SINGULAR RICCI FLOWS I

BRUCE KLEINER AND JOHN LOTT

Abstract. We introduce singular Ricci flows, which are Ricci flow spacetimes subject to certain asymptotic conditions. We consider the behavior of Ricci flow with surgery starting from a fixed initial compact Riemannian 3-manifold, as the surgery parameter varies. We prove that the flow with surgery subconverges to a singular Ricci flow as the surgery parameter tends to zero. We establish a number of geometric and analytical properties of singular Ricci flows.

Contents

1. Introduction

It has been a long-standing problem in geometric analysis to find a good notion of a Ricci flow with singularities. The motivation comes from the fact that a Ricci flow with a smooth initial condition can develop singularities without blowing up everywhere; hence one would like to continue the flow beyond the singular time. The presence of

Date: August 10, 2014.

Research supported by NSF grants DMS-1105656, DMS-1207654 and DMS-1405899, and a Simons Fellowship.

singularities in Ricci flow is one instance of the widespread phenomenon of singularities in PDE, which is often handled by developing appropriate notions of generalized solutions. For example, in the case of the minimal surface equation, one has minimizing or stationary integral currents [20, 44], while in the case of mean curvature flow one has Brakke flows [4] and level set flows [7, 18]. Generalized solutions provide a framework for studying issues such as existence, uniqueness and partial regularity.

One highly successful way to deal with Ricci flow singularities was by surgery, as developed by Hamilton [26] and Perelman [41]. Surgery served as a way to suppress singularities and avoid related analytical issues, thereby making it possible to apply Ricci flow to geometric and topological problems. Perelman's construction of Ricci flow with surgery drew attention to the problem of flowing through singularities. In [40, Sec. 13.2], Perelman wrote, "It is likely that by passing to the limit in this construction one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that."

In this paper we address the problem of flowing through singularities in the three-dimensional case by introducing singular Ricci flows, which are Ricci flow spacetimes subject to certain conditions. We show that singular Ricci flows have a number of good properties, by establishing existence and compactness results, as well as a number of structural results. We also use singular Ricci flows to partially answer Perelman's question, by showing that Ricci flow with surgery (for a fixed initial condition) subconverges to a singular Ricci flow as the surgery parameter goes to zero. These results and further results in the sequel strongly indicate that singular Ricci flows provide a natural analytical framework for three-dimensional Ricci flows with singularities.

Although it is rather different in spirit, we mention that there is earlier work in the literature on Ricci flow through singularities; see the end of the introduction for further discussion.

Convergence of Ricci flows with surgery. Before formulating our first result, we briefly recall Perelman's version of Ricci flow with surgery (which followed earlier work of Hamilton); see Appendix A.9 for more information.

Ricci flow with surgery evolves a Riemannian 3-manifold by alternating between two processes: flowing by ordinary Ricci flow until the metric goes singular, and modifying the resulting limit by surgery, so as to produce a compact smooth Riemannian manifold that serves as a new initial condition for Ricci flow. The construction is regulated by a global parameter $\epsilon > 0$ as well as decreasing parameter functions $r, \delta, \kappa : [0, \infty) \to (0, \infty)$, which have the following significance:

- The scale at which surgery occurs is bounded above in terms of δ. In particular, surgery at time t is performed by cutting along necks whose scale tends to zero as $\delta(t)$ goes to zero. Note that δ has other roles in addition to this.
- The function r defines the canonical neighborhood scale: at time t, near any point with scalar curvature at least $r(t)^{-2}$, the flow is (modulo parabolic rescaling) approximated to within error ϵ by either a κ-solution (see Appendix A.5) or a standard postsurgery model solution.

In Ricci flow with surgery, the initial conditions are assumed to be *normalized*, meaning that at each point m in the initial time slice, the eigenvalues of the curvature operator $\mathrm{Rm}(m)$ are bounded by one in absolute value, and the volume of the unit ball $B(m, 1)$ is at least half the volume of the Euclidean unit ball. By rescaling, any compact Riemannian manifold can be normalized.

Perelman showed that under certain constraints on the parameters, one can implement Ricci flow with surgery for any normalized initial condition. His constraints allow one to make δ as small as one wants. Hence one can consider the behavior of Ricci flow with surgery, for a fixed initial condition, as δ goes to zero.

In order to formulate our convergence theorem, we will use a spacetime framework. Unlike the case of general relativity, where one has a Lorentzian manifold, in our setting there is a natural foliation of spacetime by time slices, which carry Riemannian metrics. This is formalized in the following definition.

Definition 1.1. A *Ricci flow spacetime* is a tuple $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ where:

- $\mathcal{M}$ is a smooth manifold-with-boundary.
- $\mathfrak{t}$ is the *time function* – a submersion $\mathfrak{t} : \mathcal{M} \to I$ where $I \subset \mathbb{R}$ is a time interval; we will usually take $I = [0, \infty)$.
- The boundary of $\mathcal{M}$, if it is nonempty, corresponds to the endpoint(s) of the time interval: $\partial\mathcal{M} = \mathfrak{t}^{-1}(\partial I)$.
- $\partial_{\mathfrak{t}}$ is the *time vector field*, which satisfies $\partial_{\mathfrak{t}}\mathfrak{t} \equiv 1$.
- g is a smooth inner product on the spatial subbundle $\ker(d\mathfrak{t}) \subset T\mathcal{M}$, and g defines a Ricci flow: $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}(g)$.

For $0 \leq a < b$, we write $\mathcal{M}_a = \mathfrak{t}^{-1}(a)$, $\mathcal{M}_{[a,b]} = \mathfrak{t}^{-1}([a, b])$ and $\mathcal{M}_{\leq a} = \mathfrak{t}^{-1}([0, a])$. Henceforth, unless otherwise specified, when we refer to geometric quantities such as curvature, we will implicitly be referring to the metric on the time slices.

Note that near any point $m \in \mathcal{M}$, a Ricci flow spacetime $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ reduces to a Ricci flow in the usual sense, because the time function $\mathfrak{t}$ will form part of a chart $(x, \mathfrak{t})$ near m for which the coordinate vector field $\frac{\partial}{\partial t}$ coincides with $\partial_{\mathfrak{t}}$; then one has $\frac{\partial g}{\partial t} = -2 \operatorname{Ric}(g)$.

Our first result partially answers the question of Perelman alluded to above, by formalizing the notion of convergence and obtaining subsequential limits:

Theorem 1.2. *Let $\{\mathcal{M}^j\}_{j=1}^{\infty}$ be a sequence of three-dimensional Ricci flows with surgery (in the sense of Perelman) where:*

- *The initial conditions $\{\mathcal{M}^j_0\}$ are compact normalized Riemannian manifolds that lie in a compact family in the smooth topology, and*
- *If $\delta_j : [0, \infty) \to (0, \infty)$ denotes the Perelman surgery parameter for $\mathcal{M}^j$ then $\lim_{j \to \infty} \delta_j(0) = 0$.*

Then after passing to a subsequence, there is a Ricci flow spacetime $(\mathcal{M}^{\infty}, \mathfrak{t}_{\infty}, \partial_{t_{\infty}}, g_{\infty})$, and a sequence of diffeomorphisms

$$\{\mathcal{M}^j \supset U_j \stackrel{\Phi_j}{\to} V_j \subset \mathcal{M}^{\infty}\} \tag{1.3}$$

with the following properties:

1. *$U_j \subset \mathcal{M}^j$ and $V_j \subset \mathcal{M}^{\infty}$ are open subsets.*
2. *Let R_j and R_{∞} denote the scalar curvature on $\mathcal{M}^j$ and $\mathcal{M}^{\infty}$, respectively. Given $\overline{t} < \infty$ and $\overline{R} < \infty$, if j is sufficiently large then*

$$\begin{aligned} U_j &\supset \{m_j \in \mathcal{M}^j \ : \ \mathfrak{t}_j(m_j) \le \bar{t}, \ R_j(m_j) \le \overline{R}\}, \\ V_j &\supset \{m_{\infty} \in \mathcal{M}^{\infty} \ : \ \mathfrak{t}_{\infty}(m_{\infty}) \le \bar{t}, \ R_{\infty}(m_{\infty}) \le \overline{R}\}. \end{aligned} \tag{1.4}$$

3. *Φ^j is time preserving, and the sequences $\{\Phi^j_* \partial_{t_j}\}_{j=1}^{\infty}$ and $\{\Phi^j_* g_j\}_{j=1}^{\infty}$ converge smoothly on compact subsets of $\mathcal{M}^{\infty}$ to $\partial_{t_{\infty}}$ and g_{∞}, respectively.*
4. *Φ^j is asymptotically volume preserving: Let $\mathcal{V}_j, \mathcal{V}_{\infty} : [0, \infty) \to [0, \infty)$ denote the respective volume functions $\mathcal{V}_j(t) = Vol(\mathcal{M}^j_t)$ and $\mathcal{V}_{\infty}(t) = Vol(\mathcal{M}^{\infty}_t)$. Then $\mathcal{V}_{\infty} : [0, \infty) \to [0, \infty)$ is continuous and $\lim_{j \to \infty} \mathcal{V}_j = \mathcal{V}_{\infty}$, with uniform convergence on compact subsets of $[0, \infty)$.*

Furthermore:

(a) *The scalar curvature function $R_{\infty} : \mathcal{M}^{\infty}_{\le T} \to \mathbb{R}$ is bounded below and proper for all $T \ge 0$.*
(b) *$\mathcal{M}^{\infty}$ satisfies the Hamilton-Ivey pinching condition of (A.14).*

(c) $\mathcal{M}^\infty$ is κ-noncollapsed below scale ϵ and satisfies the r-canonical neighborhood assumption, where κ and r are the aforementioned parameters from Ricci flow with surgery.

Theorem 1.2 may be compared with other convergence results such as Hamilton's compactness theorem [25] and its variants [32, Appendix E], as well as analogous results for sequences of Riemannian manifolds. All of these results require uniform bounds on curvature in regions of a given size around a basepoint, which we do not have. Instead, our approach is based on the fact that in a three dimensional Ricci flow with surgery, the scalar curvature controls the local geometry. We first prove a general pointed compactness result for sequences of (possibly incomplete) Riemannian manifolds whose local geometry is governed by a control function. We then apply this general compactness result in the case when the Riemannian manifolds are the spacetimes of Ricci flows with surgery, and the control functions are constructed from the scalar curvature functions. To obtain (1)-(3) of Theorem 1.2 we have to rule out the possibility that part of the spacetime with controlled time and scalar curvature escapes to infinity, i.e. is not seen in the pointed limit. This is done by means of a new estimate on the spacetime geometry of a Ricci flow with surgery; see Proposition 3.5 below.

Motivated by the conclusion of Theorem 1.2, we make the following definition:

Definition 1.5. A Ricci flow spacetime $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is a *singular Ricci flow* if it is 4-dimensional, the initial time slice $\mathcal{M}_0$ is a compact normalized Riemannian manifold and

(a) The scalar curvature function $R : \mathcal{M}^\infty_{\leq T} \to \mathbb{R}$ is bounded below and proper for all $T \geq 0$.
(b) $\mathcal{M}$ satisfies the Hamilton-Ivey pinching condition of (A.14).
(c) For a global parameter $\epsilon > 0$ and decreasing functions $\kappa, r : [0, \infty) \to (0, \infty)$, the spacetime $\mathcal{M}$ is κ-noncollapsed below scale ϵ in the sense of Appendix A.4 and satisfies the r-canonical neighborhood assumption in the sense of Appendix A.8.

Although conditions (b) and (c) in Definition 1.5 are pointwise conditions imposed everywhere, we will show elsewhere that one obtains an equivalent definition if (b) and (c) are only assumed to hold outside of some compact subset of $\mathcal{M}_{\leq T}$, for all $T \geq 0$. Thus (b) and (c) can be viewed as asymptotic conditions at infinity for a Ricci flow defined on a noncompact spacetime.

With this definition, Perelman's existence theorem for Ricci flow with surgery and Theorem 1.2 immediately imply :

Corollary 1.6. *If (M, g_0) is a compact normalized Riemannian 3-manifold then there exists a singular Ricci flow having initial condition (M, g_0), with parameter functions κ and r as in Theorem 1.2.*

From the PDE viewpoint, flow with surgery is a regularization of Ricci flow, while singular Ricci flows may be considered to be generalized solutions to Ricci flow. In this language, Corollary 1.6 gives the existence of generalized solutions by means of a regularization procedure. One can compare this with the existence proof for Brakke flows in [4, 30] or level set flows in [7, 18].

The existence assertion in Corollary 1.6 leads to the corresponding uniqueness question:

Question 1.7. If two singular Ricci flows have isometric initial conditions, are the underlying Ricci flow spacetimes the same up to diffeomorphism?

An affirmative answer would confirm Perelman's expectation that Ricci flow with surgery should converge to a canonical flow through singularities, as it would imply that if one takes a fixed initial condition in Theorem 1.2) then one would have convergence without having to pass to a subsequence. Having such a uniqueness result, in conjunction with Theorem 1.2, would closely parallel the results of [5, 28, 29, 35] that 2-convex mean curvature flow with surgery converges to level set flow when the surgery parameters tend to zero.

The structure of singular Ricci flows. The asymptotic conditions in the definition of a singular Ricci flow have a number of implications which we analyze in this paper. In addition to clarifying the structure of limits of Ricci flows with surgery as in Theorem 1.2, the results indicate that singular Ricci flows are well behaved objects from geometric and analytical points of view.

To analyze the geometry of Ricci flow spacetimes, we use two different Riemannian metrics.

Definition 1.8. Let $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ be a Ricci flow spacetime. The *spacetime metric* on $\mathcal{M}$ is the Riemannian metric $g_{\mathcal{M}} = \hat{g} + d\mathfrak{t}^2$, where $\hat{g}$ is the extension of g to a quadratic form on $T\mathcal{M}$ such that $\partial_{\mathfrak{t}} \in \ker(\hat{g})$. The *quasiparabolic metric* on $\mathcal{M}$ is the Riemannian metric $g^{qp}_{\mathcal{M}} = (1 + R^2)^{\frac{1}{2}}\, \hat{g} + (1 + R^2) d\mathfrak{t}^2$.

For the remainder of the introduction, unless otherwise specified, $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ will denote a fixed singular Ricci flow.

Conditions (b) and (c) of Definition 1.5 imply that the scalar curvature controls the local geometry of the singular Ricci flow. This has several implications.

- (High-curvature regions in singular Ricci flows are topologically standard) For every t, the superlevel set $\{R \geq r^{-2}(t)\} \cap \mathcal{M}_t$ is contained in a disjoint union of connected components whose diffeomorphism types come from a small list of possibilities, with well-controlled local geometry. In particular, each connected component C of $\mathcal{M}_t$ has finitely many ends, and passing to the metric completion $\bar{C}$ adds at most one point for each end (Proposition 5.31).
- (Bounded geometry) The spacetime metric $g_{\mathcal{M}}$ has bounded geometry at the scale defined by the scalar curvature, while the quasiparabolic metric $g^{qp}_{\mathcal{M}}$ is complete and has bounded geometry in the usual sense — the injectivity radius is bounded below and all derivatives of curvature are uniformly bounded (Lemma 5.23).

The local control on geometry also leads to a compactness property for singular Ricci flows:

- (Compactness) If one has a sequence $\{(\mathcal{M}^j, \mathfrak{t}_j, \partial_{\mathfrak{t}_j}, g_j)\}_{j=1}^{\infty}$ of singular Ricci flows with a fixed choice of functions in Definition 1.5, and the initial metrics $\{(\mathcal{M}^j_0, g_j(0))\}_{j=1}^{\infty}$ form a precompact set in the smooth topology, then a subsequence converges in the sense of Theorem 1.2 (Proposition 5.39).

The proof of the compactness result is similar to the proof of Theorem 1.2. We also have global results concerning the scalar curvature and volume.

- (Scalar curvature and volume control) For any $T < \infty$, the scalar curvature is integrable on $\mathcal{M}_{\leq T}$. The volume function $\mathcal{V}(t) = \mathrm{vol}(\mathcal{M}_t)$ is finite and locally p-Hölder in t for some exponent $p \in (0,1)$, and has a locally bounded upper right derivative. The usual formula holds for volume evolution:

$$\mathcal{V}(t_1) - \mathcal{V}(t_0) = -\int_{\mathcal{M}_{[t_0,t_1]}} R \ \mathrm{dvol}_{g_{\mathcal{M}}} \tag{1.9}$$

  for all $0 \leq t_0 \leq t_1 < \infty$. (Propositions 5.11 and 8.15)
- (L^p bound on scalar curvature) With the same parameter $p \in (0,1)$ as above, for all t the scalar curvature is L^p on $\mathcal{M}_t$. (Proposition 8.1)

The starting point for the proof of the above results concerning volume is the fact that for Ricci flow, the time derivative of the volume form is given by minus the scalar curvature. To obtain global results, one is effectively forced to do integration by parts using the time vector field $\partial_{\mathfrak{t}}$. However, there is a substantial complication due to the potential incompleteness of the time vector field; we overcome this by using Theorem 1.12 below, together with the gradient estimate on scalar curvature.

