H^{1,p}(\mu_\infty)\) is reflexive for \(p > 1\), one can always assume that \(g_f\) is the \(L^p\)-limit of a sequence \(g_k\) where \(g_k\) is an upper gradient of a Lipschitz function \(f_k\), and where \(f_k \to f\) in \(H^{1,p}(\mu_\infty)\) and \(\text{Lip}_f \to \text{Lip}_f \mu_\infty\)-a.e. It thus suffices to show that whenever \(g\) is an upper gradient for \(f\), one has \(g \geq \text{Lip}_f \mu_\infty\)-a.e. Without loss of generality we can assume \(g \in L^1_{\text{loc}}(\mu_\infty)\). We secondly observe that, as \(x_\infty\) is a chart function, we have \(\text{Lip}_f = |D_\infty f| \mu_\infty\)-a.e. Now the Fubini representation of the measure \(\mathcal{L}^2\) on \(\mathbb{R}^2\) descends to a similar representation of the measure \(\mu_\infty\) in terms of the horizontal geodesics of \(X_\infty\). If \(p\) is a Lebesgue point for \(g\) and \(|D_\infty f|\), for any \(\varepsilon > 0\) we can find a horizontal line \(\gamma\) with \(\gamma(0)\) a Lebesgue point for \(g \circ \gamma\) and \(D_\infty f \circ \gamma\), \(d(\gamma(0), p) \leq \varepsilon\), and \(g(\gamma(0)) \in [g(p) - \varepsilon, g(p) + \varepsilon]\), and \(|D_\infty f|(\gamma(0)) \in [|D_\infty f|(\gamma(0)) - \varepsilon, |D_\infty f|(\gamma(0)) + \varepsilon]\). Applying Lebesgue differentiation to \(g \circ \gamma\) and \(D_\infty f \circ \gamma\) at 0 one gets \(g(\gamma(0)) \geq |D_\infty f|(\gamma(0))\).

9. The \(n = 2\) case of Theorem 1.1 concluded

We now verify (6) of Theorem 1.1 in the \(n = 2\) case.

Let \(\{(\lambda_k X_\infty, p_k)\}\) and \((Z, z)\) be as in the statement of Theorem 1.1. For every \(k\), the map \(\hat{\pi}_\infty : (\mathbb{R}^n, \lambda_k d_\alpha) \to X_\infty\) is a rescaling of a David-Semmes regular map, so it is a David-Semmes regular map with uniform constants. Therefore there is an \(N\) such that for every \(k\), there exists points \(\hat{p}_{k,1}, \ldots, \hat{p}_{k,\ell}\) \((\ell \leq N)\), such that \((\hat{\pi}_\infty)^{-1}(B(p_k, r)) \subset \cup_i B(p_k,i, C_r)\) for all \(r \in (0, \infty)\). After passing to a subsequence we may assume that \(\ell\) is constant, and that for each \(1 \leq i \leq \ell\), the pointed maps \(\hat{\pi}_\infty : (\mathbb{R}^n, \lambda_k d_{X_\infty}^\circ, \hat{p}_{k,i}) \to (\lambda_k X_\infty, p_k)\) Gromov-Hausdorff converge as \(k \to \infty\) to a Lipschitz map \(\phi_i : (W_i, w_i) \to (Z, z)\). Then the \(\phi_i\)'s are David-Semmes regular, and \(\cup_i \text{Im}(\phi_i) = Z\).
As $\phi_i$ is a light map, the topological dimension of $(Z, z)$ is at least $n$. Note that the properties of satisfying a $(1, 1)$-Poincaré inequality, being Ahlfors $Q$-regular, and having Assouad-Nagata dimension $\leq n$, with uniform constants, all pass to Gromov-Hausdorff limits. Therefore (2) and (3) hold.

Part (4) can be verified in several different ways.

One approach is to implement either of the arguments of Section 8 by passing the ingredients — the horizontal galleries of Corollary 7.9, the measured families of curves $\Gamma$ of Theorem 7.17 and the derivation $D_\infty$ — to the limit space $Z$. More specifically, in adapting Remark 8.24, one can associate to $\Gamma$ and $D_\infty$ normal 1-currents in the sense of Lang and use compactness of normal currents [Lan11, Thm. 5.4].

A second approach is to exploit the symmetry of $X_\infty$. Using the self-similarity of $X_\infty$ induced by the affine transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ we may assume, without loss of generality, that the scale factor $\lambda_k$ is 1 for all $k$. Also, note that the action $\mathbb{Z}^2 \acts \mathbb{R}^2$ preserves the equivalence relation $\mathcal{R}_\infty$ on the open set $\{p \in \mathbb{R}^2 \mid x(p) \not\in Z\}$, and induces an action on $\{p \in X_\infty \mid x_\infty(p) \not\in Z\}$ which is a local isometry. In particular, for every point $p \in X_\infty$ with $x_\infty(p) \not\in Z$, if $r = \text{dist}(x_\infty(p), Z)$, then the ball $B(p, r) \subset X_\infty$ is measure-preserving isometric to the ball $B(\hat{\pi}_\infty(\hat{q}), r) \subset X_\infty$ for some $\hat{q} \in (0, 1)^2$. This property passes to the Gromov-Hausdorff limit, allowing one to see that $\mathcal{H}^Q$-a.e. point of $Z$ lies in a ball that is isometric to a ball in $X_\infty$ itself, and in particular the analytic dimension is 1.

A third approach involves rescaling the direct system of cell complexes $\{X_j\}$ and passing to a pointed limit, which is another pointed direct system with similar properties. See Section 11 — especially Subsection 11.4 — for more details.

10. The $n > 2$ Case of Theorem 1.1

In this section we will discuss the higher-dimensional version of the examples treated in Sections 3-5.

Pick $n > 2$. We will imitate the $n = 2$ construction, but where the last $(n - 1)$ coordinates of $\mathbb{R}^n$ will play the rôle of the $y$-coordinate in $\mathbb{R}^2$. Thus the notation $(x, y)$ will henceforth mean that $y \in \mathbb{R}^{n-1}$. The words “horizontal” and “vertical” will have obvious interpretations in this new setup.
Let \( m = 1 + 3(n - 1) \), and pick \( m_v \gg m \). Let \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) be the linear transformation \( \Phi(x, y) = (m^{-1}x, m_v^{-1}y) \). Let \( Y_0 \) be the cell structure on \( \mathbb{R}^n \) coming from the tiling by unit cubes, and for \( j \in \mathbb{Z} \) let \( Y_j \) be the image of \( Y_0 \) under \( \Phi^j \).

Following the \( n = 2 \) construction, we define an equivalence relation \( \mathcal{R} \) on \( \mathbb{R}^n \) that is generated by the identifications of certain pairs of \((n-1)\)-cells of \( Y_1 \) by vertical translation. For each \( k \in \mathbb{Z} \), \( d \in \{2, \ldots, n\} \), we perform gluings within the three vertical hyperplanes

\[
\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = k + m^{-1}(1 + 3(d-2) + (i-1))\}_{i \in \{1,2,3\}},
\]

so as to enable horizontal galleries to “jump” in the \( x_d \)-coordinate direction. To define these gluings, for each \( k, \ell \in \mathbb{Z} \), \( 2 \leq d \leq n \), \( 1 \leq i \leq 3 \), we let

(10.1) \[
a_{k,\ell,d,i} = \left\{(x_1, \ldots, x_n) \mid x_1 = m^{-1}(1 + 3(d-2) + (i-1)), x_d \in [(3\ell + i - 1)m_v^{-1}, (3\ell + i)m_v^{-1}] \right\},
\]

(10.2) \[
a'_{k,\ell,d,i} = \left\{(x_1, \ldots, x_n) \mid x_1 = m^{-1}(1 + 3(d-2) + (i-1)), x_d \in [(3\ell + i)m_v^{-1}, (3\ell + i + 1)m_v^{-1}] \right\},
\]

and we identify \( a_{k,\ell,d,i} \) with \( a'_{k,\ell,d,i} \) by the vertical translation \( p \mapsto p + m_v^{-1}e_d \).

The definition of \( \mathcal{R}_\infty \) and the pseudodistance \( \hat{d}_\infty \) on the quotient \( X_\infty \) remain the same as before. Then \( (X_\infty, \hat{d}_\infty) \) is the metric space of Theorem 1.1 for general \( n \).

The verification of the assertions in Theorem 1.1 for \( (X_\infty, \hat{d}_\infty) \) proceeds along the same steps, with appropriate modifications, a few of which we indicate here:

- A gallery is a chain of cells where two consecutive cells share an \((n-1)\)-face.
- (cf. Lemma [3.3][1]) If \( p_0, p_1 \in X_j \) are distinct points with \( \pi_j(p_0) = \pi_j(p_1) \), then \( p_i = \hat{\pi}_j(\hat{p}_i) \) for a unique point \( \hat{p}_i \in \mathbb{R}^n \), and the pair \( \hat{p}_0, \hat{p}_i \) is contained in \( \ell \cap (\hat{\sigma}_0 \cup \hat{\sigma}_1) \), where \( \ell \subset \mathbb{R}^n \) is vertical line \( \ell \subset \mathbb{R}^n \) parallel to one the \( d^{th} \)-coordinate axis for some \( 2 \leq d \leq n \), and \( \hat{\sigma}_0, \hat{\sigma}_1 \) is a vertical gallery the \((n-1)\)-skeleton of \( Y_{j+1} \) (i.e. a pair of vertical \((n-1)\)-cells that share an \((n-2)\)-cell).
• If $\sigma$ is an $n$-cell of $X_j$ (or $Y_j$), then it has two vertical $(n-1)$-faces $\tau_0, \tau_1$, where $x_j(\tau_1) = x_j(\tau_0) + m^{-j}$; these replace the vertical 1-cells in the $n = 2$ case.

• The cosets of $R_j$ lie in orbits of $m^{-j}Z \times m_v^{-j}Z^{n-1}$, so the maps $\hat{y}_j$ and $y_j$ take values in $\mathbb{R}^{n-1}/m_v^{-j}Z^{n-1}$ instead of $\mathbb{R}/m_v^{-j}Z$.

• For the proof that $(X_\infty, \hat{d}_\infty)$ has Assouad-Nagata dimension $\leq n$, one builds $(n+1)$ “good families” of subsets by working with cells of dimension 0 through $n$.

In addition to the minor points above, there are changes to the proof of Proposition 4.12 that require more care. In Steps 1 and 2 of the proof, rather than using the distance to $0 \in S^1(m_v^{-j})$, one uses the distance to the $(n-2)$-skeleton in the torus $T^{n-1}(m_v^{-j}) = \mathbb{R}^{n-1}/m_v^{-j}Z^{n-1}$. However, the conclusion of Step 2 only says that $\pi_{i-1}(\bar{\sigma}_i)$ lies at controlled distance from the $(n-2)$-skeleton of $T^{n-1}(m_v^{-j})$. To proceed, one has to inductively exclude the possibility that $\pi_{i-1}(\bar{\sigma}_i)$ lies far from the $(n-3)$-skeleton of $T^{n-1}(m_v^{-j})$, etc. This is done using a variation on Steps 1 and 2, where one uses the distance from the $(n-k)$-skeleton of $T^{n-1}(m_v^{-j})$, for $k \geq 3$, and a variation of Lemma 4.9.

11. Generalizations

In this section we consider generalizations of the examples given in previous sections. It is not our intention to be exhaustive — it is clear that one can further generalize the construction in different ways. Our main purpose is to illustrate that the same overall scheme of proof applies to a much broader class of examples, and to clarify the logical structure of the proofs by identifying the essential properties needed.

11.1. Admissible systems. Before giving the precise definition, we begin with some observations about the examples discussed in earlier sections. These are direct limits of direct systems

$$\ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} X_{j+1} \xrightarrow{\pi_{j+1}} \ldots$$

where the $X_j$’s are cell complexes. Crucial to the analysis is the distinction between horizontal and vertical directions, and the fact that the first coordinate function on $\mathbb{R}^n$ descends to a compatible family of functions. We will axiomatize this by requiring cells to have distinguished characteristic maps that induce the horizontal/vertical structure.

Fix integers $n \geq 2$ and $2 \leq m \leq m_v$. Let $\{Y_j\}_{j \in \mathbb{Z}}$ be a family of tilings of $\mathbb{R}^n$, where $Y_j$ is a tiling by translates of the parallelopiped $[0, m^{-j}] \times [0, m_v^{-j}]^{n-1}$, and for all $j \in \mathbb{Z}$, the tiling $Y_{j+1}$ is subdivision
of $Y_j$. Let $G$ be the group of isometries $g : \mathbb{R}^n \to \mathbb{R}^n$ of the form $g = \text{id}_\mathbb{R} \times h$ for some isometry $h : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. Let $G_j \subset G$ be the subgroup that preserves the cell structure of $Y_j$; hence for all $g \in G_j$, the linear part $L(g)$ preserves the set of coordinate vectors $\{\pm e_i\}_{2 \leq i \leq n}$.

**Definition 11.1.** An **admissible direct system** is a tuple consisting of:

- A direct system of cell complexes indexed by the integers
  $$\ldots \xrightarrow{\pi_{-1}} X_0 \xrightarrow{\pi_0} \ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} \ldots$$

- A collection $\{x_j : X_j \to \mathbb{R}\}$ of continuous maps.