To describe the next results, we introduce the following definitions.

Definition 1.10. A path $\gamma : I \to \mathcal{M}$ is *time-preserving* if $\mathfrak{t}(\gamma(t)) = t$ for all $t \in I$. The *worldline* of a point $m \in \mathcal{M}$ is the maximal time-preserving integral curve $\gamma : I \to \mathcal{M}$ of the time vector field $\partial_{\mathfrak{t}}$, which passes through m.

If $\gamma : I \to \mathcal{M}$ is a worldline then we may have $\sup I < \infty$. In this case, the scalar curvature blows up along $\gamma(t)$ as $t \to \sup I$, and the worldline encounters a singularity. An example would be a shrinking round space form, or a neckpinch. A worldline may also encounter a singularity going backward in time.

Definition 1.11. A worldline $\gamma : I \to \mathcal{M}$ is *bad* if $\inf I > 0$, i.e. if it is not defined at $t = 0$.

Among our structural results, perhaps the most striking is the following:

Theorem 1.12. *Suppose that $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is a singular Ricci flow and $t \geq 0$. If C is a connected component of $\mathcal{M}_t$ then only finitely many points in C have bad worldlines.*

As an illustration of the theorem, consider a singular Ricci flow that undergoes a generic neck pinch at time t_0, so that the time slice $\mathcal{M}_{t_0}$ has two ends (ϵ-horns in Perelman's language) which are instantly capped off when $t > t_0$. In this case the theorem asserts that only finitely many (in this case two) worldlines emerge from the singularity. See Figure 1, where the bad wordlines are indicated by dashed curves. (The point in the figure where the two dashed curves meet is not in the spacetime.)

A key ingredient in the proof of Theorem 1.12 is a new stability property of neck regions in κ-solutions. We recall that κ-solutions are the class of ancient Ricci flows used to model the high curvature part of Ricci flows with surgery. We state the stability property loosely as follows, and refer the reader to Section 6 for more details:

Neck Stability. *Let $\mathcal{M}$ be a noncompact κ-solution other than the shrinking round cylinder. If $\gamma : I \to \mathcal{M}$ is a worldline and $\mathcal{M}_{t_1}$ is is sufficiently necklike at $\gamma(t_1)$, then as $t \to -\infty$, the time slice $\mathcal{M}_t$ looks*

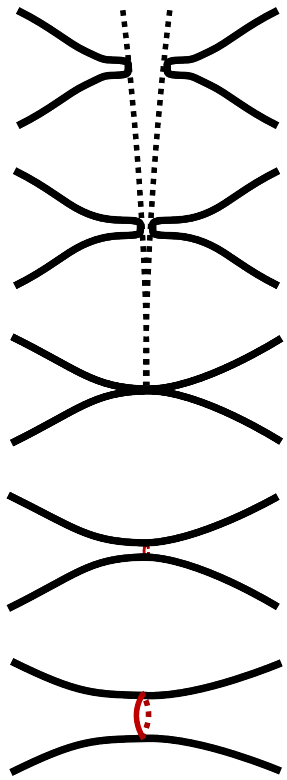

FIGURE 1.

more and more necklike at $\gamma(t)$. Here the notion of necklike is scale invariant.

An easy but illustrative case is the Bryant soliton, in which worldines other than the tip itself move away from the tip (in the scale invariant sense) as one goes backward in time.

As mentioned above, Theorem 1.12 is used in the proof of (1.9) and the properties of volume.

Finally, we mention some connectedness properties of singular Ricci flows.

- (Paths back to $\mathcal{M}_0$ avoiding high-curvature regions) Any point $m \in \mathcal{M}$ can be joined to the initial time slice $\mathcal{M}_0$ by a time-preserving curve $\gamma : [0, \mathfrak{t}(m)] \to \mathcal{M}$, along which $\max\{R(\gamma(t)) : t \in [0, \mathfrak{t}(m)]\}$ is bounded in terms of $R(m)$ and $\mathfrak{t}(m)$. (Proposition 5.38)
- (Backward stability of components) If $\gamma_0, \gamma_1 : [t_0, t_1] \to \mathcal{M}$ are time-preserving curves such that $\gamma_0(t_1)$ and $\gamma_1(t_1)$ lie in the

same connected component of $\mathcal{M}_{t_1}$, then $\gamma_0(t)$ and $\gamma_1(t)$ lie in the same component of $\mathcal{M}_t$ for all $t \in [t_0, t_1]$. (Proposition 5.32)

Related work. We now mention some other work that falls into the broad setting of Ricci flow with singular structure.

A number of authors have considered Ricci flow with low regularity initial conditions, studying existence and/or uniqueness, and instantaneous improvement of regularity [9, 22, 23, 24, 34, 43, 45, 46, 47, 50, 51]. Ricci flow with persistent singularities has been considered in the case of orbifold Ricci flow, and Ricci flow with conical singularities [8, 10, 11, 12, 15, 27, 33, 36, 37, 38, 42, 52, 53, 55, 56]

Passing to flows through singularities, Feldman-Ilmanen-Knopf noted that in the noncompact Kähler setting, there are some natural examples of flows through singularities which consist of a shrinking gradient soliton that has a conical limit at time zero, which transmutes into an expanding gradient soliton [21]. Closer in spirit to this paper, Angenent-Caputo-Knopf constructed a rotationally invariant Ricci flow through singularities starting with a metric on S^{n+1} [1]. They showed that the rotationally invariant neckpinches from [2], which have a singular limit as t approaches zero from below, may be continued as a smooth Ricci flow on two copies of S^{n+1} for $t > 0$. The paper [1] also showed that the forward evolution has a unique asymptotic profile near the singular point in spacetime.

There has been much progress on flowing through singularities in the Kähler setting. For a flow on a projective variety with log terminal singularities, Song-Tian [48] showed that the flow can be continued through the divisorial contractions and flips of the minimal model program. We refer to [48] for the precise statements. The paper [19] has related results, but uses a viscosity solution approach instead of the regularization scheme in [48]. The fact that the Kähler-Ricci flow can be reduced to a scalar equation, to which comparison principles may be applied, is an important simplifying feature of Kähler-Ricci flow that is not available in the non-Kähler case.

The work in this paper is related to the question of whether there is a good notion of a weak Ricci flow. This question is subject to different interpretations, because the term “weak” has different meanings in different settings. The results in this paper show that singular Ricci flows give a possible answer in the 3-dimensional case. With appropriate modifications, the results in this paper carry over to four-dimensional Ricci flow with nonnegative isotropic curvature; we will discuss this elsewhere. In the general higher dimensional case, however, it remains unclear if there is a good general notion of weak Ricci flow.

Organization of the paper. The remainder of the paper is broken into two parts. Part I, which is composed of Sections 2-4, is primarily concerned with the proof of the convergence result, Theorem 1.2. Part II, which is composed of Sections 5-8, deals with results on singular Ricci flows. Appendix A introduces and recalls notation and terminology for Ricci flows and Ricci flows with surgery.

We now describe the contents section by section.

Section 2 gives a general pointed compactness result for Riemannian manifolds and spacetimes, whose geometry is locally controlled as a function of some auxiliary function. Section 3 develops some properties of Ricci flow with surgery; it is aimed at showing that the Ricci flow spacetime associated with a Ricci flow with surgery has locally controlled geometry in the sense of Section 2. Section 4 applies the two preceding sections to give the proof of Theorem 1.2.

Section 5 establishes some foundational results about singular Ricci flows, concerning scalar curvature and volume, as well as some results involving the structure of the high curvature region. Section 6 proves that neck regions in κ-solutions have a stability property going backward in time. Section 7 proves Theorem 1.12, concerning the finiteness of the number of bad worldlines, and gives several applications. Section 8 contains some estimates involving volume and scalar curvature.

Appendix A collects a variety of background material; the reader may wish to quickly peruse this before proceeding to the body of the paper.

Notation and terminology. We refer the reader to Appendix A.1 for notation and terminology.

Part I

2. Compactness for spaces of locally controlled geometries

In this section we prove a compactness result, Theorem 2.5, for sequences of controlled Riemannian manifolds. In Subsection 2.4 we extend the theorem to a compactness result for spacetimes, meaning Riemannian manifolds equipped with time functions and time vector fields.

2.1. **Compactness of the space of locally controlled Riemannian manifolds.**

We will need a sequential compactness result for sequences of Riemannian manifolds which may be incomplete, but whose local geometry (injectivity radius and all derivatives of curvature) is bounded by a function of a auxiliary function $\psi : M \to [0, \infty)$. A standard case of this is sequential compactness for sequences $\{(M_j, g_j, \star_j)\}$ of Riemannian r-balls, assuming that the distance function $d_j(\star_j, \cdot) : M_j \to [0, r)$ is proper, and that the geometry is bounded in terms of $d_j(\star_j, \cdot)$.

Fix a smooth decreasing function $\mathfrak{r} : [0, \infty) \to (0, 1]$ and smooth increasing functions $C_k : [0, \infty) \to [1, \infty)$ for all $k \geq 0$.

Definition 2.1. Suppose that (M, g) is a Riemannian manifold equipped with a function $\psi : M \to [0, \infty)$.

Given $A \in [0, \infty]$, a tensor field ξ is (ψ, A)-*controlled* if for all $m \in \psi^{-1}([0, A))$ and $k \geq 0$, we have

$$\|\nabla^k \xi(m)\| \leq C_k(\psi(m)). \tag{2.2}$$

If in addition ψ is smooth then we say that the tuple (M, g, ψ) is (ψ, A)-*controlled* if

(1) The injectivity radius of M at m is at least $\mathfrak{r}(\psi(m))$ for all $m \in \psi^{-1}([0, A))$, and
(2) The tensor field Rm, and ψ itself, are (ψ, A)-controlled.

Note that if $A = \infty$ then $\psi^{-1}([0, A))$ is all of M, so there are quantitative bounds on the geometry at each point of M. Note also that the value of a control function may not reflect the actual bounds on the geometry, in the sense that the geometry may be more regular near m than the value of $\psi(m)$ suggests. This creates flexibility in choosing a control function, which is useful in applications below.

Example 2.3. Suppose that $(M, g, \star)$ is a complete pointed Riemannian manifold. Put $\psi(m) = d(\star, m)$. Then Rm is (ψ, A)-controlled if and only if for all $r \in (0, A)$ and $k \geq 0$, we have $\|\nabla^k \operatorname{Rm}\| \leq C_k(r)$ on $B(\star, r)$.

Example 2.4. Suppose that ψ has constant value $c > 0$. If $A \leq c$ then (M, g, ψ) is vacuously (ψ, A)-controlled. If $A > c$ then (M, g, ψ) is (ψ, A)-controlled if and only if for all $m \in M$, we have $\operatorname{inj}(m) \geq \mathfrak{r}(c)$ and $\|\nabla^k \operatorname{Rm}(m)\| \leq C_k(c)$.

There are compactness results in Riemannian geometry saying that one can extract a subsequential limit from a sequence of complete pointed Riemannian manifolds having uniform local geometry. This last condition means that for each $r > 0$, one has quantitative uniform bounds on the geometry of the r-ball around the basepoint; c.f. [25, Theorem 2.3]. In such a case, one can think of the distance from the basepoint as a control function. We will give a compactness theorem for Riemannian manifolds (possibly incomplete) equipped with more general control functions.

For notation, if $\Phi : U \to V$ is a diffeomorphism then we will write Φ_* for both the pushforward action of Φ on contravariant tensor fields on U, and the pullback action of Φ^{-1} on covariant tensor fields on U.

Theorem 2.5 (Compactness for controlled manifolds)**.** *Let*

$$\{(M_j, g_j, \star_j, \psi_j)\}_{j=1}^{\infty} \tag{2.6}$$

be a sequence of pointed tuples which are (ψ_j, A_j)-controlled, where $\lim_{j\to\infty} A_j = \infty$ and $\sup_j \psi_j(\star_j) < \infty$. Then after passing to a subsequence, there is a pointed (ψ_∞, ∞)-controlled tuple

$$(M_\infty, g_\infty, \star_\infty, \psi_\infty) \tag{2.7}$$

and a sequence of diffeomorphisms $\{M_j \supset U_j \xrightarrow{\Phi^j} V_j \subset M_\infty\}_{j=1}^{\infty}$ such that

(1) Given $A, r < \infty$, for all sufficiently large j the open set U_j contains the ball $B(\star_j, r)$ in the Riemannian manifold $(\psi_j^{-1}([0,A)), g_j)$ and likewise V_j contains the ball $B(\star_\infty, r)$ in the Riemannian manifold $(\psi_\infty^{-1}([0,A)), g_\infty)$.

(2) Given $\epsilon > 0$ and $k \geq 0$, for all sufficiently large j we have

$$\|\Phi^j_* g_j - g_\infty\|_{C^k(V_j)} < \epsilon \tag{2.8}$$

and

$$\|\Phi^j_* \psi_j - \psi_\infty\|_{C^k(V_j)} < \epsilon. \tag{2.9}$$

(3) M_∞ is connected and, in particular, every $x \in M_\infty$ belongs to V_j for j large.

We will give the proof of Theorem 2.5 in Subsection 2.3. We first describe some general results about controlled Riemannian manifolds.

2.2. **Some properties of controlled Riemannian manifolds.** One approach to proving Theorem 2.5 would be to imitate what one does when one has curvature and injectivity radius bounds on r-balls, replacing the control function based on distance to the basepoint by the control function ψ. While this could be done, it would be somewhat involved. Instead, we will perform a conformal change on the Riemannian manifolds in order to put ourselves in a situation where the geometry is indeed controlled by the distance from the basepoint. We then take a subsequential limit of the conformally changed metrics, and at the end perform another conformal change to get a subsequential limit of the original sequence.

Let (M, g, ψ) be (ψ, A)-controlled. Put

$$\widetilde{g} = \left(\frac{C_1}{\mathfrak{r}} \circ \psi \right)^2 g. \tag{2.10}$$

We will only consider $\widetilde{g}$ on the subset $\psi^{-1}([0, A))$, where it is smooth. The next two lemmas are about g-balls and $\widetilde{g}$-balls.

Lemma 2.11. *For each finite $a \in (0, A]$, each $\star \in \psi^{-1}([0, a))$ and each $r < \infty$, there is some $R = R(a, r) < \infty$ so that the ball $B_g(\star, r)$ in the Riemannian manifold $(\psi^{-1}([0, a)), g)$ is contained in the ball $B_{\widetilde{g}}(\star, R)$ in the Riemannian manifold $(\psi^{-1}([0, a)), \widetilde{g})$.*

Proof. On $\psi^{-1}([0, a))$ we have $\widetilde{g} \leq \left(\frac{C_1(a)}{\mathfrak{r}(a)} \right)^2 g$. Therefore any path in $\psi^{-1}([0, a))$ with g-length at most r has $\widetilde{g}$-length at most $\frac{C_1(a)}{\mathfrak{r}(a)} r$, so we may take $R = \frac{C_1(a)}{\mathfrak{r}(a)} r$. □

Let $\star$ be a basepoint in $\psi^{-1}([0, A))$.

Lemma 2.12. *For all $R \in (0, A - \psi(\star))$, the ball $B_{\widetilde{g}}(\star, R)$ in the Riemannian manifold $(\psi^{-1}([0, A)), \widetilde{g})$ is contained in $\psi^{-1}([0, \psi(\star) + R))$.*

Proof. Given $m \in B_{\widetilde{g}}(\star, R)$, let $\gamma : [0, L] \to \psi^{-1}([0, A))$ be a smooth path from $\star$ to m with unit $\widetilde{g}$-speed and $\widetilde{g}$-length $L \in (0, R)$. Then

$$\begin{aligned} \psi(m) - \psi(\star) &= \int_0^L \frac{d}{dt} \psi(\gamma(t))\, dt \leq \int_0^L |d\psi|_{\widetilde{g}} (\gamma(t))\, dt \\ &= \int_0^L \frac{\mathfrak{r}}{C_1} (\psi(\gamma(t))) \cdot |d\psi|_g (\gamma(t))\, dt \\ &\leq \int_0^L \mathfrak{r}(\psi(\gamma(t)))\, dt \leq \int_0^L 1\, dt = L. \end{aligned} \tag{2.13}$$

The lemma follows. □

We now look at completeness properties of $\widetilde{g}$-balls.