- For each cell $\sigma$ of $X_j$, a collection $\Phi_\sigma$ of distinguished characteristic maps $\phi : \hat{\sigma} \to \sigma$, where $\hat{\sigma}$ is a cell of $Y_j$.

- A Radon measure $\mu_j$ on $X_j$ for all $j \in \mathbb{Z}$.

The tuple is required to satisfy the following conditions for some constants $\Delta, H$:

**Ax1**: $X_j$ is connected, is a union of its closed $n$-cells, and all links have cardinality at most $\Delta$.

**Ax2**: Compatibility of distinguished characteristic maps:

**Ax2a**: (Compatibility with $G$) For any cell $\sigma$ of $X_j$, any two elements $\phi_0 : \hat{\sigma}_0 \to \sigma, \phi_1 : \hat{\sigma}_1 \to \sigma$ of $\Phi_\sigma$ agree up to precomposition with some $g \in G_j$ such that $g(\hat{\sigma}_0) = \hat{\sigma}_1$. In particular, the notions of vertical and horizontal cells in $Y_j$ descend to well-defined notions for cells in $X_j$.

**Ax2b**: (Compatibility with face restrictions) The restriction of any distinguished characteristic map $\phi : \hat{\sigma} \to \sigma$ to a face of $\hat{\sigma}$ is a distinguished characteristic map.

**Ax2c**: (Compatibility with subdivision) If $\sigma$ is a cell of $X_j$ and $\phi : \hat{\sigma} \to \sigma$ belongs to $\Phi_\sigma$, then for any cell $\hat{\sigma}'$ of $Y_{j+1}$ contained in $\hat{\sigma}$, the composition $\pi_j \circ \phi \big|_{\hat{\sigma}'} : \hat{\sigma}' \to \sigma' \subset X_{j+1}$ is a distinguished characteristic map of some cell $\sigma'$ of $X_{j+1}$. Moreover all distinguished characteristic maps of $X_{j+1}$ may be obtained in this way.

**Ax2d**: (Compatibility with $x_j$) For every cell $\sigma$ of $X_j$ and every $\phi : \hat{\sigma} \to \sigma$ in $\Phi_\sigma$, $x_j \circ \phi = x$, where $x : \mathbb{R}^n \to \mathbb{R}$ is the first coordinate function.

**Ax3**: $\pi_j : X_j^{(1)} \to X_{j+1}$ is a surjective cellular map, where $X_j^{(1)}$ denotes the subdivision of $X_j$ defined by restricting distinguished characteristic maps $\phi : \hat{\sigma} \to \sigma$ to cells $\hat{\sigma}' \in Y_{j+1}$. 
(Ax4): (Fibers have controlled vertical diameter) For every j, and every 0-cell \( \bar{v} \) of \( X_{j+1} \), any two elements \( v_0, v_1 \) of the inverse image \( \pi_{j}^{-1}(\bar{v}) \) are contained in a vertical edge path of \( X_{j}^{(1)} \) of combinatorial length at most \( H \).

(Ax5): (Gallery accessibility) For every j, and any two adjacent n-cells \( \sigma_0, \sigma_1 \) of \( X_j \), there is a horizontal gallery from \( \sigma_0 \) to \( \sigma_1 \) of combinatorial length at most \( \Delta \).

(Ax6): Compatibility of measures:
  (Ax6a): \((\pi_j)_# \mu_j = \mu_{j+1}\) for all \( j \in \mathbb{Z}\).
  (Ax6b): (Lebesgue measure on cells) For every \( j \), every n-cell \( \sigma \) of \( X_j \), and every \( \phi_\sigma \in \Phi_\sigma \), the restriction \( \mu_j|_\sigma \) agrees with the pushforward of the Lebesgue measure, \((\phi_\sigma)_# \mathcal{L}^n\), up to a constant \( w_\sigma \).
  (Ax6c): (Doubling) For any two adjacent n-cells \( \sigma, \sigma' \) of \( X_j \), the weights \( w_\sigma, w_\sigma' \) agree to within a factor of at most \( \Delta \).

Remark 11.2. As with the examples from Sections [2] and [10] one way of constructing an admissible direct system is by defining an increasing sequence \( \cdots \subset R_{j-1} \subset R_j \subset R_{j+1} \subset \cdots \) of equivalence relations on \( \mathbb{R}^n \), such that \( R_j \) respects the cell structure on \( Y_j \) and the quotients \( X_j = \mathbb{R}^n/R_j \) satisfy Definition 11.1.

For any admissible system we define a pseudodistance \( \hat{d}_\infty \) on the direct limit \( X_\infty \) as follows. We let \( \hat{d}_\infty \) be the largest pseudodistance on \( X_\infty \) such that, for every \( j \in \mathbb{Z} \) and every cell \( \hat{\sigma} \) of \( X_j \), the projection to \( X_\infty \) has \( \hat{d}_\infty \)-diameter at most \( m^{-j} \). In general this need not define a metric (see the next example), so we form a metric space \((\hat{X}_\infty, \hat{d}_\infty)\), the limit space of the admissible system, by collapsing subsets of \( X_\infty \) of zero \( d_\infty \)-diameter to points. We denote by \( \hat{\pi}_\infty^j \) the composition \( X_j \xrightarrow{\pi} X_\infty \to \hat{X}_\infty \). The pushforward of \( \mu_j \) under \( \hat{\pi}_\infty^j \) is independent of \( j \) by Axiom (Ax6), and it thus defines the natural measure \( \mu_\infty \) on \( \hat{X}_\infty \).

Example 11.3. To illustrate how the pseudodistance \( \hat{d}_\infty \) can fail to be a distance, we construct a direct system \( \{X_j\} \) that satisfies all the conditions of Definition 11.1 except (Ax5), such that the pseudodistance \( \hat{d}_\infty \) is not a distance. It is not hard to modify this example to obtain an admissible system with the same property.

Let \( n = 2, m = 3, m_v = 6 \). For every \( k, \ell \in \mathbb{Z}^2 \), we define a pair of 2-cells
\[
a_{k,\ell} = \left[ \frac{1}{3} + k, \frac{2}{3} + k \right] \times \left[ \frac{1}{6} + \ell, \frac{2}{6} + \ell \right],
\]
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\[ a'_{k,\ell} = \left[ \frac{1}{3} + k \cdot \frac{2}{3} + k \right] \times \left[ \frac{4}{6} + \ell \cdot \frac{5}{6} + \ell \right]. \]

Now let \( R \) be the equivalence relation on \( \mathbb{R}^2 \) obtained by identifying \( a_{k,\ell} \) with \( a'_{k,\ell} \) by the vertical translation \( p \mapsto p + \frac{1}{2} e_2 \), for every \( k, \ell \). We now define \( R_j, X_j, \hat{\pi}^j \), etc. as in Section 3. Then \( \{ X_j = \mathbb{R}^2 / R_j \} \) is the desired direct system. Note that there is a Cantor set in the vertical line \( \{ x = \frac{1}{2} \} \) that maps to a subset of \( X_{\infty} \) with zero \( \hat{d}_{\infty} \)-diameter.

On the other hand if \( \hat{\pi} \in R_{\infty}, p = \hat{\pi}(p) \), then the set \( (\hat{\pi})^{-1}(\hat{\pi}(\hat{p})) \) is countable, being a countable union of the sets \( (\hat{\pi}^j)^{-1}(\hat{\pi}^j(\hat{p})) \), each of which is finite.

Most of Theorem 1.1 generalizes verbatim to admissible systems:

**Theorem 11.4.** For every \( n, m, H \), there is an \( m_v = m_v(n, m, H) \) such that if \( m_v \geq m_v \), then:

1. \((X_{\infty}, \hat{d}_{\infty}, \mu_{\infty})\) is a complete doubling metric measure space satisfying a \((1,1)\)-Poincaré inequality.
2. \( X_{\infty} \) has topological and Assouad-Nagata dimension \( n \).
3. \((X_{\infty}, \hat{d}_{\infty}, \mu_{\infty})\) has analytic dimension 1.
4. (1)-(3) also hold for any pointed measured Gromov-Hausdorff limit of any sequence of rescalings of \((X_{\infty}, \hat{d}_{\infty}, \mu_{\infty})\).

Part (1) of Theorem 1.1 does not generalize directly to admissible systems, because, in particular, \((X_{\infty}, \hat{d}_{\infty})\) is not Ahlfors-regular in general. To formulate a modified statement, we introduce another metric on the \( X_j \)'s, which plays the role of the metric \( d_\alpha \) on \( \mathbb{R}^n \).

**Definition 11.5.** For all \( j \in \mathbb{Z} \), let \( d_j \) be the largest pseudodistance on \( X_j \) such that \( d_j \leq \hat{d}_j \), and every cell of \( X_{j+1}^{(k)} \) has \( d_j \)-diameter at most \( m^{-2(j+k)} \).

**Theorem 11.6.** For every \( j \in \mathbb{Z} \):

1. \((X_j, d_j, \mu_j)\) is a doubling metric measure space.
2. \( \pi_j^\infty : (X_j, d_j) \to (X_{\infty}, \hat{d}_{\infty}) \) is a Lipschitz light map (see Definition 11.7); in particular, the point inverses of \( \pi_j^\infty \) are uniformly totally disconnected (i.e. have Assouad-Nagata dimension 0).
3. Let \( Q = 1 + \frac{(n-1) \log m_v}{\log m} \). If for every \( j \), the inverse image under \( \pi_j : X_j \to X_{j+1} \) of every open \( n \)-cell of \( X_{j+1} \) is a single open \( n \)-cell of \( X_j^{(1)} \), then \((X_j, d_j)\) is Ahlfors \( Q \)-regular, and \( \pi_j^\infty : (X_j, d_j) \to (X_{\infty}, \hat{d}_{\infty}) \) is a David-Semmes regular map.
We recall here the notion of Lipschitz light map \cite[Def. 1.14]{CK13}.

**Definition 11.7.** A Lipschitz map $f : X \to Y$ is **Lipschitz light** if there is some $C > 0$ such that for every bounded subset $W \subset Y$ the diam(W)-components of $f^{-1}(W)$ have diameter at most $C \text{diam}(W)$. Note that a Lipschitz light map is also a continuous light map.

11.2. **The proof of Theorem 11.4.** Overall, the proof of parts (1)-(3) of Theorem 11.4 follows closely that of Theorem 1.1. However, some modifications are needed and these are explained in this Subsection. In Subsection 11.4 we give the proof of item (4), i.e. the stability under taking weak tangents. Note that even for the specific examples considered in the previous sections a cleaner treatment is obtained in the more general context of admissible systems.

The proof of Proposition 4.12 remains valid using a map $y_j$ which takes values in the quotient $\tilde{T}_j$ of $\mathbb{R}^{n-1} \simeq \{(0) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n \}$ by the action of $G_j$, which is a quotient of the torus $T^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ by the finite group of orthogonal transformations of $\mathbb{R}^{n-1}$ that preserves the subset $\{\pm e_d\}_{2 \leq d \leq n}$. Note that in the case $n \geq 3$ one must also modify Steps 1 and 2 of that argument by using the distances from the $\{0, 1, \cdots, n-2\}$-skeleta of $\tilde{T}_j$ (compare Section 10).

The proof that the Assouad-Nagata dimension of $\bar{X}_\infty$ is at most $n$ can be carried out as in Theorem 6.2 by working with the cells of $X_j$ of dimensions $1, \cdots, n$. The fact that the topological dimension (and hence the Assouad-Nagata dimension) is at least $n$ can be deduced by adapting either of the arguments in Lemma 5.2.

To prove the Poincaré inequality, one can essentially follow the argument of Theorem 7.35.

We now turn to the proof of assertion (3) in Theorem 11.4, i.e. that $(\bar{X}_\infty, \bar{d}_\infty, \mu_\infty)$ has analytic dimension 1. We first define a direct system of derivations \{\$D_j : \text{Lip}_b(X_j, d_j) \to L^\infty(X_j, \mu_j)\}$ as follows. For every bounded $d_j$-Lipschitz function $u_j : X_j \to \mathbb{R}$ we let $D_ju_j$ be the function in $L^\infty(\mu_j)$ such that, for every $n$-cell $\sigma$ of $X_j$, we have

$$D_j(u_j \circ \phi_\sigma) = \frac{\partial(u_j \circ \phi_\sigma)}{\partial x}$$

for every distinguished characteristic map $\phi_\sigma \in \Phi_\sigma$; this is well-defined because of (Ax2a). Note that the family \{\$D_j\$\} is compatible with the projections \{\$\pi_j^k\$\}, and by pushforward we get a well-defined derivation $D_\infty : \text{Lip}_b(\bar{X}_\infty) \to L^\infty(\bar{X}_\infty, \mu_\infty)$. With this setup, the first part of the proof of Theorem 8.5 (or using Weaver derivations as sketched
in Remark 8.24) carries over. The justification after (8.13) requires modifications for general admissible systems, however, as it is based on the fact that for \( \mu_\infty \)-a.e. point, the point inverse \((\bar{\pi}_j^\infty)^{-1}(p)\) contains only one point. Instead, one may use the fact that for a Lipschitz function \( f : \bar{X}_\infty \to \mathbb{R} \), the function \( D_j(f \circ \bar{\pi}_j^\infty) \) is a.e. constant on the fibers of \( \bar{\pi}_j^\infty \), which is established in Lemmas 11.8, 11.13 and 11.19.