Lemma 2.14. *For all $R \in (0, A - \psi(\star))$, the ball $B_{\widetilde{g}}(\star, R)$ in the Riemannian manifold $(\psi^{-1}([0, A)), \widetilde{g})$ has compact closure in $\psi^{-1}([0, A))$.*

Proof. Choose $R' \in (R, A - \psi(\star))$. From [3, Chapter 1, Theorem 2.4], it suffices to show that any $\widetilde{g}$-unit speed geodesic $\gamma : [0, L) \to \psi^{-1}([0, A))$ with $\gamma(0) = \star$, having $\widetilde{g}$-length $L \in (0, R')$, can be extended to $[0, L]$. From Lemma 2.12, $\gamma([0, L)) \subset \psi^{-1}([0, \psi(\star) + R'))$. Hence the g-injectivity radius along $\gamma([0, L))$ is bounded below by $\mathfrak{r}(\psi(\star) + R'))$. For large K, the points $\{\gamma(L - \frac{1}{k})\}_{k=K}^{\infty}$ form a Cauchy sequence in

$(\psi^{-1}([0,A), d_{\widetilde{g}})$. As $d_{\widetilde{g}}$ and d_g are biLipschitz on $\psi^{-1}([0,\psi(\star)+R'))$, the sequence is also Cauchy in $(\psi^{-1}([0,A)), d_g)$. From the uniform positive lower bound on the g-injectivity radius at $\gamma(L-\frac{1}{k})$, there is a limit in $\psi^{-1}([0,A))$. The lemma follows. □

Corollary 2.15. *If $A=\infty$ then $(M,\widetilde{g})$ is complete.*

Proof. This follows from Lemma 2.14 and [3, Chapter 1, Theorem 2.4]. □

Finally, we give bounds on the geometry of $\widetilde{g}$-balls.

Lemma 2.16. *Given $\mathfrak{r}$, $\{C_k\}_{k=1}^{\infty}$ and $S<\infty$, there exists*

(1) A smooth decreasing function $\widetilde{\mathfrak{r}}:[0,\infty)\to(0,1]$, and

(2) Smooth increasing functions $\widetilde{C}_k:[0,\infty)\to[1,\infty)$, $k\geq 0$,

with the following properties. Suppose that (M,g,ψ) is (ψ,A)-controlled and $\psi(\star)\leq S$. Then

(a) $\mathrm{Rm}_{\widetilde{g}}$ and ψ are $(d_{\widetilde{g}}(\star,\cdot), A-S)$-controlled on the Riemannian manifold $(\psi^{-1}([0,A)),\widetilde{g})$ (in terms of the functions $\{\widetilde{C}_k\}_{k=1}^{\infty}$).

(b) If $R<A-S-1$ then $\mathrm{inj}_{\widetilde{g}}\geq\widetilde{\mathfrak{r}}(R)$ pointwise on $B_{\widetilde{g}}(\star,R)$.

Proof. Conclusion (a) (along with the concomitant functions $\{\widetilde{C}_k\}_{k=1}^{\infty}$) follows from Lemma 2.12, the assumption that Rm_g and ψ are (ψ,A)-controlled, and the formula for the Riemannian curvature of a conformally changed metric.

To prove conclusion (b), suppose that $R<A-S-1$ and $m\in B_{\widetilde{g}}(\star,R)$. Since $d_{\widetilde{g}}$ and d_g are biLipschitz on the ball $B_{\widetilde{g}}(\star,R+1)$ in the Riemannian manifold $(\psi^{-1}([0,R+1+S)),\widetilde{g})$, we can find $\epsilon=\epsilon(R,S,\{C_k\})>0$ so that $B_g(m,\epsilon)\subset B_{\widetilde{g}}(m,1)\subset B_{\widetilde{g}}(\star,R+1)$. Since we have a g-curvature bound on $B_{\widetilde{g}}(\star,R+1)$, and a lower g-injectivity radius bound at m, we obtain a lower volume bound $\mathrm{vol}(B_g(m,\epsilon),g)\geq v_0=v_0(R,S,\mathfrak{r},\{C_k\})>0$. Since g and $\widetilde{g}$ are relatively bounded on $B_g(m,\epsilon)$, this gives a lower volume bound $\mathrm{vol}(B_{\widetilde{g}}(m,1),\widetilde{g})\geq v_1=v_1(R,S,\mathfrak{r},\{C_k\})>0$. Using the curvature bound of part (a) and [6, Theorem 4.7], we obtain a lower bound $\mathrm{inj}_{\widetilde{g}}(m)\geq i_0=i_0(R,S,\mathfrak{r},\{C_k\})>0$. This proves the lemma. □

2.3. **Proof of Theorem 2.5.** Put $\widetilde{g}_j=\left(\frac{C_1}{\mathfrak{r}}\circ\psi_j\right)^2 g_j$. Consider the tuple $(M_j,\widetilde{g}_j,\star_j,\psi_j)$. Recall that $\lim_{j\to\infty}A_j=\infty$.

For the moment, we replace the index j by the index l. Using Lemma 2.14, Lemma 2.16 and a standard compactness theorem [25, Theorem 2.3], after passing to a subsequence we can find a complete pointed Riemannian manifold $(M_\infty,g_\infty,\star_\infty)$, domains $\widetilde{U}_l\subset M_l$ and $\widetilde{V}_l\subset M_\infty$, and diffeomorphisms $\widetilde{\Phi}^l:\widetilde{U}_l\to\widetilde{V}_l$ so that

(a) $\star_l \in \widetilde{U}_l$.
(b) $\star_\infty \in \widetilde{V}_l$.
(c) $\widetilde{V}_l$ has compact closure.
(d) For any compact set $K \subset M_\infty$, we have $K \subset \widetilde{V}_l$ for all sufficiently large l.
(e) Given $\epsilon > 0$ and $k \geq 0$, we have

$$\| \widetilde{\Phi}^l_* \widetilde{g}_l - \widetilde{g}_\infty \|_{C^k(\widetilde{V}_l, \widetilde{g}_\infty)} < \epsilon \tag{2.17}$$

for all sufficiently large l.

Using Lemma 2.16 again, after passing to a further subsequence if necessary, we can assume that there is a smooth ψ_∞ on M_∞ so that for all $\epsilon > 0$ and $k \geq 0$, we have

$$\| \widetilde{\Phi}^l_* \psi_l - \psi_\infty \|_{C^k(\widetilde{V}_l, \widetilde{g}_\infty)} < \epsilon \tag{2.18}$$

for all sufficiently large l. Put $g_\infty = \left(\frac{C_1}{\mathfrak{r}} \circ \psi_\infty \right)^{-2} \widetilde{g}_\infty$.

We claim that if the sequence $\{l_j\}_{j=1}^\infty$ increases rapidly enough then the conclusions of the theorem can be made to hold with $V_j = B_{\widetilde{g}_\infty}(\star_\infty, j) \subset M^\infty$, $U_j = \left(\widetilde{\Phi}^{l_j} \right)^{-1} (V_j) \subset M^{l_j}$ and $\Phi^j = \widetilde{\Phi}^{l_j} \Big|_{U_j}$. To see this, we note that

- Given $A, r < \infty$, Lemma 2.11 implies that the ball $B(\star_\infty, r)$ in the Riemannian manifold $(\psi_\infty^{-1}([0, A)), g_\infty)$ will be contained in V_j for all sufficiently large j.
- If the sequence $\{l_j\}_{j=1}^\infty$ increases rapidly enough then for large j, the map Φ^j is arbitrarily close to an isometry and $(\Phi^j)_* \psi_j$ is arbitrarily close to ψ_∞ on V_j. Hence given $A, r < \infty$, the ball $B(\star_j, r)$ in the Riemannian manifold $(\psi_j^{-1}([0, A)), g_j)$ will be contained in U_j for all sufficiently large j, so conclusion (1) of the theorem holds.
- The metrics g_∞ and $\widetilde{g}_\infty$ are biLipschitz on V_j. Then if the sequence $\{l_j\}_{j=1}^\infty$ increases rapidly enough, conclusion (2) of the theorem can be made to hold.
- Conclusion (3) of the theorem follows from the definition of V_j.

This proves Theorem 2.5. □

2.4. **Compactness of the space of locally controlled spacetimes.**

We now apply Theorem 2.5 to prove a compactness result for spacetimes.

Definition 2.19. A *spacetime* is a Riemannian manifold $(\mathcal{M}, g_\mathcal{M})$ equipped with a submersion $\mathfrak{t} : \mathcal{M} \to \mathbb{R}$ and a smooth vector field $\partial_\mathfrak{t}$ such that $d\mathfrak{t}(\partial_\mathfrak{t}) \equiv 1$.

Given $A \in [0,\infty]$ and $\psi \in C^\infty(\mathcal{M})$, we say that the tuple $(\mathcal{M}, g_{\mathcal{M}}, \mathfrak{t}, \partial_{\mathfrak{t}}, \psi)$ is (ψ, A)-*controlled* if $(\mathcal{M}, g_{\mathcal{M}}, \psi)$ is (ψ, A)-controlled in the sense of Definition 2.1 and, in addition, the tensor fields $\mathfrak{t}$ and $\partial_{\mathfrak{t}}$ are (ψ, A)-controlled.

Theorem 2.20 (Compactness for controlled spacetimes)**.** *Let*

$$\left\{ \left(\mathcal{M}^j, g_{\mathcal{M}^j}, \mathfrak{t}_j, (\partial_{\mathfrak{t}})_j, \star_j, \psi_j \right) \right\}_{j=1}^{\infty} \tag{2.21}$$

be a sequence of pointed tuples which are (ψ_j, A_j)*-controlled, where* $\lim_{j\to\infty} A_j = \infty$ *and* $\sup_j \psi_j(\star_j) < \infty$*. Then after passing to a subsequence, there is a pointed* (ψ_∞, ∞)*-controlled tuple*

$$(\mathcal{M}^\infty, g_{\mathcal{M}^\infty}, \mathfrak{t}_\infty, (\partial_{\mathfrak{t}})_\infty, \star_\infty, \psi_\infty) \tag{2.22}$$

and a sequence of diffeomorphisms

$$\{ \mathcal{M}^j \supset U_j \xrightarrow{\Phi^j} V_j \subset \mathcal{M}^\infty \}_{j=1}^{\infty} \tag{2.23}$$

of open sets such that

(1) *Given* $A < \infty$ *and* $r < \infty$*, for all sufficiently large* j*, the open set* U_j *contains the ball* $B(\star_j, r)$ *in the Riemannian manifold* $\left(\psi_j^{-1}([0,A)), g_{\mathcal{M}^j}\right)$ *and likewise* V_j *contains the ball* $B(\star_\infty, r) \subset (\psi_\infty^{-1}([0,A)), g_{\mathcal{M}^\infty})$*.*

(2) Φ^j *exactly preserves the time functions:*

$$\mathfrak{t}_\infty \circ \Phi^j = \mathfrak{t}_j \text{ for all } j\,. \tag{2.24}$$

(3) Φ^j *asymptotically preserves the tensor fields* $g_{\mathcal{M}^j}$*,* $(\partial_{\mathfrak{t}})_j$*, and* ψ_j*: if* $\xi_j \in \left\{ g_{\mathcal{M}^j}, (\partial_{\mathfrak{t}})_j, \psi_j \right\}$ *and* ξ_∞ *is the corresponding element of* $\{g_{\mathcal{M}^\infty}, (\partial_{\mathfrak{t}})_\infty, \psi_\infty\}$*, then for all* $\epsilon > 0$ *and* $k \geq 0$*, we have*

$$\| \Phi^j_* \xi_j - \xi_\infty \|_{C^k(V_j)} < \epsilon \tag{2.25}$$

for all sufficiently large j*.*

(4) $\mathcal{M}^\infty$ *is connected and, in particular, every* $x \in \mathcal{M}^\infty$ *belongs to* V_j *for large* j*.*

Proof. Put $\widetilde{g}_{\mathcal{M}^j} = \left(\frac{C_1}{\mathfrak{r}} \circ \psi_j\right)^2 g_{\mathcal{M}^j}$. Consider the pointed spacetime $(\mathcal{M}^j, \widetilde{g}_{\mathcal{M}^j}, \mathfrak{t}_j, (\partial_{\mathfrak{t}})_j, \star_j, \psi_j)$.

Lemma 2.26. *For every* $A < \infty$*, the tensor fields* $\mathrm{Rm}_{\widetilde{g}_j}$*,* $\mathfrak{t}_j$*,* $(\partial_t)_j$*, and* ψ_j *are all* $\left(d_{\widetilde{g}_{\mathcal{M}^j}}(\star_j, \cdot), A\right)$*-controlled for all large* j*, in the sense of Lemma 2.16.*

Proof. This follows as in the proof of Lemma 2.16. □

We follow the proof of Theorem 2.5 up to the construction of $\widetilde{\Phi}^l : \widetilde{U}_l \to \widetilde{V}_l$ and ψ_∞. Using Lemma 2.26, after passing to a further subsequence if necessary, we can assume that there are smooth $\mathfrak{t}_\infty$ and $(\partial_\mathfrak{t})_\infty$ on $\mathcal{M}^\infty$ so that if $\xi_l \in \{\mathfrak{t}_l, (\partial_\mathfrak{t})_l, \psi_l\}$ then for all $\epsilon > 0$ and $k \geq 0$, we have

$$\| \widetilde{\Phi}^l_* \xi_l - \xi_\infty \|_{C^k(\widetilde{V}_l, \widetilde{g}_{\mathcal{M}^\infty})} < \epsilon \tag{2.27}$$

for all sufficiently large l. Again, $\mathfrak{t}_\infty : \mathcal{M}^\infty \to \mathbb{R}$ is a submersion and $(\partial_\mathfrak{t})_\infty \mathfrak{t}_\infty = 1$. As before, put $g_{\mathcal{M}^\infty} = \left(\frac{C_1}{\mathfrak{r}} \circ \psi_\infty \right)^2 \widetilde{g}_{\mathcal{M}^\infty}$.

Let $\{\phi_s\}$ be the flow generated by $(\partial_\mathfrak{t})_\infty$; this exists for at least a small time interval if the starting point is in a given compact subset of M_∞. Put $V'_j = B_{\widetilde{g}_\infty}(\star_\infty, j)$. Then there is some $\Delta_j > 0$ so that $\{\phi_s\}$ exists on V'_j for $|s| < \Delta_j$. Given $l_j \gg 0$, put $U_j = (\widetilde{\Phi}^{l_j})^{-1}(V'_j)$. Assuming that l_j is large enough, we can define $\Phi^j : U_j \to \mathcal{M}^\infty$ by

$$\Phi^j(m) = \phi_{\mathfrak{t}_j(m) - \mathfrak{t}_\infty(\widetilde{\Phi}^{l_j}(m))} (\widetilde{\Phi}^{l_j}(m)). \tag{2.28}$$

By construction, $\mathfrak{t}_\infty(\Phi^j(m)) = \mathfrak{t}_j(m)$, and so conclusion (2) of the theorem holds. If l_j is large then Φ^j will be a diffeomorphism to its image. Putting $V_j = \Phi^j(U_j)$, if l_j is large enough then V_j can be made arbitrarily close to V'_j. It follows that conclusions (1), (3) and (4) of the theorem hold. □

Remark 2.29. In Hamilton's compactness theorem, [25], the comparison map Φ^j preserves both the time function and the time vector field. In Theorem 2.20 the comparison map Φ^j preserves the time function, but the time vector field is only preserved asymptotically. This is good enough for our purposes.

3. Properties of Ricci flows with surgery

In this section we prove several estimates for Ricci flows with surgery. These will be used in the proof of Theorem 1.2, to show that the sequence of Ricci flow spacetimes has the local control required for the application of the spacetime compactness theorem, Theorem 2.20.

The arguments in this section require familiarity with some basic properties of Ricci flow with surgery. For the reader's convenience, we have collected these properties in Appendix A.8. The reader may wish to review this material before proceeding.

Let $\mathcal{M}$ be a Ricci flow with surgery in the sense of Perelman [32, Section 68]. As recalled in Appendix A.9, there are parameters — or more precisely positive decreasing parameter functions — associated with $\mathcal{M}$:

- The canonical neighborhood scale function $r(t) > 0$. We can assume that $r(0) < \frac{1}{10}$.
- The noncollapsing function $\kappa(t) > 0$.
- The parameter $\delta(t) > 0$. This has a dual role: it is the quality of the surgery neck, and it enforces a scale buffer between the canonical neighborhood scale r, the intermediate scale ρ and the surgery scale h.
- The intermediate scale $\rho(t) = \delta(t)r(t)$, which defines the threshold for discarding entire connected components at the singular time.
- The surgery scale $h(t) < \delta^2(t)r(t)$.
- The global parameter $\epsilon > 0$. This enters in the definition of a canonical neighborhood. For the Ricci flow with surgery to exist, a necessary condition is that ϵ be small enough.

In this section, canonical neighborhoods are those defined for Ricci flows with surgery, as in [32, Definition 69.1]. The next lemma gives a sufficient condition for parabolic neighborhoods to be unscathed.

Lemma 3.1. *Let $\mathcal{M}$ be a Ricci flow with surgery, with normalized initial condition. Given $T > \frac{1}{100}$, there are numbers $\mu = \mu(T) \in (0,1)$, $\sigma = \sigma(T) \in (0,1)$, $i_0 = i_0(T) > 0$ and $A_k = A_k(T) < \infty$, $k \geq 0$, with the following property. If $t \in \left(\frac{1}{100}, T\right]$ and $|R(x,t)| < \mu\rho(0)^{-2} - r(T)^{-2}$, put $Q = |R(x,t)| + r(t)^{-2}$. Then*

(1) The forward parabolic ball $P_+(x,t,\sigma Q^{-\frac{1}{2}})$ and the backward parabolic ball $P_-(x,t,\sigma Q^{-\frac{1}{2}})$ are unscathed.

(2) $|\operatorname{Rm}| \leq A_0 Q$, $\operatorname{inj} \geq i_0 Q^{-\frac{1}{2}}$ and $|\nabla^k \operatorname{Rm}| \leq A_k Q^{1+\frac{k}{2}}$ on the union $P_+(x,t,\sigma Q^{-\frac{1}{2}}) \cup P_-(x,t,\sigma Q^{-\frac{1}{2}})$ of the forward and backward parabolic balls.

Proof. By [32, Lemma 70.1], we have $R(x',t') \leq 8Q$ for all $(x',t') \in P_-(x,t,\eta^{-1}Q^{-\frac{1}{2}})$. where $\eta < \infty$ is a universal constant. The same argument works for $P_+(x,t,\eta^{-1}Q^{-\frac{1}{2}})$. Since R is proper on time slices (c.f. [32, Lemma 67.9]), it follows that $B(x,t,\eta^{-1}Q^{-\frac{1}{2}})$ has compact closure in its time slice.

If $\mu \leq \frac{1}{8}$ then

$$8Q = 8\left(|R(x,t)| + r(t)^{-2}\right) \leq 8\left(|R(x,t)| + r(T)^{-2}\right) \leq 8\mu\rho(0)^{-2} \leq \rho(t')^{-2} \tag{3.2}$$

for all $t' \in [t - \eta^{-2}Q^{-1}, t + \eta^{-2}Q^{-1}]$. Hence if $\sigma < \eta^{-1}$ then the forward and backward parabolic balls $P_{\pm}(x,t,\sigma Q^{-\frac{1}{2}})$ do not intersect the surgery regions.

To show that the balls $P_\pm(x,t,\sigma Q^{-\frac{1}{2}})$ are unscathed, for an appropriate value of σ, it remains to show that $P_-(x,t,\sigma Q^{-\frac{1}{2}})$ does not intersect the time-zero slice. We have $t - \sigma^2 Q^{-1} \geq t - \sigma^2 r(t)^2$. Since $t > \frac{1}{100}$ and $r(t) < r(0) < \frac{1}{10}$, if $\sigma < \frac{1}{2}$ then $t - \sigma^2 Q^{-1} \geq \frac{1}{200}$ and the balls $P_\pm(x,t,\sigma Q^{-\frac{1}{2}})$ are unscathed.

The Hamilton-Ivey estimate of (A.14) gives an explicit upper bound $|\operatorname{Rm}| \leq A_0 Q$ on $P_\pm(x,t,\sigma Q^{-\frac{1}{2}})$. Using the distance distortion estimates for Ricci flow [32, Section 27], there is a universal constant $\alpha > 0$ so that whenever $(x',t') \in P_\pm(x,t,\frac{1}{2}\sigma Q^{-\frac{1}{2}})$, we have $P_-(x',t',\alpha Q^{-\frac{1}{2}}) \subset P_+(x,t,\sigma Q^{-\frac{1}{2}}) \cup P_-(x,t,\sigma Q^{-\frac{1}{2}})$. Then Shi's local derivative estimates [32, Appendix D] give estimates $|\nabla^k \operatorname{Rm}| \leq A_k Q^{1+\frac{k}{2}}$ on $P_\pm(x,t,\frac{1}{2}\sigma Q^{-\frac{1}{2}})$.

Since $t \leq T$, we have $\kappa(t) \geq \kappa(T)$. The κ-noncollapsing statement gives an explicit lower bound $\operatorname{inj} \geq i_0 Q^{-\frac{1}{2}}$ on a slightly smaller parabolic ball, which after reducing σ, we can take to be of the form $P_\pm(x,t,\sigma Q^{-\frac{1}{2}})$. □

If $\mathcal{M}$ is a Ricci flow solution and $\gamma : [a,b] \to \mathcal{M}$ is a time-preserving spacetime curve then we define $\operatorname{length}_{g_\mathcal{M}}(\gamma)$ using the spacetime metric $g_\mathcal{M} = dt^2 + g(t)$. The next lemma says that given a point (x_0,t_0) in a κ-solution (in the sense of Appendix A.5), it has a large backward parabolic neighborhood so that any point (x_1,t_1) in the parabolic neighborhood can be connected to (x_0,t_0) by a time-preserving curve whose length is controlled by $R(x_0,t_0)$, and along which the scalar curvature is controlled by $R(x_0,t_0)$.

Lemma 3.3. *Given $\kappa > 0$, there exist $A = A(\kappa) < \infty$ and $C = C(\kappa) < \infty$ with the following property. If $\mathcal{M}$ is a κ-solution and $(x_0,t_0) \in \mathcal{M}$ then*

(1) There is some $(x_1,t_1) \in P_-(x_0,t_0,\frac{1}{2}AR(x_0,t_0)^{-\frac{1}{2}})$ with $R(x_1,t_1) \leq \frac{1}{3}R(x_0,t_0)$.

(2) The scalar curvature on $P_-(x_0,t_0,2AR(x_0,t_0)^{-\frac{1}{2}})$ is at most $\frac{1}{2}CR(x_0,t_0)$, and

(3) Given $(x_1,t_1) \in P_-(x_0,t_0,\frac{1}{2}AR(x_0,t_0)^{-\frac{1}{2}})$, there is a time-preserving curve $\gamma : [t_1,t_0] \to P_-\left(x_0,t_0,\frac{3}{4}AR(x_0,t_0)^{-\frac{1}{2}}\right)$ from (x_1,t_1) to (x_0,t_0) with

$$\operatorname{length}_{g_\mathcal{M}}(\gamma) \leq \frac{1}{2}C\left(R(x_0,t_0)^{-\frac{1}{2}} + R(x_0,t_0)^{-1}\right). \tag{3.4}$$

Proof. To prove (1), suppose, by way of contradiction, that for each $j \in \mathbb{Z}^+$ there is a κ-solution $\mathcal{M}^j$ and some $(x_0^j,t_0^j) \in \mathcal{M}^j$ so that $R > \frac{1}{3}R(x_0^j,t_0^j)$ on $P_-(x_0^j,t_0^j,\frac{1}{2}j)$. By compactness of the space of pointed normalized κ-solutions (see Appendix A.5), after normalizing so that $R(x_0^j, t_0^j) = 1$ and passing to a subsequence, there is a limiting κ-solution $\mathcal{M}'$, defined for $t \le 0$, with $R \ge \frac{1}{3}$ everywhere. By the weak maximum principle for complete noncompact manifolds [32, Theorem A.3] and the evolution equation for scalar curvature, there is a universal constant $\Delta > 0$ (whose exact value isn't important) so that if $R \ge \frac{1}{3}$ on a time-t slice then there is a singularity by time $t + \Delta$. Applying this with $t = -2\Delta$ gives a contradiction.

Part (2) of the lemma, for some value of C, follows from the compactness of the space of pointed normalized κ-solutions.

To prove (3), the curve which starts as a worldline from (x_1, t_1) to (x_1, t_0), and then moves as a minimal geodesic from (x_1, t_0) to (x_0, t_0) in the time-t_0 slice, has $g_{\mathcal{M}}$-length at most $\frac{1}{2} A R(x_0, t_0)^{-\frac{1}{2}} + \frac{1}{4} A^2 R(x_0, t_0)^{-1}$. By a slight perturbation to make it time-preserving, we can construct γ in $P_-\left(x_0, t_0, \frac{3}{4} A R(x_0, t_0)^{-\frac{1}{2}}\right)$ with length at most $\frac{1}{2}(A+1) R(x_0, t_0)^{-\frac{1}{2}} + \frac{1}{4}(A+1)^2 R(x_0, t_0)^{-1}$. After redefining C, this proves the lemma. □

The next proposition extends the preceding lemma from κ-solutions to points in Ricci flows with surgery. Recall that ϵ is the global parameter in the definition of Ricci flow with surgery.

Proposition 3.5. *There is an $\epsilon_0 > 0$ so that if $\epsilon < \epsilon_0$ then the following holds. Given $T < \infty$, suppose that $\rho(0) \le \frac{r(T)}{\sqrt{C}}$, where C is the constant from Lemma 3.3. Then for any $R_0 < \frac{1}{C}\rho(0)^{-2}$, there are $L = L(R_0, T) < \infty$ and $R_1 = R_1(R_0, T) < \infty$ with the following property. Let $\mathcal{M}$ be a Ricci flow with surgery having normalized initial conditions. Given $(x_0, t_0) \in \mathcal{M}$ with $t_0 \le T$, suppose that $R(x_0, t_0) \le R_0$. Then there is a time preserving curve $\gamma : [0, t_0] \to \mathcal{M}$ with $\gamma(t_0) = (x_0, t_0)$ and* $\text{length}_{g_{\mathcal{M}}}(\gamma) \le L$ *so that $R(\gamma(t)) \le R_1$ for all $t \in [0, t_0]$.*

Proof. We begin by noting that we can find $\epsilon_0 > 0$ so that if $\epsilon < \epsilon_0$ then for any $(x, t) \in \mathcal{M}$ with $t \le T$ which is in a canonical neighborhood, Lemma 3.3(2) implies that $R \le C R(x, t)$ on $P_-(x, t, A R(x, t)^{-\frac{1}{2}})$. If in addition $R(x, t) \le \frac{1}{C}\rho(0)^{-2}$ then for any $(x', t') \in P_-(x, t, A R(x, t)^{-\frac{1}{2}})$, we have

$$R(x', t') \le \rho(0)^{-2} \le \rho(t')^{-2}. \tag{3.6}$$

Hence the parabolic neighborhood does not intersect the surgery region.

To prove the proposition, we start with (x_0, t_0), and inductively form a sequence of points (x_i, t_i) and the curve γ as follows, starting with

$i = 1$.
Step 1 : If $R(x_{i-1}, t_{i-1}) \geq r(t_{i-1})^{-2}$ then go to Substep A. If $R(x_{i-1}, t_{i-1}) < r(t_{i-1})^{-2}$ then go to Substep B.
Substep A : Since $R(x_{i-1}, t_{i-1}) \geq r(t_{i-1})^{-2}$, the point (x_{i-1}, t_{i-1}) is in a canonical neighborhood. As will be explained, the backward parabolic ball $P_-(x_{i-1}, t_{i-1}, AR(x_{i-1}, t_{i-1})^{-\frac{1}{2}})$ does not intersect the surgery region. Applying Lemma 3.3 and taking ϵ_0 small, we can find $(x_i, t_i) \in P_-(x_{i-1}, t_{i-1}, AR(x_{i-1}, t_{i-1})^{-\frac{1}{2}})$ with $R(x_i, t_i) \leq \frac{1}{2} R(x_{i-1}, t_{i-1})$, and a time-preserving curve

$$\gamma : [t_i, t_{i-1}] \to P_-(x_{i-1}, t_{i-1}, AR(x_{i-1}, t_{i-1})^{-\frac{1}{2}}) \tag{3.7}$$

from (x_i, t_i) to (x_{i-1}, t_{i-1}) whose length is at most

$$C \left(R(x_{i-1}, t_{i-1})^{-\frac{1}{2}} + R(x_{i-1}, t_{i-1})^{-1} \right), \tag{3.8}$$

along which the scalar curvature is at most $CR(x_{i-1}, t_{i-1})$. If $t_i > 0$ then go to Step 2. If $t_i = 0$ then the process is terminated.
Substep B : Since $R(x_{i-1}, t_{i-1}) < r(t_{i-1})^{-2}$, put $x_i = x_{i-1}$ and $t_i = \inf\{t : R(x_{i-1}, s) \leq r(s)^{-2} \text{ for all } s \in [t, t_{i-1}]\}$. Define $\gamma : [t_i, t_{i-1}] \to \mathcal{M}$ to be the worldline $\gamma(s) = (x_{i-1}, s)$.

If $t_i > 0$ then go to Step 2. (Note that $R(x_i, t_i) = r(t_i)^{-2}$.) If $t_i = 0$ then the process is terminated.
Step 2 : Increase i by one and go to Step 1.

To recapitulate the iterative process, if R_0 is large then there may initially be a sequence of Substep A's. Since the curvature decreases by a factor of at least two for each of these, the number of these initial substeps is bounded above by $\log_2 \left(\frac{R_0}{r(0)^{-2}} \right)$. Thereafter, there is some (x_{i-1}, t_{i-1}) so that $R(x_{i-1}, t_{i-1}) < r(t_{i-1})^{-2}$. We then go backward in time along a segment of a worldline until we either hit a point (x_i, t_i) with $R(x_i, t_i) = r(t_i)^{-2}$, or we hit time zero. If we hit (x_i, t_i) then we go back to Substep A, which produces a point (x_{i+1}, t_{i+1}) with at most half as much scalar curvature, etc.

We now check the claim in Substep A that the backward parabolic ball $P_-(x_{i-1}, t_{i-1}, AR(x_{i-1}, t_{i-1})^{-\frac{1}{2}})$ does not intersect the surgery region. In the initial sequence of Substep A's, we always have $R(x_{i-1}, t_{i-1}) \leq R_0 < \frac{1}{C} \rho(0)^{-2}$. If we return to Substep A sometime after the initial sequence, we have

$$R(x_{i-1}, t_{i-1}) = r(t_{i-1})^{-2} \leq r(T)^{-2} \leq \frac{1}{C} \rho(0)^{-2}. \tag{3.9}$$

Either way, from the first paragraph of the proof, we conclude that $P_-(x_{i-1}, t_{i-1}, AR(x_{i-1}, t_{i-1})^{-\frac{1}{2}})^{-1})$ does not intersect the surgery region.

We claim that the iterative process terminates. If not, the decreasing sequence (t_i) approaches some $t_\infty > 0$. For an infinite number of i, we must have $R(x_i, t_i) = r(t_i)^{-2}$, which converges to $r(t_\infty)^{-2}$. Consider a large i with $R(x_i, t_i) = r(t_i)^{-2}$. The result of Substep A is a point (x_{i+1}, t_{i+1}) with $R(x_{i+1}, t_{i+1}) \leq \frac{1}{2} R(x_i, t_i) = \frac{1}{2} r(t_i)^{-2} \sim \frac{1}{2} r(t_\infty)^{-2}$. If i is large then this is less than $r(t_{i+1})^{-2} \sim r(t_\infty)^{-2}$. Hence one goes to Substep B to find (x_{i+1}, t_{i+2}) with $R(x_{i+1}, t_{i+2}) = r(t_{i+2})^{-2}$. However, there is a double-sided bound on $\frac{\partial R}{\partial t}(x_{i+1}, t)$ for $t \in [t_{i+2}, t_{i+1}]$, coming from the curvature bound on a backward parabolic ball in [32, Lemma 70.1] and Shi's local estimates [32, Appendix D]. This bound implies that the amount of backward time required to go from a point with scalar curvature $R(x_{i+1}, t_{i+1}) \leq \frac{1}{2} r(t_i)^{-2} \sim \frac{1}{2} r(t_\infty)^{-2}$ to a point with scalar curvature $R(x_{i+1}, t_{i+2}) = r(t_{i+2})^{-2} \sim r(t_\infty)^{-2}$ satisfies

$$t_{i+1} - t_{i+2} \geq \text{const.}\, r(t_\infty)^2. \tag{3.10}$$

This contradicts the fact that $\lim_{i \to \infty} t_i = t_\infty$.

We note that the preceding argument can be made effective. This gives a upper bound N on the number of points (x_i, t_i), of the form $N = N(R_0, T)$. We now estimate the length of γ. The contribution to the length from segments arising from Substep B is at most T. The contribution from segments arising from Substep A is bounded above by $NC(r(0) + r(0)^2)$.

It remains to estimate the scalar curvature along γ. Along a portion of γ arising from Substep A, the scalar curvature is bounded above by $CR(x_{i-1}, t_{i-1}) \leq C \max(R_0, r(T)^{-2})$. Along a portion of γ arising from Substep B, the scalar curvature is bounded above by $r(T)^{-2}$. Thus we can take $R_1 = (C+1)(R_0 + r(T)^{-2})$. This proves the proposition. □

Finally, we give an estimate on the volume of the high-curvature region in a Ricci flow with surgery. This estimate will be used to prove the volume convergence statement in Theorem 1.2.