**Lemma 11.8.** Fix \( p_\infty \in \bar{X}_\infty \) and \( k \geq j \); let \( S = \{\sigma_0, \ldots, \sigma_L\} \) be a chain of \( n \)-cells of \( X_j^{(k-j)} \) such that each cell of \( S \) intersects \((\bar{\pi}_j^\infty)^{-1}(p_\infty)\).
Assume that:

\begin{equation}
(11.9) \quad p_j \in \sigma_0 \cap (\bar{\pi}_j^\infty)^{-1}(p_\infty);
\end{equation}

if \( m_v \) is sufficiently large there is a universal constant \( C_0 \) such that:

\begin{equation}
(11.10) \quad d_j(p_j, \sigma_L) \leq 3C_0m^{-k}.
\end{equation}

Moreover, let \( k \geq j \) and denote by \( \sigma_k \) the \( n \)-cell of \( X_k^{(1)} \) containing \( \pi_j^k(p_j) \). Assume that any vertical gallery in \( X_k^{(1)} \) from \( \sigma_k \) to the horizontal \((n-1)\)-skeleton of \( X_k \) has length at least 10H. Then \((\bar{\pi}_j^\infty)^{-1}(p_\infty)\) is entirely contained in the open \( n \)-cell of \( X_k \) that contains \( \pi_j^k(p_j) \).

**Proof.** Let \( S = \{\sigma_0, \ldots, \sigma_L\} \) be a chain of \( n \)-cells of \( X_j^{(k-j)} \) such that each cell of \( S \) intersects \((\bar{\pi}_j^\infty)^{-1}(p_\infty)\). Fix \( p_j \in \sigma_0 \cap (\bar{\pi}_j^\infty)^{-1}(p_\infty) \). By Corollary 4.19 \( \pi_j^k(S) \) lies in the star of a cell of \( X_k \); by Axiom (Ax4) we conclude that \( \pi_j^{k-1}(S) \) lies in a combinatorial ball of radius \( C_0 \) in \( X_{k-1}^{(1)} \) which is centered on an \( n \)-cell of \( X_{k-1}^{(1)} \). Consider now an \( n \)-cell \( \sigma \) of \( X_j^{(k-1-j)} \); by Axiom (Ax2c) the map \( \pi_j^{k-1} : X_j^{(k-j)} \to X_{k-1}^{(1)} \) is a combinatorial isomorphism when restricted to \( \sigma \). Thus, for \( 0 \leq i \leq L \) we have:

\begin{equation}
(11.11) \quad d_2(x_j(\sigma_0), x_j(\sigma_i)) \leq 2m^{-k}
\end{equation}

\begin{equation}
(11.12) \quad d_{\bar{f}_j}(y_j(\sigma_0), y_j(\sigma_i)) \leq 2C_0m_v^{-k};
\end{equation}

therefore, if \( m_v \) is sufficiently large compared to \( C_0 \) and \( m \) we must have:

\begin{equation}
(11.13) \quad d_j(\sigma_0, \sigma_i) \leq 3C_0m^{-k}.
\end{equation}

Let now \( \sigma_k \) denote the \( n \)-cell of \( X_k^{(1)} \) containing \( \pi_j^k(p_j) \) and assume that any vertical gallery in \( X_k^{(1)} \) from \( \sigma_k \) to the horizontal \((n-1)\)-skeleton of \( X_k^{(1)} \) has length at least 10H. Let \( q \in (\bar{\pi}_k^\infty)^{-1}(p_\infty) \) and choose a cell \( \sigma_k' \) of \( X_k^{(1)} \) which contains \( q \) and intersects \((\bar{\pi}_k^\infty)^{-1}(p_\infty)\). As both cells \( \pi_k(\sigma_k) \) and \( \pi_k(\sigma_k') \) meet \((\bar{\pi}_{k+1}^\infty)^{-1}(p_\infty)\), by Corollary 4.19 they
must be adjacent. However, as \( \sigma^j_k \) intersects \( (\pi^\infty_k)^{-1}(p_\infty) \), by Axiom (Ax4) there is a vertical gallery in \( X_k^{(1)} \) of length at most \( H \) from \( \sigma_k \) to a cell \( \tau \) adjacent to \( \sigma^j_k \). Thus \( q \) lies in the open cell of \( X_k \) containing \( \pi^j_k(p_j) \).

**Lemma 11.13.** Let \( \{g_j : X_j \to \mathbb{R}\}_{j \in \mathbb{Z}} \) be a family of bounded measurable functions, and suppose that \( \{g_j\} \) is compatible with projection in the sense that, for all \( j \leq k \), one has \( g_k \circ \pi^j_k = g_j \) \( \mu_j \)-a.e. Moreover, let \( g_\infty \in L^\infty(\bar{X}_\infty, \mu_\infty) \) be the function satisfying:

\[
\bar{\pi}^\infty_j(g_j \mu_j) = g_\infty \mu_\infty.
\]

Then for every \( j \in \mathbb{Z} \), and \( \mu_\infty \)-a.e. \( p \in \bar{X}_\infty \), if \( \tilde{\mu}_j(p) \) is the disintegration of \( \mu_j \) with respect to \( \bar{\pi}^\infty_j \), then \( g_j \) is \( \tilde{\mu}_j(p) \)-a.e. equal to \( g_\infty(p) \) on \( (\bar{\pi}^\infty_j)^{-1}(p) \).

**Proof.** For \( \eta \in [1, \lfloor \frac{m}{3H} \rfloor] \) and \( k \geq j \) let \( S_{\eta,k} \) be the set of points \( q \in X_j \) such that:

(a) \( q \) does not belong to the \((n-1)\)-skeleton of \( X_j^{(k-j)} \) for any \( k \geq j \).

(b) If \( \sigma \) is the \( n \)-cell of \( X_k^{(1)} \) containing \( \pi^j_k(q) \), then any vertical gallery in \( X_k^{(1)} \) from \( \sigma \) to the horizontal \((n-1)\)-skeleton of \( X_k \) has length at least \( \eta H \).

If \( q \in S_{\eta,k} \) and \( \eta \geq 10 \), by Lemma 11.8 the fibre \((\bar{\pi}^\infty_k)^{-1}(\bar{\pi}^\infty_j(q))\) is entirely contained in the open \( n \)-cell of \( X_k \) which contains \( \pi^j_k(q) \).

We now look at the set of points \( S \) in \( X_j \) such that (b) occurs infinitely often:

\[
S_\eta = \bigcap_{k \geq 1} \bigcup_{k' \geq k} S_{\eta,k'};
\]

then \( \mu_j(S_\eta^c) = 0 \). Consider now the subset \( \Omega \subset \bar{X}_\infty \):

\[
\Omega = \{ p \in \bar{X}_\infty : (\bar{\pi}^\infty_j)^{-1}(p) \cap S_\eta^c \neq \emptyset \};
\]

then by Lemma 11.8 \((\bar{\pi}^\infty_j)^{-1}(\Omega) \subset S_\eta^c \) and so \( \mu_\infty(\Omega) = 0 \). Thus for \( \mu_\infty \)-a.e. \( p \in \bar{X}_\infty \) we can assume that \((\bar{\pi}^\infty_j)^{-1}(p) \subset S_{10} \).