Proposition 3.11. *Given $T < \infty$, there are functions $\epsilon^T_{1,2} : [0, \infty) \to [0, \infty)$, with $\lim_{\overline{R} \to \infty} \epsilon^T_1(\overline{R}) = 0$ and $\lim_{\delta \to 0} \epsilon^T_2(\delta) = 0$, having the following property. Let $\mathcal{M}$ be a Ricci flow with surgery, with normalized initial condition. Let $\mathcal{V}(0)$ denote its initial volume. Given $\overline{R} > r(T)^{-2}$, if t is not a surgery time then let $\mathcal{V}^{\geq \overline{R}}(t)$ be the volume of the corresponding superlevel set of R in $\mathcal{M}_t$. If t is a surgery time, let $\mathcal{V}^{\geq \overline{R}}(t)$ be the volume of the corresponding superlevel set of R in the postsurgery* *manifold* $\mathcal{M}_t^+$. *Then if* $t \in [0,T]$, *we have*

$$\mathcal{V}^{\geq \overline{R}}(t) \leq \left(\epsilon_1^T(\overline{R}) + \epsilon_2^T(\delta(0)) \right) \mathcal{V}(0). \tag{3.12}$$

Proof. Suppose first that $\mathcal{M}$ is a smooth Ricci flow. Given $t \in [0,T]$, let $i_t : \mathcal{M}_0 \to \mathcal{M}_t$ be the identity map. For $x \in \mathcal{M}_0$, put

$$J_t(x) = \frac{i_t^* \, \mathrm{dvol}_{g(t)}}{\mathrm{dvol}_{g(0)}}(x). \tag{3.13}$$

Let $\gamma_x : [0,t] \to \mathcal{M}$ be the worldline of x. From the Ricci flow equation,

$$J_t(x) = e^{-\int_0^t R(\gamma_x(s)) \, ds}. \tag{3.14}$$

Suppose that $m \in \mathcal{M}_t$ satisfies $R(m) \geq \overline{R}$. Then $R(m) > r(t)^{-2}$. Let $x \in \mathcal{M}_0$ be the point where the worldline of m hits $\mathcal{M}_0$. From (A.9) we have

$$\frac{dR(\gamma_x(s))}{ds} \geq -\eta R(\gamma_x(s))^2 \tag{3.15}$$

as long as $R(\gamma_x(s)) \geq r(s)^{-2}$. Let t_1 be the smallest number so that $R(\gamma_x(s)) \geq r(s)^{-2}$ for all $s \in [t_1, t]$. Since $r(0) < \frac{1}{10}$, the normalized initial conditions imply that $t_1 > 0$. From (3.15), if $s \in [t_1, t]$ then

$$R(\gamma_x(s)) \geq \frac{1}{R(m)^{-1} + \eta(t-s)}. \tag{3.16}$$

In particular,

$$r(t_1)^{-2} = R(t_1) \geq \frac{1}{R(m)^{-1} + \eta(t-t_1)}. \tag{3.17}$$

From (3.16),

$$\int_{t_1}^t R(\gamma_x(s)) \, ds \geq -\frac{1}{\eta} \log \frac{R^{-1}(m)}{R^{-1}(m) + \eta(t-t_1)}. \tag{3.18}$$

Hence

$$\begin{aligned} e^{-\int_{t_1}^t R(\gamma_x(s)) \, ds} &\leq \left(\frac{R^{-1}(m)}{R^{-1}(m) + \eta(t-t_1)} \right)^{\frac{1}{\eta}} \\ &\leq \left(R(m) r(t_1)^2 \right)^{-\frac{1}{\eta}} \leq \left(R(m) r(T)^2 \right)^{-\frac{1}{\eta}}. \end{aligned} \tag{3.19}$$

On the other hand, for all $s \geq 0$, equation (A.11) gives

$$R(\gamma_x(s)) \ \geq \ - \frac{3}{1+2s}, \tag{3.20}$$

so

$$e^{-\int_0^{t_1} R(\gamma_x(s)) \, ds} \leq (1+2t_1)^{\frac{3}{2}}. \tag{3.21}$$

Thus

$$J_t(x) \le (1+2T)^{\frac{3}{2}} \left(R(m)r(T)^2\right)^{-\frac{1}{\eta}}. \tag{3.22}$$

Integrating over such $x \in \mathcal{M}_0$, arising from worldlines emanating from $\{m \in \mathcal{M}_t \,:\, R(m) \ge \overline{R}\}$, we conclude that

$$\mathcal{V}^{\ge \overline{R}}(t) \le (1+2T)^{\frac{3}{2}} \left(\overline{R}\, r(T)^2\right)^{-\frac{1}{\eta}} \mathcal{V}(0), \tag{3.23}$$

so the conclusion of the proposition holds in this case with

$$\epsilon_1^T(\overline{R}) = (1+2T)^{\frac{3}{2}} \left(\overline{R}\, r(T)^2\right)^{-\frac{1}{\eta}}. \tag{3.24}$$

Now suppose that $\mathcal{M}$ has surgeries. For simplicity of notation, we assume that t is not a surgery time; otherwise we replace $\mathcal{M}_t$ by $\mathcal{M}_t^+$. We can first apply the preceding argument to the subset of $\{m \in \mathcal{M}_t \,:\, R(m) \ge \overline{R}\}$ consisting of points whose worldline goes back to $\mathcal{M}_0$. The conclusion is that the volume of this subset is bounded above by $\epsilon_1^T(\overline{R})\mathcal{V}(0)$, where $\epsilon_1^T(\overline{R})$ is the same as in (3.24). Now consider the subset of $\{m \in \mathcal{M}_t \,:\, R(m) \ge \overline{R}\}$ consisting of points whose worldline does not go back to $\mathcal{M}_0$. We can cover such points by the forward images of surgery caps (or rather the subsets thereof which go forward to time t), for surgeries that occur at times $t_\alpha \le t$. Let $\mathcal{V}_{t_\alpha}^{cap}$ be the total volume of the surgery caps for surgeries that occur at time t_α. Let $\mathcal{V}_{t_\alpha}^{remove}$ be the total volume that is removed at time t_α by the surgery process. From the nature of the surgery process [32, Section 72], there is an increasing function $\delta' : (0,\infty) \to (0,\infty)$, with $\lim_{\delta\to 0} \delta'(\delta) = 0$, so that

$$\frac{\mathcal{V}_{t_\alpha}^{cap}}{\mathcal{V}_{t_\alpha}^{remove}} \le \delta'(\delta(0)). \tag{3.25}$$

This is essentially because the surgery procedure removes a long capped tube, whose length (relative to $h(t)$) is large if $\delta(t)$ is small, and replaces it by a hemispherical cap.

On the other hand, using (A.12),

$$\sum_{t_\alpha \le t} \mathcal{V}_{t_\alpha}^{remove} \le (1+2T)^{\frac{3}{2}} \mathcal{V}(0), \tag{3.26}$$

since surgeries up to time t cannot remove more volume than was initially present or generated by the Ricci flow. The time-t volume coming from forward worldlines of surgery caps is at most $(1+2T)^{\frac{3}{2}} \sum_{t_\alpha \le t} \mathcal{V}_{t_\alpha}^{cap}$. Hence the proposition is true if we take

$$\epsilon_2^T(\delta) = (1+2T)^3 \delta'(\delta). \tag{3.27}$$

□

4. The main convergence result

In this section we prove Theorem 4.1 except for the statement about the continuity of $\mathcal{V}_\infty$, which will be proved in Corollary 7.11.

The convergence assertion in Theorem 4.1 involves a sequence $\{\mathcal{M}^j\}$ of Ricci flows with surgery, where the functions r and κ are fixed, but $\delta_j \to 0$; hence ρ_j and h_j also go to zero. We will conflate these Ricci flows with surgery with their associated Ricci flow spacetimes; see Appendix A.9.

Theorem 4.1. *Let $\{\mathcal{M}^j\}_{j=1}^\infty$ be a sequence of Ricci flows with surgery with normalized initial conditions such that:*

- *The time-zero slices $\{\mathcal{M}^j_0\}$ are compact manifolds that lie in a compact family in the smooth topology.*
- $\lim_{j\to\infty} \delta_j(0) = 0$.

Then after passing to a subsequence, there is a singular Ricci flow $(\mathcal{M}^\infty, \mathfrak{t}_\infty, \partial_{\mathfrak{t}_\infty}, g_\infty)$, and a sequence of diffeomorphisms

$$\{\mathcal{M}^j \supset U_j \stackrel{\Phi^j}{\longrightarrow} V_j \subset \mathcal{M}^\infty\}, \tag{4.2}$$

so that

(1) $U_j \subset \mathcal{M}^j$ and $V_j \subset \mathcal{M}^\infty$ are open subsets.
(2) Given $\bar{t} > 0$ and $\overline{R} < \infty$, we have

$$U_j \supset \{m \in \mathcal{M}^j \mid \mathfrak{t}_j(m) \le \bar{t},\ R_j(m) \le \overline{R}\} \tag{4.3}$$

and

$$V_j \supset \{m \in \mathcal{M}^\infty \mid \mathfrak{t}_\infty(m) \le \bar{t},\ R_\infty(m) \le \overline{R}\} \tag{4.4}$$

for all sufficiently large j.
(3) Φ^j is time preserving, and the sequences $\{\Phi^j_* \partial_{\mathfrak{t}_j}\}_{j=1}^\infty$ and $\{\Phi^j_* g_j\}_{j=1}^\infty$ converge smoothly on compact subsets of $\mathcal{M}^\infty$ to $\partial_{\mathfrak{t}_\infty}$ and g_∞, respectively. (Note that by (4.4) any compact set $K \subset \mathcal{M}^\infty$ will lie in the interior of V_j, for all sufficiently large j.)
(4) For every $t \ge 0$, we have

$$\inf_{\mathcal{M}^\infty_t} R \ \ge \ -\frac{3}{1+2t} \tag{4.5}$$

and

$$\max(\mathcal{V}_j(t), \mathcal{V}_\infty(t)) \le V_0\,(1+2t)^{\frac{3}{2}}, \tag{4.6}$$

where $\mathcal{V}_j(t) = \operatorname{vol}(\mathcal{M}^j_t)$, $\mathcal{V}_\infty(t) = \operatorname{vol}(\mathcal{M}^\infty_t)$, and $V_0 = \sup_j \mathcal{V}_j(0) < \infty$.

(5) Φ^j *is asymptotically volume preserving : if* $\mathcal{V}_j, \mathcal{V}_\infty : [0,\infty) \to [0,\infty)$ *denote the respective volume functions* $\mathcal{V}_j(t) = \mathrm{vol}(\mathcal{M}^j_t)$ *and* $\mathcal{V}_\infty(t) = \mathrm{vol}(\mathcal{M}^\infty_t)$*, then* $\lim_{j\to\infty} \mathcal{V}_j = \mathcal{V}_\infty$ *uniformly on compact subsets of* $[0,\infty)$*.*

Proof. Because of the normalized initial conditions, the Ricci flow solution g_j is smooth on the time interval $\left[0, \frac{1}{100}\right]$. As a technical device, we first extend g_j backward in time to a family of metrics $g_j(t)$ which is smooth for $t \in \left(-\infty, \frac{1}{100}\right]$. To do this, using [16], there is an explicit smooth extension h_j of $g_j(t) - g_j(0)$ to the time interval $t \in \left(-\infty, \frac{1}{100}\right]$, with values in smooth covariant 2-tensor fields. The extension on $(-\infty, 0]$ depends on the time-derivatives of $g_j(t) - g_j(0)$ at $t = 0$ which, in turn, can be expressed in terms of $g_j(0)$ by means of repeated differentiation of the Ricci flow equation. Let $\phi : [0,\infty) \to [0,1]$ be a fixed nonincreasing smooth function with $\phi\Big|_{[0,1]} = 1$ and $\phi\Big|_{[2,\infty)} = 0$. Given $\epsilon > 0$, for $t < 0$ put

$$g_j(t) = g_j(0) + \phi\left(-\frac{t}{\epsilon}\right) h_j(t). \tag{4.7}$$

Using the precompactness of the space of initial conditions, we can choose ϵ small enough so that $g_j(t)$ is a Riemannian metric for all j and all $t \leq 0$. Then

(1) $g_j(t)$ is smooth in $t \in \left(-\infty, \frac{1}{100}\right]$,
(2) $g_j(t)$ is constant in t for $t \leq -2\epsilon$, and
(3) For $t \leq 0$, $g_j(t)$ has uniformly bounded curvature and curvature derivatives, independent of j.

Let $g_{\mathcal{M}^j}$ be the spacetime Riemannian metric on $\mathcal{M}^j$ (see Definition 1.8). Choose a basepoint $\star_j \in \mathcal{M}^j_0$.

After passing to a subsequence, we can assume that for all j,

$$\rho_j(0) \leq \sqrt{\frac{\mu}{j + r(j)^{-2}}}, \tag{4.8}$$

where $\mu = \mu(j)$ is the parameter of Lemma 3.1. Put $\psi_j(x,t) = R_j(x,t)^2 + t^2$, where R_j is the scalar curvature function of the Ricci flow metric g_j. Put $A_j = j^2$. If $(x,t) \in \psi_j^{-1}([0,A_j))$ then $|R_j(x,t)| \leq j$ and $|t| \leq j$. In particular,

$$|R_j(x,t)| \leq j \leq \mu\rho_j(0)^{-2} - r(j)^{-2} \leq \rho_j(0)^{-2} \leq \rho_j(t)^{-2}. \tag{4.9}$$

We claim that there are functions $\mathfrak{r}$, C_k so that Definition 2.19 holds for all j. Suppose that $\psi_j(x,t) \leq A_j$. When $t \leq 0$ there is nothing to prove, so we assume that $t \geq 0$. If $t \in \left[0, \frac{1}{100}\right]$ then the normalized initial conditions give uniform control on $g_j(t)$. If $t \geq \frac{1}{100}$ then

$$t\, r(t)^{-2} \geq \frac{1}{100} r(0)^{-2} > 1 > \sigma^2, \tag{4.10}$$

where $\sigma = \sigma(j)$ is the parameter from Lemma 3.1 and we use the assumption that $r(0) < \frac{1}{10}$ from the beginning of Section 3. Then

$$t > \sigma^2 r(t)^2 \geq \sigma^2 \frac{1}{|R(x,t)| + r(t)^{-2}} = \sigma^2 Q^{-1}, \tag{4.11}$$

so the parabolic ball $P_-(x, t, \sigma Q^{-\frac{1}{2}})$ of Lemma 3.1 does not intersect the initial time slice. As $|R_j(x,t)| \leq \mu \rho_j(0)^{-2} - r(j)^{-2}$ from (4.9), we can apply Lemma 3.1 to show that $(\mathcal{M}^j, g_{\mathcal{M}^j}, \psi_j)$ is (ψ_j, A_j)-controlled in the sense of Definition 2.1. Note that Lemma 3.1 gives bounds on the spatial and time derivatives of the curvature tensor of $g_j(t)$, which implies bounds on the derivatives of the curvature tensor of the spacetime metric $g_{\mathcal{M}_j}$.

Finally, the function $\mathfrak{t}_j$ and the vector field $\partial_{\mathfrak{t}_j}$, along with their covariant derivatives, are trivially bounded in terms of $g_{\mathcal{M}^j}$.

After passing to a subsequence we may assume that the number N of connected components of the initial time slice $\mathcal{M}^j_0$ is independent of j. Then the theorem follows from the special case when the initial times slices are connected, since we may apply it to the components separately. Therefore we are reduced to proving the theorem under the assumption that $\mathcal{M}^j_0$ is connected.

After passing to a subsequence, Theorem 2.20 now gives a pointed (ψ_∞, ∞)-controlled tuple

$$(\mathcal{M}^\infty, g_{\mathcal{M}^\infty}, \mathfrak{t}_\infty, (\partial_{\mathfrak{t}})_\infty, \star_\infty, \psi_\infty) \tag{4.12}$$

satisfying (1)-(4) of Theorem 2.20. We truncate $\mathcal{M}^\infty$ to the subset $\mathfrak{t}_\infty^{-1}([0,\infty))$. We now verify the claims of Theorem 4.1.

Part (1) of Theorem 4.1 follows from the statement of Theorem 2.20. Given $\overline{t} > 0$ and $\overline{R} < \infty$, Proposition 3.5 implies that there are $r = r(\overline{R}, \overline{t}) < \infty$ and $A = A(\overline{R}, \overline{t}) < \infty$ so that the set $\{m \in \mathcal{M}^j_{\leq \overline{t}} : R(m) \leq \overline{R}\}$ is contained in the metric ball $B(\star_j, r)$ in the Riemannian manifold $(\psi_j^{-1}([0,A)), g_{\mathcal{M}^j})$. Hence by the definition of ψ_j, and part (1) of Theorem 2.20, we get (4.3).