As the Lebesgue Differentiation Theorem holds in \( X_j \), for \( \mu_\infty \)-a.e. \( p \in \bar{X}_\infty \) there is a subset \( T_p \subset (\bar{\pi}^\infty_j)^{-1}(p) \) of full \( \tilde{\mu}_j(p) \) measure such that \( T_p \subset S_{10} \) and every \( q \in T_p \) is an approximate continuity point of \( g_j \). Let \( q, q' \in T_p \), and \( \varepsilon > 0 \). For any \( k_0 \geq j \) we can find \( k \geq k_0 \) such that \( q \in S_{10,k} \); let \( \sigma, \sigma' \) be the \( n \)-cells of \( X_j^{(k-j)} \) which contain \( q, q' \).
respectively. Provided \( k_0 \) is sufficiently large, we will have

\[
(11.17) \quad \max \left( \left| \int_\sigma g_j \, d\mu_j - g_j(q) \right| , \left| \int_{\sigma'} g_j \, d\mu_j - g_j(q) \right| \right) < \varepsilon.
\]

As \( q \in S_{10,k} \), we have \( \pi_j^k(\dot{\sigma}) = \pi_j^k(\dot{\sigma}') \), and then

\[
(11.18) \quad \int_\sigma g_j \, d\mu_j = \int_{\pi_j^k(\dot{\sigma})} g_k \, d\mu_k = \int_{\pi_j^k(\dot{\sigma}')} g_k \, d\mu_k = \int_{\sigma'} g_j \, d\mu_j.
\]

Therefore \( |g_j(q) - g_j(q')| < 2\varepsilon \), and as \( \varepsilon \) was arbitrary, we have \( g_j(q) = g_j(q') \). Now the lemma follows because (11.14) and the Disintegration Theorem imply that for \( \mu_\infty \text{-a.e.} \ p \in X_\infty \), \( g_\infty(p) \) is the \( \bar{\mu}_j(p) \) average of \( g_j \).

**Lemma 11.19.** For \( j \in \mathbb{Z} \) and \( p \in X_\infty \), we let \( \bar{\mu}_j(p) \) denote the disintegration of \( \mu_j \) with respect to \( \pi_j^\infty \). Let \( u : \tilde{X}_\infty \rightarrow \mathbb{R} \) be Lipschitz, and \( u_j = u \circ \pi_j^\infty \). Let \( D_\infty u \in L^\infty(X_\infty, \mu_\infty) \) be the function satisfying:

\[
(11.20) \quad \bar{\pi}_j^\infty(D_j u_j \cdot \mu_j) = D_\infty u \cdot \mu_\infty.
\]

Then for \( \mu_\infty \text{-a.e.} \ p \in X_\infty \), one has:

\[
(11.21) \quad D_j u_j = D_\infty u(p) \quad (\bar{\mu}_j(p) \text{-a.e.})
\]

**Proof.** We apply Lemma 11.13 thus it suffices to show that for \( k \geq j \) one has:

\[
(11.22) \quad D_k u_k \circ \pi_j^k = D_j u_j \quad (\mu_j \text{-a.e.})
\]

Let \( S \) be the set of points \( q \in X_j \) which do not belong to the \((n-1)\)-skeleton of \( X_j^{(m-j)} \) for any \( m \geq j \). Note that

\[
(11.23) \quad \mu_j(S) = \mu_k(\pi_j^k(S)) = 0.
\]

Now, for \( \mu_j \text{-a.e.} \ q \), \( D_j u_j(q) \) equals \( (u_j \circ \gamma_q)'(0) \) where \( \gamma_q \) is a unit-speed horizontal segment with \( \gamma_q(0) = q \) and along which \( x_j \circ \gamma_q \) is non-decreasing. As the fibres of \( \pi_j^k \) are finite, we also have that for \( \mu_j \text{-a.e.} \ q \), \( D_k u_k(\pi_j^k(q)) \) equals \( (u_k \circ \gamma_{\pi_j^k(q)})'(0) \) where \( \gamma_{\pi_j^k(q)} \) is a unit-speed horizontal segment with \( \gamma_{\pi_j^k(q)}(0) = \pi_j^k(q) \), and along which \( x_k \circ \gamma_{\pi_j^k(q)} \) is non-decreasing. If \( q \in S \) for some \( \delta > 0 \) one has:

\[
(11.24) \quad \pi_j^k(\gamma_q(t)) = \gamma_{\pi_j^k(q)}(t) \quad (|t| \leq \delta);
\]

as \( u_j = u_k \circ \pi_j^k \) we conclude that (11.22) holds.
Proof of Theorem 11.6

We first prove (1). Note that Axioms (Ax6), (Ax1) imply the existence of a \( C = C(\Delta, n) \) (independent of \( j \)) such that:

\[
(11.25) \quad \mu_j(B_{X_j}(p_j, r)) \geq C^{-1}\mu_j(B_{X_j}(p_j, 2r)) \quad (\forall r \leq 4m^{-j+1}, p_j \in X_j).
\]

Assume that \( r > m^{-j} \), and let \( k \) be such that \( m^{-k} < r \leq m^{-k+1} \); and choose a point \( p_{k-1} \in (\pi_{k-1}^j)^{-1}(p_j) \). As \( \pi_{k-1}^j \) is 1-Lipschitz:

\[
(11.26) \quad B_{X_{k-1}}(p_{k-1}, r) \subset (\pi_{k-1}^j)^{-1}(B_{X_j}(p_j, r));
\]

on the other hand, Proposition 4.12 implies that:

\[
(11.27) \quad (\pi_{k-1}^j)^{-1}(B_{X_j}(p_j, 2r)) \subset (\pi_{k-1}^j)^{-1}(B_{X_{k-1}}(p_{k-1}, 4m^{-k+1}));
\]

thus, combining (11.26), (11.27) one gets:

\[
\mu_j(B_j(p_j, r)) \geq \mu_{k-1}(B_{X_{k-1}}(p_{k-1}, r)) \geq C^{-2\log m}\mu_{k-1}(B_{X_{k-1}}(p_{k-1}, 4m^{-k+1})) \geq C^{-2\log m}\mu_j(B_j(p_j, 2r)).
\]

We now prove (2), i.e. that \( \pi_j^\infty \) is a Lipschitz-light map. We will show that there is a universal constant \( C \) such that for each \( (k, j, p_\infty) \in \mathbb{Z}^2 \times \bar{X}_\infty \), \( (\pi_j^\infty)^{-1}(p_\infty) \) can be covered by a family of sets \( \{\Omega_\alpha\}_\alpha \) such that:

\[
(11.29) \quad \text{diam } \Omega_\alpha \leq Cm^{-k},
\]

and

\[
(11.30) \quad d(\Omega_\alpha, \Omega_\beta) \geq m^{-k} \quad (\alpha \neq \beta).
\]

Note that by Corollary 4.19 any two points of \( (\pi_j^\infty)^{-1}(p_\infty) \) must belong to adjacent cells of \( X_j \), so the case of interest is \( k > j \).