Pick $m_\infty \in \mathcal{M}^\infty$. Put $t_\infty = \mathfrak{t}(m_\infty)$. By part (4) of Theorem 2.20, we know that m_∞ belongs to V_j for large j. Now part (3) of Theorem 2.20 and the definition of ψ_j imply that

$$\psi_\infty(m_\infty) = R(m_\infty)^2 + t_\infty^2\,. \tag{4.13}$$

If j is large then it makes sense to define $(x_j, t_\infty) = (\Phi^j)^{-1}(m_\infty) \in \mathcal{M}^j$. Then for j large, $R(x_j, t_\infty) < R(m_\infty) + 1$ and so by Proposition 3.5, there is a time-preserving curve $\gamma_j : [0, t_\infty] \to \mathcal{M}^j$ such that

$$\max\left(\operatorname{length}_{g_{\mathcal{M}^j}}(\gamma_j), \max_{t \in [0,t_\infty]} R(\gamma_j(t))\right) < C = C(R(m_\infty), t_\infty)\,. \tag{4.14}$$

By part (1) of Theorem 2.20 we know that $\operatorname{Im}(\gamma_j) \subset U_j$ for large j. By part (2) of Theorem 2.20, for large j the map Φ^j is an almost-isometry. Hence there are $r = r(R(m_\infty), t_\infty) < \infty$ and $A = A(R(m_\infty), t_\infty) < \infty$ so that m_∞ is contained in the metric ball $B(\star_\infty, r)$ in the Riemannian manifold $((\psi_\infty)^{-1}([0, A)), g_{\mathcal{M}^\infty})$ Combining this with (4.13) and part (1) of Theorem 2.20 yields (4.4). This proves part (2) of Theorem 4.1.

Part (3) of Theorem 4.1 now follows from part (2) of the theorem and parts (2) and (3) of Theorem 2.20.

Equation (4.5) follows from (A.11) and the smooth approximation in part (3) of Theorem 4.1. Let $\mathcal{V}_j^{<\overline{R}}(t)$ be the volume of the $\overline{R}$-sublevel set for the scalar curvature function on $\mathcal{M}^j_t$. (If t is a surgery time, we replace $\mathcal{M}_t$ by $\mathcal{M}_t^+$.) Let $\mathcal{V}_\infty^{<\overline{R}}(t)$ be the volume of the $\overline{R}$-sublevel set for the scalar curvature function on $\mathcal{M}^\infty_t$. Then

$$\mathcal{V}_\infty(t) = \lim_{\overline{R}\to\infty} \mathcal{V}_\infty^{<\overline{R}}(t) = \lim_{\overline{R}\to\infty} \lim_{j\to\infty} \mathcal{V}_j^{<\overline{R}}(t) \le \limsup_{j\to\infty} \mathcal{V}_j(t). \tag{4.15}$$

Part (4) of Theorem 4.1 now follows from combining this with (A.12).

Next, we verify the volume convergence assertion (5). Let $\|\cdot\|_T$ denote the sup norm on $L^\infty([0,T])$. By parts (2) and (3) of Theorem 4.1, there is some $\epsilon_3^T(j, \overline{R}) > 0$, with $\lim_{j\to\infty} \epsilon_3^T(j, \overline{R}) = 0$, so that

$$\|\mathcal{V}_\infty^{<\overline{R}} - \mathcal{V}_j^{<\overline{R}}\|_T \le \epsilon_3^T(j, \overline{R}). \tag{4.16}$$

Also, if $\overline{S} > \overline{R}$ then

$$\|\mathcal{V}_\infty^{<\overline{S}} - \mathcal{V}_\infty^{<\overline{R}}\|_T = \lim_{j\to\infty} \|\mathcal{V}_j^{<\overline{S}} - \mathcal{V}_j^{<\overline{R}}\|_T \le \limsup_{j\to\infty} \|\mathcal{V}_j - \mathcal{V}_j^{<\overline{R}}\|_T. \tag{4.17}$$

Proposition 3.11 implies

$$\|\mathcal{V}_j - \mathcal{V}_j^{<\overline{R}}\|_T \le \left(\epsilon_1^T(\overline{R}) + \epsilon_2^T(\delta_j(0))\right) V_0. \tag{4.18}$$

Combining (4.17) and (4.18), and taking $\overline{S} \to \infty$, gives

$$\|\mathcal{V}_\infty - \mathcal{V}_\infty^{<\overline{R}}\|_T \le \epsilon_1^T(\overline{R})\, V_0. \tag{4.19}$$

Combining (4.16), (4.18) and (4.19) yields

$$\|\mathcal{V}_j - \mathcal{V}_\infty\|_T \le \left(2\epsilon_1^T(\overline{R}) + \epsilon_2^T(\delta_j(0))\right) V_0 + \epsilon_3^T(j, \overline{R}). \tag{4.20}$$

Given $\sigma > 0$, we can choose $\overline{R} < \infty$ so that $2\epsilon_1^T(\overline{R})V_0 < \frac{1}{2}\sigma$. Given this value of $\overline{R}$, we can choose J so that if $j \geq J$ then

$$\epsilon_2^T(\delta_j(0))V_0 + \epsilon_3^T(j, \overline{R}) < \frac{1}{2}\sigma. \tag{4.21}$$

Hence if $j \geq J$ then $\|\mathcal{V}_j - \mathcal{V}_\infty\|_T < \sigma$. This shows that $\lim_{j\to\infty} \mathcal{V}_j = \mathcal{V}_\infty$, uniformly on compact subsets of $[0, \infty)$, and proves part (5) of Theorem Theorem 4.1.

Finally, we check that $\mathcal{M}^\infty$ is a Ricci flow spacetime in the sense of Definition 1.5. Using part (3) of Theorem 4.1 one can pass the Hamilton-Ivey pinching condition, canonical neighborhoods, and the noncollapsing condition from the $\mathcal{M}^j$'s to $\mathcal{M}^\infty$, and so parts (b) and (c) of Definition 1.5 hold.

We now verify part (a) of Definition 1.5. We start with a statement about parabolic neighborhoods in $\mathcal{M}^\infty$. Given $T > 0$ and $\overline{R} < \infty$, suppose that $m_\infty \in \mathcal{M}^\infty$ has $\mathfrak{t}(m_\infty) \leq T$ and $R(m_\infty) \leq \overline{R}$. For large j, put $\widehat{m}_j = (\Phi^j)^{-1}(m_\infty) \in \mathcal{M}^j$. Lemma 3.1 supplies parabolic regions centered at the $\widehat{m}_j$'s which pass to $\mathcal{M}^\infty$. Hence for some $r = r(T, \overline{R}) > 0$, the forward and backward parabolic regions $P_+(m_\infty, r)$ and $P_-(m_\infty, r)$ are unscathed. There is some $K = K(T, \overline{R}) < \infty$ so that when equipped with the spacetime metric $g_{\mathcal{M}^\infty}$, the union $P_+(m_\infty, r) \cup P_-(m_\infty, r)$ is K-bilipschitz homeomorphic to a Euclidean parabolic region.

From (4.5), we know that R is bounded below on $\mathcal{M}^\infty_{[0,T]}$. In order to show that R is proper on $\mathcal{M}^\infty_{[0,T]}$, we need to show that any sequence in a sublevel set of R has a convergent subsequence. Suppose that $\{m_k\}_{k=1}^\infty \subset \mathcal{M}^\infty$ is a sequence with $\mathfrak{t}(m_k) \leq T$ and $R(m_k) \leq \overline{R}$. After passing to a subsequence, we may assume that $\mathfrak{t}(m_k) \to t_\infty \in [0, \infty)$. Then the regions $P(m_k, \frac{r}{100}, r^2) \cup P(m_k, \frac{r}{100}, -r^2)$ will intersect the time slice $\mathcal{M}_{t_\infty}$ in regions whose volume is bounded below by const. r^3. By the volume bound in part (4) of Theorem 4.1, only finitely many of these can be disjoint. Therefore, after passing to a subsequence, $\{m_k\}_{k=1}^\infty$ is contained in $P_+(m_\ell, \frac{r}{2}) \cup P_-(m_\ell, \frac{r}{2})$ for some ℓ. As $P_+(m_\ell, \frac{r}{2}) \cup P_-(m_\ell, \frac{r}{2})$ has compact closure in $P_+(m_\ell, r) \cup P_-(m_\ell, r)$, a subsequence of $\{m_k\}_{k=1}^\infty$ converges. This verifies part (a) of Definition 1.5. □

Part II

5. Basic properties of singular Ricci flows

In this section we prove some initial structural properties of Ricci flow spacetimes and singular Ricci flows. In Subsection 5.1 we justify the maximum principle on a Ricci flow spacetime and apply it to a get a lower scalar curvature bound. In Subsection 5.2 we prove some results about volume evolution for Ricci flow spacetimes which satisfy certain assumptions, that are satisfied in particular for singular Ricci flows. The main result in Subsection 5.3 says that if $\mathcal{M}$ is a singular Ricci flow and $\gamma_0, \gamma_1 : [t_0, t_1] \to \mathcal{M}$ are two time-preserving curves, such that $\gamma_0(t_1)$ and $\gamma_1(t_1)$ are in the same connected component of $\mathcal{M}_{t_1}$, then $\gamma_0(t)$ and $\gamma_1(t)$ are in the same connected component of $\mathcal{M}_t$ for all $t \in [t_0, t_1]$.

We recall the notion of a Ricci flow spacetime from Definition 1.1. In this section, we will consider it to only be defined for nonnegative time, i.e. $\mathfrak{t}$ takes value in $[0, \infty)$. We also recall the metrics $g_{\mathcal{M}}$ and $g^{qp}_{\mathcal{M}}$ from Definition 1.8. Let $n+1$ be the dimension of $\mathcal{M}$. Our notation is

- $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$,
- $\mathcal{M}_{[a,b]} = \mathfrak{t}^{-1}([a, b])$ and
- $\mathcal{M}_{\leq T} = \mathfrak{t}^{-1}([0, T])$.

5.1. **Maximum principle and scalar curvature.** In this subsection we prove a maximum principle on Ricci flow spacetimes and apply it to get a lower bound on scalar curvature.

Lemma 5.1. *Let $\mathcal{M}$ be a Ricci flow spacetime. Given $T \in (0, \infty)$, let X be a smooth vector field on $\mathcal{M}_{\leq T}$ with $X\mathfrak{t} = 0$. Given a smooth function $F : \mathbb{R} \times [0, T] \to \mathbb{R}$, suppose that $u \in C^\infty(\mathcal{M}_{\leq T})$ is a proper function, bounded above, which satisfies*

$$\partial_{\mathfrak{t}} u \leq \triangle_{g(\mathfrak{t})} u + Xu + F(u, \mathfrak{t}). \tag{5.2}$$

Suppose further that $\phi : [0, T] \to \mathbb{R}$ satifies

$$\partial_{\mathfrak{t}} \phi = F(\phi(\mathfrak{t}), \mathfrak{t}) \tag{5.3}$$

with initial condition $\phi(0) = \alpha \in \mathbb{R}$. If $u \leq \alpha$ on $\mathcal{M}_0$ then $u \leq \phi \circ \mathfrak{t}$ on $\mathcal{M}_{\leq T}$.

Proof. As in [49, Pf. of Theorem 3.1.1], for $\epsilon > 0$, we consider the ODE

$$\partial_t \phi_\epsilon = F(\phi_\epsilon(t), t) + \epsilon \tag{5.4}$$

with initial condition $\phi_\epsilon(0) = \alpha + \epsilon$. It suffices to show that for all small ϵ we have $u < \phi_\epsilon$ on $\mathcal{M}_{\leq T}$.

If not then we can find some $\epsilon > 0$ so that the property $u < \phi_\epsilon$ fails on $\mathcal{M}_{\leq T}$. As u is proper and bounded above, there is a first time t_0 so that the property fails on $\mathcal{M}_{t_0}$, and an $m \in \mathcal{M}_{t_0}$ so that $u(m) = \phi_\epsilon(t_0)$. The rest of the argument is the same as in [49, Pf. of Theorem 3.1.1]. $\square$

Lemma 5.5. *Let $\mathcal{M}$ be a Ricci flow spacetime. Suppose that for each $T \geq 0$, the scalar curvature R is proper and bounded below on $\mathcal{M}_{\leq T}$. Suppose that the initial scalar curvature bounded below by $-C$, for some $C \geq 0$. Then*

$$R(m) \ \geq \ - \frac{C}{1 + \frac{2}{n} C \mathfrak{t}(m)}. \tag{5.6}$$

Proof. Since R is proper on $\mathcal{M}_{\leq T}$, we can apply Lemma 5.1 to the evolution equation for $-R$ and follow the standard proof to get (5.6). □

5.2. **Volume.** In this subsection we first justify a Fubini-type statement for Ricci flow spacetimes. Then we show that certain standard volume estimates for smooth Ricci flows extend to the setting of Ricci flow spacetimes under two assumptions : first that the quasiparabolic metric is complete along worldlines that do not terminate at the time-zero slice, and second that in any time slice almost all points have worldlines that extend backward to time zero.

The Fubini-type statement is the following.

Lemma 5.7. *Let $\mathcal{M}$ be a Ricci flow spacetime. Given $0 < t_1 < t_2 < \infty$, suppose that $F : \mathcal{M}_{[t_1,t_2]} \to \mathbb{R}$ is measurable and bounded below. Suppose that $\mathcal{M}_{[t_1,t_2]}$ has finite volume with respect to $g_{\mathcal{M}}$. Then*

$$\int_{\mathcal{M}_{[t_1,t_2]}} F \ \mathrm{dvol}_{g_{\mathcal{M}}} = \int_{t_1}^{t_2} \int_{M_t} F \ \mathrm{dvol}_{g(t)} \ dt. \tag{5.8}$$

Proof. As F is bounded below on $\mathcal{M}_{[t_1,t_2]}$, for the purposes of the proof we can add a constant to F and assume that it is positive. Given $m \in \mathcal{M}_{[t_1,t_2]}$, there is an time-preserving embedding $e : (a,b) \times X \to \mathcal{M}$ with $e_*(\partial_s) = \partial_{\mathfrak{t}}$ (where $s \in (a,b)$) whose image is a neighborhood of m. We can cover $\mathcal{M}_{[t_1,t_2]}$ by a countable collection $\{P_i\}$ of such neighborhoods, with a subordinate partition of unity $\{\phi_i\}$. Let $e_i : (a_i, b_i) \times X_i \to P_i$ be the corresponding map. As

$$\int_{\mathcal{M}_{[t_1,t_2]}} \phi_i \, F \ \mathrm{dvol}_{g_{\mathcal{M}}} = \int_{t_1}^{t_2} \int_{e(X_i,t)} \phi_i \, F \ \mathrm{dvol}_{g(t)} \ dt, \tag{5.9}$$

we obtain

$$\int_{\mathcal{M}_{[t_1,t_2]}} F \ \mathrm{dvol}_{g_{\mathcal{M}}} = \sum_i \int_{\mathcal{M}_{[t_1,t_2]}} \phi_i \, F \ \mathrm{dvol}_{g_{\mathcal{M}}} \tag{5.10}$$
$$= \sum_i \int_{t_1}^{t_2} \int_{e_i(X_i,t)} \phi_i \, F \ \mathrm{dvol}_{g(t)} \ dt$$
$$= \int_{t_1}^{t_2} \sum_i \int_{e_i(X_i,t)} \phi_i \, F \ \mathrm{dvol}_{g(t)} \ dt$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{M}_t} F \ \mathrm{dvol}_{g(t)} \ dt.$$

This proves the lemma. □

We now prove some results about the behavior of volume in Ricci flow spacetimes.

Proposition 5.11. *Let $\mathcal{M}$ be a Ricci flow spacetime. Suppose that*

(a) The quasiparabolic metric $g^{qp}_{\mathcal{M}}$ of Definition 1.8 is complete along worldlines that do not terminate at the time-zero slice.
(b) If $B_t \subset \mathcal{M}_t$ is the set of points whose maximal worldline does not extend backward to time zero, then B_t has measure zero with respect to $\mathrm{dvol}_{g(t)}$, for each $t \geq 0$.
(c) The initial time slice $\mathcal{M}_0$ has volume $\mathcal{V}(0) < \infty$.
(d) The scalar curvature is proper and bounded below on time slabs $\mathcal{M}_{\leq T}$, and the initial time slice has scalar curvature bounded below by $-C$, with $C \geq 0$.

Let $\mathcal{V}(t)$ be the volume of $\mathcal{M}_t$. Then

(1) $\mathcal{V}(t) \leq \mathcal{V}(0)\left(1 + \frac{2}{n}Ct\right)^{\frac{n}{2}}$.
(2) R is integrable on $\mathcal{M}_{[t_1,t_2]}$.
(3) For all $t_1 < t_2$,

$$\mathcal{V}(t_2) - \mathcal{V}(t_1) = -\int_{\mathcal{M}_{[t_1,t_2]}} R \ \mathrm{dvol}_{g_{\mathcal{M}}} . \tag{5.12}$$

(4) The volume function $\mathcal{V}(t)$ is absolutely continuous.
(5) Given $0 \leq t_1 \leq t_2 < \infty$, we have

$$\mathcal{V}(t_2) - \mathcal{V}(t_1) \leq \frac{C}{1 + \frac{2}{n}Ct_1}\left(1 + \frac{2}{n}Ct_2\right)^{\frac{n}{2}} \mathcal{V}(0) \cdot (t_2 - t_1). \tag{5.13}$$

Proof. Given $0 \leq t_1 < t_2 < \infty$, there is a partition $\mathcal{M}_{[t_1,t_2]} = \mathcal{M}'_{[t_1,t_2]} \cup \mathcal{M}''_{[t_1,t_2]} \cup \mathcal{M}'''_{[t_1,t_2]}$, where

(1) A point in $\mathcal{M}'_{[t_1,t_2]}$ has a worldline that intersects both M_{t_1} and M_{t_2}.
(2) A point in $\mathcal{M}''_{[t_1,t_2]}$ has a worldline that intersects M_{t_1} but not M_{t_2}.
(3) A point in $\mathcal{M}'''_{[t_1,t_2]}$ has a worldline that does not intersect M_{t_1}.