For each \( n \)-cell \( \sigma \) of \( X_j^{(k-j)} \) intersecting \( (\pi_j^\infty)^{-1}(p_\infty) \) let \( \Omega_\sigma \) denote the set of points of \( (\pi_j^\infty)^{-1}(p_\infty) \) that can be connected to \( \sigma \) using a chain \( S = \{\sigma_0, \cdots, \sigma_L\} \) of cells of \( X_j^{(k-j)} \), such that each \( \sigma_i \) intersects \( (\pi_j^\infty)^{-1}(p_\infty) \). From (11.10) in Lemma 11.8 we conclude that:

\[
(11.31) \quad \text{diam } \Omega_\sigma \leq 6Cn^{-k}.
\]

On the other hand, if \( \Omega_\sigma \neq \Omega_{\sigma'} \), then \( \Omega_\sigma \) and \( \Omega_{\sigma'} \) do not intersect adjacent cells of \( X_j^{(k-j)} \) and hence:

\[
(11.32) \quad d_j(\Omega_\sigma, \Omega_{\sigma'}) \geq m^{-k}.
\]
We now turn to the proof of (3). We first observe that if \( \sigma \) is an \( n \)-cell of \( X_j \), one has:
\[
\mu_j(\sigma) = w_\sigma m^{-jQ},
\]
where \( w_\sigma \) is the weight in Axiom (Ax6b). Let \( \sigma_0, \sigma_1 \) be \( n \)-cells of \( X_j \). If
\[
d_j(\sigma_0, \sigma_1) \leq m^{-j},
\]
the cells \( \sigma_0, \sigma_1 \) are adjacent and hence
\[
w_{\sigma_0} \leq \Delta
\]
by Axiom (Ax6c). Assume now that \( d_j(\sigma_0, \sigma_1) > m^{-j} \). As \( X_j \) is connected ((Ax1)) there is an \( N \geq 2 \) such that:
\[
(N - 1)m^{-j} < d_j(\sigma_0, \sigma_1) \leq Nm^{-j};
\]
let \( k < 0 \) be such that \( m^{-k-1} < N \leq m^{-k} \). Using Proposition 4.12 we see that:
\[
d_k((\pi_j^{(-k)})^{-1}(\sigma_0), (\pi_j^{(-k)})^{-1}(\sigma_1)) \leq 3m^{-j-k}.
\]
Now by Axiom (Ax3)
\[
\pi_j^{(-k)} : X_j^{(-k)} \to X_j
\]
is a surjective cellular map; moreover, we are assuming that \( \pi_j^{(-k)} \) is injective on the complement of the \( (n-1) \)-skeleton of \( X_j^{(-k)} \). Thus, if \( \sigma_i \) denotes the interior of \( \sigma_i \), there is a unique \( n \)-cell \( \bar{\sigma}_i \) of \( X_j^{(-k)} \) such that:
\[
(\pi_j^{(-k)})^{-1}(\sigma_i) \cap \bar{\sigma}_i \neq \emptyset.
\]
By (11.37) \( \bar{\sigma}_0, \bar{\sigma}_1 \) are at combinatorial distance at most 3 and hence:
\[
w_{\bar{\sigma}_0} \leq \Delta^3;
\]
but \( w_{\sigma_i} = w_{\bar{\sigma}_i} \), and so the weights \( w_{\sigma_0} \) and \( w_{\sigma_1} \) are comparable up to a uniformly bounded multiplicative factor. We therefore find a universal constant \( C \) such that, for each \( j \in \mathbb{Z} \), each \( p_j \in X_j \) and each \( r \leq 3m^{-j+1} \) one has:
\[
\mu_j(B_{X_j}(p_j, r)) \approx C r^{-jQ}.
\]
Suppose now that \( m^{-k} < r \leq m^{-k+1} \) and let \( p_k \in (\pi_j^{(-k)})^{-1}(p_j) \); then, arguing as in the proof of (1), we obtain:
\[
B_{X_{k-1}}(p_{k-1}, m^{-k}) \subset (\pi_{k-1}^{(-k)})^{-1}(B_{X_j}(p_j, r)) \subset B_{X_{k-1}}(p_{k-1}, 3m^{-k+1});
\]
therefore, by enlarging $C$, we have that (11.41) holds also for $r > 3m^{-j+1}$. This proves that each $(X_j, d_j, \mu_j)$ is Ahlfors-regular, where the constant in the Ahlfors-regularity condition is independent of $j$.

We now show that $\bar{\pi}_j^\infty$ is David-Semmes regular (where the constants again do not depend on $j$). Let $k < 0$; as the map in (11.38) is injective on the complement of the $(n-1)$-skeleton of $X_j^{(k)}$, and as by (Ax1) there is a uniform bound $\Delta$ on the cardinality of each link of $X_j$, we conclude that for each $p_j \in X_j$ one has that $(\pi_j^{j+k})^{-1}(p_j)$ has cardinality at most $C$. As $k$ and $j$ are arbitrary, we conclude that for $p_\infty \in X_\infty$ also $(\pi_j^\infty)^{-1}(p_\infty)$ has cardinality at most $C$. Fix now $\bar{p}_\infty \in \bar{X}_\infty$ and let $\{\Omega_\alpha\}$ be a family of subsets of $X_j$ which cover $(\pi_j^\infty)^{-1}(\bar{p}_\infty)$, which and were obtained in the proof of (2). In constructing the $\{\Omega_\alpha\}_\alpha$ there was the freedom to choose a scale $m^{-k}$, which in this case we take to be $m^{-j}$, so that (11.29), (11.30) hold with $k = j$. For each $\Omega_\alpha$ there is a $p_\infty, \sigma \in [\bar{p}_\infty]$ such that:

(11.43) $(\pi_j^\infty)^{-1}(p_\infty, \sigma) \cap \Omega_\alpha \neq \emptyset$.

Fix now one $p_\infty \in [\bar{p}_\infty]$. Then by Corollary 4.19 the sets $(\pi_j^\infty)^{-1}(p_\infty)$ and $(\pi_j^\infty)^{-1}(p_\infty, \sigma)$ must intersect adjacent cells of $X_j$ and so $(\pi_j^\infty)^{-1}(p_\infty) \cap \Omega_\alpha \neq \emptyset$. As distinct $\Omega_\alpha$’s are disjoint, we conclude that the cardinality of the set $\{\Omega_\alpha\}$ is at most $C$. Let now \(\bar{q}_\infty \in B_{\bar{X}_\infty}(\bar{p}_\infty, m^{-j})\); then $(\pi_j^\infty)^{-1}(\bar{q}_\infty)$ is contained in a $3m^{-j}$-neighbourhood of $\bigcup_\alpha \Omega_\alpha$; as each set $\Omega_\alpha$ intersects $(\pi_j^\infty)^{-1}(p_\infty)$ we conclude that:

(11.44) $(\pi_j^\infty)^{-1}(\bar{q}_\infty) \subset \bigcup_{p \in (\pi_j^\infty)^{-1}(p_\infty)} B(p, 6(C_0 + 1)m^{-j})$;  

thus the David-Semmes regularity condition holds with constants $3(C_0 + 1)$ and $C$. \qed

11.4. Preservation of admissibility under limits.

We now turn to the proof of (4) in Theorem 11.4, i.e. the stability of assertions (1)–(3) under the operation of taking weak tangents. This is an immediate consequence of the following lemma and (1)-(3) of Theorem 11.4.

**Lemma 11.45.** Any weak tangent of an admissible system is, modulo rescaling, measure-preserving isometric to the limit space of some admissible system.
Proof. Let \((Y, \nu, p)\) be a weak tangent of \((X_\infty, \mu_\infty)\). Thus there exists a sequence of basepoints \(\{p_\alpha\}\) in \(X_\infty\), as well as sequences \(\{\lambda_\alpha\}, \{\lambda'_\alpha\}\) of scale factors, such that the sequence \(\{(\lambda_\alpha X_\infty, \lambda'_\alpha \mu_\infty, p_\alpha)\}\) converges in the pointed measured Gromov-Hausdorff topology to the pointed (doubling) metric measure space \((Y, \nu, p)\).

For each \(\alpha\), we have \(\lambda_\alpha = a_\alpha m^{j_\alpha}\) for unique elements \(j_\alpha \in \mathbb{Z}\) and \(a_\alpha \in [1, m)\). After passing to a subsequence, we may assume that \(a_\alpha\) converges to some \(a_\infty \in [1, m]\). Without loss of generality, we may replace \(\lambda_\alpha\) with \(m^{j_\alpha}\), since the resulting sequence will converge to the same limit, modulo rescaling by \(a_\infty^{-1}\).

Now observe that for every \(\alpha\), the rescaled metric measure space \((\lambda_\alpha X_\infty, \lambda'_\alpha \mu_\infty)\) is measure-preserving isometric to the limit space of an admissible system obtained from \(\{(X_j, \mu_j)\}\) by shifting the indices by \(j_\alpha\), and rescaling the measures; moreover this new admissible system satisfies Definition 11.1 where the constants \(m, m_v, H, \Delta\) are independent of \(\alpha\). Thus Lemma 11.45 is reduced to the following lemma. \(\square\)

**Lemma 11.46.** Suppose \(\{(X_{j,\alpha}, \mu_{j,\alpha})\}_{\alpha \in \mathbb{N}}\) is a sequence of admissible systems with uniform constants, and \(q_{j,\alpha} \in X_{j,\alpha}\) are projection compatible basepoints chosen such that

\[
0 < \liminf_{\alpha} \mu_{0,\alpha}(B(q_{0,\alpha}, 1)) \leq \limsup_{\alpha} \mu_{0,\alpha}(B(q_{0,\alpha}, 1)) < \infty.
\]

Then after passing to a subsequence, the sequences of pointed admissible systems converge in a natural sense to a pointed admissible system \((X_{j,\infty}, \mu_{j,\infty}, q_{j,\infty})\) whose limit space \((X_\infty, \mu_\infty, q_\infty)\) is measure-preserving isometric to the pointed measured Gromov-Hausdorff limit of the sequence of pointed limit spaces \(\{(X_{\infty,\alpha}, \mu_{\infty,\alpha}, q_{\infty,\alpha})\}\).

**Proof.** This is a consequence of standard finiteness/compactness arguments applied to the controlled-geometry complexes of the admissible systems, so we will be brief.

Pick \(j \in \mathbb{Z}\). Since the parameter \(\Delta\) is independent of \(\alpha\), for any \(N\), there are only finitely many possibilities for the combinatorial \(N\)-ball centered at \(q_{j,\alpha}\), up to a homeomorphism preserving the collection of distinguished characteristic maps. Therefore, after passing to a subsequence, there is a pointed cell-complex \((X_{j,\infty}, q_{j,\infty})\) with a collection of distinguished characteristic maps, such that for all \(N\) and large \(\alpha\), there is a homeomorphism \(\Psi_{j,\alpha, N}\) from the combinatorial \(N\)-ball in \(X_{j,\alpha}\) centered at \(q_{j,\alpha}\) to the combinatorial \(N\)-ball in \(X_{j,\infty}\) centered at \(q_{j,\infty}\), such that \(\Psi_{j,\alpha, N}\) respects distinguished characteristic maps,
\( \Psi_{j,\alpha,N}(q_{j,\alpha}) \to q_{j,\infty} \) as \( \alpha \to \infty \), and for all \( N' \geq N \) the maps \( \Psi_{j,\alpha,N} \), \( \Psi_{j,\alpha,N'} \) are compatible on the \( N \)-balls for large \( \alpha \).

Using (11.47), we get that the \( \mu_{j,\alpha} \)-measure of any \( n \)-cell of \( X_{j,\alpha} \) containing \( q_{j,\alpha} \) is controlled. Therefore, passing to a subsequence again, there is a family of measures \( \{\mu_{j,\infty}\} \) and a compatible system of projection maps \( \{\pi_{j,\infty}^k : X_{j,\infty} \to X_{k,\infty}\} \) which define an admissible system, such that the maps \( \{\Psi_{j,\alpha,N}\} \) are asymptotically measure-preserving and compatible with projection.

It follows from Proposition 4.12 that for all \( R \), the \( \delta_{j,\alpha} \)-ball \( B(q_{j,\alpha},R) \subset X_{j,\alpha} \) is a Gromov-Hausdorff approximation to within error \( \lesssim m^{-j} \) of the \( \delta_{\infty,\alpha} \)-ball \( B(q_{\infty,\alpha},R) \subset X_{\infty,\alpha} \), so the maps \( \{\Psi_{j,\alpha,N}\} \) induce the pointed measured Gromov-Hausdorff convergence

\[
(X_{\infty,\alpha}, \delta_{\infty,\alpha}, \mu_{\infty,\alpha}, q_{\infty,\alpha}) \longrightarrow (X_{\infty,\infty}, \delta_{\infty,\infty}, \mu_{\infty,\infty}, q_{\infty,\infty})
\].

\[\square\]

**References**