By our assumptions, $\mathcal{M}'''_{[t_1,t_2]} \subset \bigcup_{t\in[t_1,t_2]} B_t$ has measure zero with respect to $\operatorname{dvol}_{g_{\mathcal{M}}}$; c.f. the proof of Lemma 5.7. Put $X_1 = \mathcal{M}'_{[t_1,t_2]} \cap M_{t_1}$ and $X_2 = \mathcal{M}''_{[t_1,t_2]} \cap M_{t_1}$. For $s \in [t_1, t_2]$, there is a natural embedding $i_s : X_1 \to \mathcal{M}_s$ coming from flowing along worldlines. The complement $M_{t_2} - i_{t_2}(X_1)$ has measure zero. Thus

$$\mathcal{V}(t_2) = \int_{M_{t_2}} \operatorname{dvol}_{g(t_2)} = \int_{X_1} i^*_{t_2} \operatorname{dvol}_{g(t_2)} . \tag{5.14}$$

and

$$\mathcal{V}(t_1) = \int_{X_1} \operatorname{dvol}_{g(t_1)} + \int_{X_2} \operatorname{dvol}_{g(t_1)} \tag{5.15}$$

Given $x \in X_1$, let $\gamma_x : [t_1, t_2] \to \mathcal{M}_{[t_1,t_2]}$ be its worldline. Put

$$J_s(x) = \frac{i^*_s \operatorname{dvol}_{g(s)}}{\operatorname{dvol}_{g(t_1)}}(x). \tag{5.16}$$

From the Ricci flow equation,

$$J_s(x) = e^{-\int_{t_1}^{s} R(\gamma_x(u))\, du}. \tag{5.17}$$

Using Lemma 5.5,

$$\begin{aligned} \mathcal{V}(t_2) &= \int_{X_1} J_{t_2}(x) \operatorname{dvol}_{g(t_1)}(x) \le \int_{X_1} e^{\int_{t_1}^{t_2} \frac{C}{1+\frac{2C}{n}u}\, du} \operatorname{dvol}_{g(t_1)} \\ &= \mathcal{V}(t_1) \left(\frac{1 + \frac{2C}{n} t_2}{1 + \frac{2C}{n} t_1} \right)^{\frac{n}{2}} . \end{aligned} \tag{5.18}$$

When $t_1 = 0$, this proves part (1) of the proposition.

Next,

$$\begin{aligned}(5.19)\quad \int_{X_1}(i_{t_2}^*\,\mathrm{dvol}_{g(t_2)}-\mathrm{dvol}_{g(t_1)}) &= \int_{X_1}(J_{t_2}(x)-1)\,\mathrm{dvol}_{g(t_1)}(x)\\ &= \int_{X_1}\int_{t_1}^{t_2}\frac{dJ_s(x)}{ds}\,ds\ \mathrm{dvol}_{g(t_1)}(x)\\ &= -\int_{X_1}\int_{t_1}^{t_2}R(\gamma_x(s))\,J_s(x)\,ds\ \mathrm{dvol}_{g(t_1)}(x)\\ &= -\int_{t_1}^{t_2}\int_{X_1}R\,i_s^*\,\mathrm{dvol}_{g(s)}\ ds\\ &= -\int_{\mathcal{M}'_{[t_1,t_2]}}R\ \mathrm{dvol}_{g_{\mathcal{M}}},\end{aligned}$$

where we applied Lemma 5.7 with $F = R\,1_{\mathcal{M}'_{[t_1,t_2]}}$ in the last step.

Given $x\in X_2$, let $e(x)\in(t_1,t_2)$ be the supremal extension time of its worldline. From the completeness of $g^{qp}_{\mathcal{M}}$ on worldlines that do not terminate at the time-zero slice,

$$\int_{t_1}^{e(x)}R(\gamma_x(u))\,du=\infty. \tag{5.20}$$

Thus $\lim_{s\to e(x)}J_s(x)=0$, so

$$\begin{aligned}(5.21)\quad -\int_{X_2}\mathrm{dvol}_{g(t_1)} &= \int_{X_2}\int_{t_1}^{e(x)}\frac{dJ_s(x)}{ds}\,ds\ \mathrm{dvol}_{g(t_1)}(x)\\ &= -\int_{X_2}\int_{t_1}^{e(x)}R(\gamma_x(s))\,J_s(x)\,ds\ \mathrm{dvol}_{g(t_1)}(x)\\ &= -\int_{X_2}\int_{t_1}^{e(x)}R\,ds\,i_s^*\,\mathrm{dvol}_{g(s)}\\ &= -\int_{\mathcal{M}''_{[t_1,t_2]}}R\ \mathrm{dvol}_{g_{\mathcal{M}}}.\end{aligned}$$

Part (3) of the proposition follows from combining equations (5.14), (5.15), (5.19) and (5.21). Part (2) of the proposition is now an immediate consequence.

By Lemma 5.7 and part (3) of the proposition, the function $t\mapsto \int_{\mathcal{M}_t}R\,\mathrm{dvol}$ is locally-L^1 on $[0,\infty)$ with respect to Lebesgue measure. This implies part (4) of the proposition.

To prove part (5) of the proposition, using Lemma 5.5 and parts (1) and (3) of the proposition, we have

$$\begin{aligned}
(5.22) \qquad \mathcal{V}(t_2) - \mathcal{V}(t_1) &= - \int_{t_1}^{t_2} \int_{\mathcal{M}_t} R \ \mathrm{dvol}_{g(t)} \ dt \\
&\le \int_{t_1}^{t_2} \frac{C}{1 + \frac{2}{n} C t} \mathcal{V}(t) \, dt \\
&\le \frac{C}{1 + \frac{2}{n} C t_1} \left(1 + \frac{2}{n} C t_2 \right)^{\frac{n}{2}} \mathcal{V}(0) \cdot (t_2 - t_1).
\end{aligned}$$

This proves the proposition. □

5.3. **Basic structural properties of singular Ricci flows.** In this subsection we collect a number of properties of singular Ricci flows, the latter being in the sense of Definition 1.5. We first show the completeness of the quasiparabolic metric.

Lemma 5.23. *If $\mathcal{M}$ is a singular Ricci flow then the quasiparabolic metric $g^{qp}_{\mathcal{M}}$ of Definition 1.8 is complete away from the time-zero slice.*

Proof. Suppose that $\gamma : [0, \infty) \to \mathcal{M}$ is a curve that goes to infinity in $\mathcal{M}$, with $\mathfrak{t} \circ \gamma$ bounded away from zero. We want to show that its quasiparabolic length is infinite. If $\mathfrak{t} \circ \gamma$ is not bounded then the quasiparabolic length of γ is infinite from the definition of $g^{qp}_{\mathcal{M}}$, so we can assume that $\mathfrak{t} \circ \gamma$ takes value in some interval $[0, T]$. Since R is proper and bounded below on $\mathcal{M}_{\le T}$, we have $\lim_{s \to \infty} R(\gamma(s)) = \infty$. After truncating the initial part of γ, we can assume that $R(\gamma(s)) \ge r(T)^{-2}$ for all s. In particular, each point $\gamma(s)$ is in a canonical neighborhood. Now

$$(5.24) \qquad \frac{dR(\gamma(s))}{ds} = \frac{\partial R}{\partial t} \frac{dt}{ds} + \langle \nabla R, \gamma' \rangle_g,$$

so

$$(5.25) \qquad \left| \frac{dR(\gamma(s))}{ds} \right|^2 \le 2 \left(\left| \frac{\partial R}{\partial t} \right|^2 \left| \frac{dt}{ds} \right|^2 + |\nabla R|_g^2 \, |\gamma'|_g^2 \right).$$

The gradient estimates in (A.9), of the form

$$(5.26) \qquad |\nabla R| < \mathrm{const.} \, R^{\frac{3}{2}}, \qquad |\partial_t R| < \mathrm{const.} \, R^2,$$

are valid for points in a canonical neighborhood of a singular Ricci flow solutions. Then

$$(5.27) \qquad \left| \frac{dR(\gamma(s))}{ds} \right|^2 \le C R^2 \left| \frac{d\gamma}{ds} \right|^2_{g^{qp}_{\mathcal{M}}}$$

for some universal $C < \infty$. We deduce that

$$\int_0^\infty \left| R^{-1} \frac{dR}{ds} \right| (\gamma(s))\, ds \;\leq\; C^{\frac{1}{2}} \int_0^\infty \left| \frac{d\gamma}{ds} \right|_{g^{qp}_{\mathcal{M}}} ds. \tag{5.28}$$

Since the left-hand side is infinite, the quasiparabolic length of γ must be infinite. This proves the lemma. □

The next lemma gives the existence of unscathed forward and backward parabolic neighborhoods of a certain size around a point, along with geometric bounds on those neighborhoods.

Lemma 5.29. *Let $\mathcal{M}$ be a singular Ricci flow. Given $T < \infty$, there are numbers $\sigma = \sigma(T) > 0$, $i_0 = i_0(T) > 0$ and $A_k = A_k(T) < \infty$, $k \geq 0$, with the following property. If $m \in \mathcal{M}$ and $\mathfrak{t}(m) \leq T$, put $Q = |R(m)| + r(\mathfrak{t}(m))^{-2}$. Then*

(1) The forward parabolic ball $P_+(m, \sigma Q^{-\frac{1}{2}})$ and the backward parabolic ball $P_-(m, \sigma Q^{-\frac{1}{2}})$ are unscathed.
(2) $|\operatorname{Rm}| \leq A_0 Q$, $\operatorname{inj} \geq i_0 Q^{-\frac{1}{2}}$ and $|\nabla^k \operatorname{Rm}| \leq A_k Q^{1+\frac{k}{2}}$ on the union $P_+(m, \sigma Q^{-\frac{1}{2}}) \cup P_-(m, \sigma Q^{-\frac{1}{2}})$ of the forward and backward parabolic balls.

Proof. The proof is the same as that of Lemma 3.1. □

The next two propositions characterize the high-scalar-curvature part of a time slice.

Proposition 5.30. *Let $\mathcal{M}$ be a singular Ricci flow. For all $\epsilon_1 > 0$, there is a scale function $r_1 : [0, \infty) \to (0, \infty)$ with $r_1(t) \leq r(t)$, such that for every point $(x, t) \in \mathcal{M}$ with $R(x, t) > r_1(t)^{-2}$ the ϵ_1-canonical neighborhood assumption holds, and moreover $(\mathcal{M}, (x, t))$ is ϵ_1-modelled on a κ-solution. (Recall that here $\kappa = \kappa(t)$, i.e. we are suppressing the time dependence in our notation.)*

Proof. The proof is similar to the proof of [32, Theorem 52.7]. The main difference is that there are several places in the proof where one applies Lemma 5.29, because time slices are not assumed to be compact as in [32, Theorem 52.7], and are therefore not necessarily complete. □

Proposition 5.31. *Let $\mathcal{M}$ be a singular Ricci flow. For any $T < \infty$ and $\hat{\epsilon} > 0$, there exist $C_1 = C_1(\hat{\epsilon}, T) < \infty$ and $\overline{R} = \overline{R}(\hat{\epsilon}, T) < \infty$ such that for every $t \leq T$, each connected component of the time slice $\mathcal{M}_t$ has finitely many ends, each of which is an $\hat{\epsilon}$-horn. Morever for every $\overline{R}' \geq \overline{R}$, the superlevel set $\mathcal{M}_t^{>\overline{R}'} = \{m \in \mathcal{M}_t \;:\; R(m) > \overline{R}'\}$ is contained in a finite disjoint union of properly embedded three-dimensional submanifolds-with-boundary $\{N_i\}_{i=1}^k$ such that*

(1) Each N_i is contained in the superlevel set $\mathcal{M}_t^{>C_1^{-1}\overline{R}'}$.
(2) The boundary ∂N_i has scalar curvature in the interval $(C_1^{-1}\overline{R}, C_1\overline{R})$.
(3) For each i one of the following holds:
  (a) N_i is diffeomorphic to $S^1 \times S^2$ or $I \times S^2$ and consists of $\hat{\epsilon}$-neck points. Note that here the interval I can be open (a double horn), closed (a tube) or half-open (a horn).
  (b) N_i is diffeomorphic to $D^3 = \overline{B^3}$ or $\mathbb{R}P^3 - B^3$ and its boundary $\partial N_i \simeq S^2$ consists of $\hat{\epsilon}$-neck points.
  (c) N_i is diffeomorphic to S^3, $\mathbb{R}P^3$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.
  (d) N_i is diffeomorphic to a spherical space form other than S^3 or $\mathbb{R}P^3$.
(4) In cases (b) and (c) of (3), if $S_i \subset N_i$ is a subset consisting of non-$\hat{\epsilon}$-neck points, such that for any two distinct elements $s_1, s_2 \in S_i$ we have $d_{\mathcal{M}_t}(s_1, s_2) > C_1 R^{-\frac{1}{2}}(s_1)$, then the cardinality $|S_i|$ is at most 1 in case (b) and at most 2 in case (c).
(5) Each N_i with nonempty boundary has volume at least $C_1^{-1}\left(\overline{R}'\right)^{-\frac{3}{2}}$.

Proof. The proof is the same as in [32, Section 67]. □

We now prove a statement about preservation of connected components when going backwards in time.

Proposition 5.32. *Let $\mathcal{M}$ be a singular Ricci flow. If $\gamma_0, \gamma_1 : [t_0, t_1] \to \mathcal{M}$ are time-preserving curves, and $\gamma_0(t_1)$, $\gamma_1(t_1)$ lie in the same connected component of $\mathcal{M}_{t_1}$, then $\gamma_0(t)$, $\gamma_1(t)$ lie in the same connected component of $\mathcal{M}_t$ for every $t \in [t_0, t_1]$.*

Proof. The idea of the proof is to consider the values of t for which $\gamma_0(t)$ can be joined to $\gamma_1(t)$ in $\mathcal{M}_t$, and the possible curves c_t in $\mathcal{M}_t$ that join them. Among all such curves c_t, we look at one which minimizes the maximum value of scalar curvature along the curve. Call this threshold value of scalar curvature $R_{crit}(t)$. We will argue that c_t can only intersect the high-scalar-curvature part of $\mathcal{M}_t$ in its neck-like regions. But the scalar curvature in a neck-like region is strictly decreasing when one goes backward in time; this will imply that $R_{crit}(t)$, when large, is decreasing when one goes backward in time, from which the lemma will follow.

To begin the formal proof, suppose that the lemma is false. Let $S \subset [t_0, t_1]$ be the set of times $t \in [t_0, t_1]$ such that $\gamma_0(t)$ and $\gamma_1(t)$ lie in the same connected component of $\mathcal{M}_t$; note that S is open. Put $\hat{t} = \inf\{t \mid [t, t_1] \subset S\}$. Then $\hat{t} > \frac{1}{100}$ since $\mathcal{M}_{[0, \frac{1}{100}]}$ is a product. Also, for $i \in \{0, 1\}$ and $t \in [\hat{t}, t_1]$ close to $\hat{t}$, if $\hat{\gamma}_i(t) \in \mathcal{M}_t$ is the worldline of $\gamma_i(\hat{t})$ at time t then $\hat{\gamma}_i(t)$ lies in the same connected component of $\mathcal{M}_t$ as $\gamma_i(t)$. Therefore, after reducing t_1 if necessary, we may assume without loss of generality that γ_i is a worldline.

For $t \in [t_0, t_1]$ and $\overline{R} < \infty$, put $\mathcal{M}_t^{\leq \overline{R}} = \{m \in \mathcal{M}_t \mid R(m) \leq \overline{R}\}$. For $t \in (\hat{t}, t_1]$, let $\mathcal{R}_t$ be the set of $\overline{R} \in \mathbb{R}$ such that $\gamma_0(t)$ and $\gamma_1(t)$ lie in the same connected component of $\mathcal{M}_t^{\leq \overline{R}}$. Put $R_{crit}(t) = \inf \mathcal{R}_t$. Since $R : \mathcal{M}_t \to \mathbb{R}$ is proper, the sets $\{\mathcal{M}_t^{\leq \overline{R}}\}_{\overline{R} > R_{crit}(t)}$ are compact and nested, which implies that $\gamma_0(t)$ and $\gamma_1(t)$ lie in the same component of $\mathcal{M}_t^{\leq R_{crit}(t)} = \bigcap_{\overline{R} > R_{crit}(t)} \mathcal{M}_t^{\leq \overline{R}}$, i.e. $R_{crit}(t) \in \mathcal{R}_t$.

By Lemma 5.29, there is a $C < \infty$ such that for any $t \in (\frac{1}{100}, t_1]$ and any $m \in \mathcal{M}_t$ there is a $\tau = \tau(t_1, R(m)) > 0$, where τ is a continuous function that is nonincreasing in $R(m)$, such that the worldline γ_m of m is defined and satisfies

$$\left| \frac{\partial R}{\partial t} \right| (\gamma_m(t)) < C \cdot R(m)^2 \tag{5.33}$$

in the time interval $(t - \tau, t + \tau)$. Now for $t \in (\hat{t}, t_1]$ and t' satisfying $|t' - t| < \tau(t_1, R_{crit}(t))$, let $Z_{t,t'} \subset \mathcal{M}_{t'}$ denote the result of flowing $\mathcal{M}_t^{\leq R_{crit}(t)}$ under $\partial_{\mathfrak{t}}$ for an elapsed time $t' - t$. Then $Z_{t,t'}$ is well-defined and contains $\gamma_0(t')$ and $\gamma_1(t')$ in the same component, so

$$R_{crit}(t') \leq \max_{Z_{t,t'}} R \leq R_{crit}(t) + C \cdot R_{crit}(t)^2 \cdot |t - t'|. \tag{5.34}$$

This implies that $R_{crit} : (\hat{t}, t_1] \to \mathbb{R}$ is locally Lipschitz (in particular continuous) and that

$$R_{crit}(t) \to \infty \tag{5.35}$$

as $t \to \hat{t}$ from the right; otherwise there would be a sequence $t_i \to \hat{t}$ along which R_{crit} is uniformly bounded above by some $\hat{R} < \infty$, which would allow us to construct $Z_{t_i,\hat{t}}$ whenever $|\hat{t} - t_i| < \tau(t_1, \hat{R})$, contradicting the definition of $\hat{t}$.

We now concentrate on t close to $\hat{t}$. Suppose that for some $t \in (\hat{t}, t_1]$ we have

$$R_{crit}(t) \gg \max(r^{-2}(t_1), \max_{[t_0,t_1]} R \circ \gamma_0, \max_{[t_0,t_1]} R \circ \gamma_1) \,. \tag{5.36}$$

Then by Proposition 5.31 the superlevel set $\mathcal{M}_t^{>(R_{crit}(t)-1)}$ is contained in a finite union of components $\{N_{i,t}\}_{i=1}^{k_t}$, each diffeomorphic to one of the possibilities (a)-(d) in the statement of the proposition. Let X_t be the result of removing from $\mathcal{M}_t$ the interior of each $N_{i,t}$ that is not of type (a). Since $\gamma_0(t)$ and $\gamma_1(t)$ lie outside $\bigcup_{i=1}^{k_t} N_{i,t}$, and each N_i of type (b)-(d) has at most one boundary component, it follows that $\gamma_0(t)$ and $\gamma_1(t)$ lie in the same connected component of X_t.

For $t' < t$ close to t, let $X_{t'} \subset \mathcal{M}_{t'}$ be the result of flowing X_t under $\partial_{\mathfrak{t}}$. Now $Y_t = X_t \cap \mathcal{M}_t^{>(R_{crit}(t)-1)}$ consists of ϵ-neck points, and at such a point the scalar curvature is strictly increasing as a function of time. Hence there is a $\tau_1 > 0$ such that the worldline $\gamma_m : [t - \tau_1, t] \to \mathcal{M}$ of any $m \in Y_t$ satisfies

$$R(\gamma_m(t')) < R(m) \leq R_{crit}(t) \tag{5.37}$$

for $t' \in [t - \tau_1, t)$. This implies that $R_{crit}(t') < R_{crit}(t)$ when $t' < t$ is close to t, again under the assumption (5.36), which contradicts (5.35). □

We now state a result about connecting a point in a singular Ricci flow to the time-zero slice by a curve whose length is quantitatively bounded, and along which the scalar curvature is quantitatively bounded.

Proposition 5.38. *Let $\mathcal{M}$ be a singular Ricci flow. Given $T, R_0 < \infty$, there are $L = L(R_0, T) < \infty$ and $R_1 = R_1(R_0, T) < \infty$ with the following property. Suppose that $R(m_0) \leq R_0$, with $t_0 = \mathfrak{t}(m_0) \leq T$. Then there is a time preserving curve $\gamma : [0, t_0] \to \mathcal{M}$ with $\gamma(t_0) = m_0$ with* $\mathrm{length}(\gamma) \leq L$ *so that $R(\gamma(t)) \leq R_1$ for all $t \in [0, t_0]$.*

Proof. The proof is the same as that of Proposition 3.5. □

Finally, we give a compactness result for the space of singular Ricci flows.

Proposition 5.39. *Let $\{\mathcal{M}^i\}_{i=1}^\infty$ be a sequence of singular Ricci flows with a fixed choice of parameters ϵ, r and κ whose initial conditions $\{\mathcal{M}_0^i\}_{i=1}^\infty$ lie in a compact family in the smooth topology. Then a subsequence of $\{\mathcal{M}^i\}_{i=1}^\infty$ converges, in the sense of Theorem 4.1, to a singular Ricci flow $\mathcal{M}^\infty$.*

Proof. Using Proposition 5.38, the proof is the same as that of Proposition 4.1. □

Remark 5.40. In the setting of Proposition 5.39, if we instead assume that the (normalized) initial conditions have a uniform upper volume bound then we can again take a convergent subsequence to get a limit Ricci flow spacetime $\mathcal{M}^\infty$. In this case the time-zero slice $\mathcal{M}_0^\infty$ will generally only be $C^{1,\alpha}$-regular, but $\mathcal{M}^\infty$ will be smooth on $\mathfrak{t}^{-1}(0, \infty)$.

6. Stability of necks

Fix $\kappa > 0$. We recall the notion of a κ-solution from Appendix A.5. In this section we establish a new dynamical stability property of caps and necks in κ-solutions, which we will use in Section 7 to show that a bad worldline $\gamma : I \to \mathcal{M}$ is confined to a cap region as t approaches the blow-up time $\inf I$.

Conceptually speaking, the stability assertion is that among pointed κ-solutions, the round cylinder is an attractor under backward flow; similarly, under forward flow, non-neck points form an attractor. The rough idea of the argument is as follows. Suppose that $\delta \ll 1$ and $(x_0, 0)$ is a δ-neck in a κ-solution $\mathcal{M}$. Then (x_0, t) will also be neck-like as long as $t < 0$ is not too negative. One also knows that $\mathcal{M}$ has an asymptotic soliton $\hat{\mathcal{M}}$ as $t \to -\infty$, which is a shrinking round cylinder. Thus $\mathcal{M}$ tends toward neck-like geometry as t approaches $-\infty$, which is the desired stability property. However, there is a catch here: the asymptotic soliton is a pointed limit of a sequence where the basepoints are not fixed, and so a priori it says nothing about the asymptotic geometry near (x_0, t) as $t \to -\infty$. To address this we exploit the behavior of the l-function.

6.1. **The main stability asssertion.** We recall the notation $\hat{\mathcal{M}}(t)$ from Section A.1 for the parabolic rescaling of a Ricci flow spacetime $\mathcal{M}$. We recall the notation *Cyl* and *Sphere* from Section A.1 for the standard Ricci flow solutions. We also recall the notion of one Ricci flow spacetime being ϵ-modelled on another one, from Appendix A.2.

Theorem 6.1. *There is a $\delta_{neck} = \delta_{neck}(\kappa) > 0$, and for all $\delta_0, \delta_1 \leq \delta_{neck}$ there is a $T = T(\delta_0, \delta_1, \kappa) \in (-\infty, 0)$ with the following property. Suppose that*

(a) $\mathcal{M}$ is a κ-solution with noncompact time slices,
(b) $(x_0, 0) \in \mathcal{M}$,
(c) $R(x_0, 0) = 1$, and
(d) $(\mathcal{M}, (x_0, 0))$ is a δ_0-neck.

Then either $\mathcal{M}$ is isometric to the $\mathbb{Z}_2$-quotient of a shrinking round cylinder, or for all $t \in (-\infty, T]$,

(1) $(\mathcal{M}, (x_0, t))$ is a δ_1-neck and
(2) $\left(\hat{\mathcal{M}}(-t), g, (x_0, -1)\right)$ is δ_1-close to $(Cyl, (y_0, -1))$, where $y_0 \in S^2 \times \mathbb{R}$.

We recall the notion of a generalized neck from Appendix A.2.

Corollary 6.2. *If $\delta_{neck} = \delta_{neck}(\kappa)$ as in the previous theorem, then there is a $T = T(\kappa) < (-\infty, 0)$ such that if $\mathcal{M}$ is a κ-solution with noncompact time slices, $(x_0, 0) \in \mathcal{M}$, $R(x_0, 0) = 1$, and $(x_0, 0)$ is a generalized δ_{neck}-neck, then (x_0, T) is a generalized $\frac{\delta_{neck}}{4}$-neck, and $R(x_0, T) < \frac{1}{4}$.*

6.2. **Convergence of l-functions, asymptotic l-functions and asymptotic solitons.** In this subsection we prove some preparatory results about the convergence of l-functions with regard to a convergent sequence of κ-solutions.

Suppose that we are given that $R(x_1, t_1) \leq C$ at some point (x_1, t_1) in a κ-solution. By compactness of the space of pointed normalized κ-solutions (see Appendix A.5), we obtain $R(x, t_1) \leq F(C, d_{t_1}(x, x_1))$ for some universal function F. Since R is pointwise nondecreasing in time in a κ-solution, we also have $R(x, t) \leq F(C, d_{t_1}(x, x_1)$ whenever $t \leq t_1$.

Lemma 6.3 (Curvature bound at basepoint)**.** *Let $\{(\mathcal{M}^j, (x_j, 0))\}$ be a sequence of pointed κ-solutions, with $\sup_j R(x_j, 0) < \infty$, and let $l_j : \mathcal{M}^j_{<0} \to (0, \infty)$ be the reduced distance function with spacetime basepoint at $(x_j, 0)$. Then after passing to a subsequence, we have convergence of tuples*

$$(\mathcal{M}^j, (x_j, 0), l_j) \to (\mathcal{M}^\infty, (x_\infty, 0), l_\infty)\,. \tag{6.4}$$

Here the Ricci flow convergence is the usual smooth convergence on parabolic balls, and l_j converges to a locally Lipschitz function l_∞ uniformly on compact subsets of $\mathcal{M}^\infty_{<0}$, after composition with the comparison map.

Proof. Since $R(x_j, 0)$ is uniformly bounded above, the compactness of the space of pointed normalized κ-solutions implies that a subsequence, which we relabel as $\{(\mathcal{M}^j, (x_j, 0), l_j)\}$, converges in the pointed smooth topology. Along the curve $\{x_j\} \times (-\infty, 0) \subset \mathcal{M}^j$, the scalar curvature is bounded above by $R(x_j, 0)$. For any $a < b < 0$, this gives a uniform upper bound on l_j on $\{x_j\} \times [a, b] \subset \mathcal{M}^j$. From (A.6), we get that l is uniformly bounded on balls of the form $B(x_j, b, r) \subset \mathcal{M}^j_b$. Then from (A.7), we get that l is uniformly bounded on parabolic balls of the form $P(x_j, b, r, a - b) \subset \mathcal{M}^j$. Using (A.5) and passing to a subsequence, we get convergence of $\{l_j\}$ to some l_∞, uniformly on compact subsets of $\mathcal{M}^\infty_{<0}$. $\square$

Proposition 6.5 (No curvature bound at basepoint)**.** *Let $\{(\mathcal{M}^j, (x_j, 0))\}$ be a sequence of pointed κ-solutions, and let $l_j : \mathcal{M}^j_{<0} \to (0, \infty)$ be the reduced distance function with spacetime basepoint at $(x_j, 0)$. Suppose* *that* $\{(y_j,-1)\in \mathcal{M}^j_{-1}\}$ *is a sequence satisfying* $\sup_j l_j(y_j,-1)<\infty$. *Then after passing to a subsequence, the tuples* $(\mathcal{M}^j,(y_j,-1),l_j)$ *converge to a tuple* $(\mathcal{M}^\infty,(y_\infty,-1),l_\infty)$ *where:*

(1) $\mathcal{M}^\infty$ *is a* κ*-solution defined on* $(-\infty,0)$*, and the convergence* $(\mathcal{M}^j,(y_j,-1))\to(\mathcal{M}^\infty,(y_\infty,-1))$ *is smooth on compact subsets of* $\mathcal{M}^\infty_{<0}$.

(2) $\{l_j\}$ *converges to a locally Lipschitz function* l_∞ *uniformly on compact subsets of* $\mathcal{M}^\infty_{<0}$*, after composition with the comparison diffeomorphisms.*

(3) *The reduced volume functions* $\tilde V_j:(-\infty,0)\to(0,\infty)$ *converge uniformly on compact sets to the function* $\tilde V_\infty:(-\infty,0)\to(0,\infty)$*, where*

$$\tilde V_\infty(t)=(-t)^{-\frac{n}{2}}\int_{\mathcal{M}^\infty_t} e^{-l_\infty}\,\mathrm{dvol}_{g(t)}\,. \tag{6.6}$$

(4) *The function* l_∞ *satisfies the differential inequalities (24.6) and (24.8) of* [32]*. If* $\tilde V_\infty$ *is constant on some time interval* $[t_0,t_1]\subset(-\infty,0)$ *then* $\mathcal{M}^\infty$ *is a gradient shrinking shrinking soliton with potential* l_∞.

Proof. From (A.7), given $a<b<0$, there is a uniform bound for l_j on $\{y_j\}\times[a,b]$. From (A.4), there are bounds for R_j on $\{y_j\}\times[a,b]$. Then we get a uniform scalar curvature bound on the balls $B(y_j,b,r)\subset\mathcal{M}^j_b$, and hence on the parabolic neighborhoods $P(y_j,b,r,-\Delta t)\subset\mathcal{M}^j$. Taking $a\to-\infty$ and $b\to 0$, and applying a diagonal argument, we can assume that $\{(\mathcal{M}^j,(y_j,-1))\}$ converges to a κ-solution $(\mathcal{M}^\infty,(y_\infty,-1))$. As in the proof of Lemma 6.3, after passing to a subsequence we get $l_j\to l_\infty$ for some l_∞ defined on $\mathcal{M}^\infty_{<0}$. The rest is as in [54]. □

Lemma 6.7. *Let* $(\mathcal{M},(x,0),l)$ *be a shrinking round cylinder with* $R\equiv 1$ *at* $t=0$*, where the* l*-function has spacetime basepoint* $(x,0)$*. Then:*

(1) *For every* $t\in(-\infty,0)$ *the* l*-function* $l:\mathcal{M}_t\to\mathbb{R}$ *attains its minimum uniquely at* (x,t).

(2) $\lim_{t\to-\infty}(\hat{\mathcal{M}}(-t),(x,-1),l)=(Cyl,(x_\infty,-1),l_\infty)$*, where the coordinate* z *for the* $\mathbb{R}$*-factor in* Cyl *satisfies* $z(x_\infty)=0$*, and*

$$l_\infty=1+\frac{z^2}{-4t}\,. \tag{6.8}$$

(3) $\lim_{t\to-\infty} l(x,t)=1$.

Proof. Part (1) follows from the formula for $\mathcal{L}$-length:

$$\mathcal{L}(\gamma)=\int_{\bar t}^0\sqrt{-t}(R+|\gamma'(t)|^2)dt\ge\int_{\bar t}^0\sqrt{-t}\,R\,dt \tag{6.9}$$

with equality if and only if $\gamma(t) = (x, t)$ for all $t \in [\bar{t}, 0]$. Part (2) follows from applying Lemma A.1 in Section A.6 to parabolic rescalings of $\mathcal{M}$. Part (3) is now immediate. $\square$

From (6.8), we get that l_∞ is strictly decreasing along backward worldlines, except at its minimum value 1. In particular, if $l_\infty(x, t_0) \leq 1+\epsilon$ then $l_\infty(x, t_1) < 1+\epsilon$ for all $t_1 < t_0$. By compactness, this stability property is inherited by any tuple which approximates the shrinking round cylinder.

Lemma 6.10. *For all $\epsilon > 0$ and $A \in (0, 1)$, there is a $\bar{\mu} > 0$ with the following property. Suppose that $(\bar{\mathcal{M}}, (x, -1), l)$ is a κ-solution, with l-function based at $(x, 0)$, so that*

(1) $(\bar{\mathcal{M}}, (x, -1), l)$ is $\bar{\mu}$-close to $(Cyl, (y, -1), l_\infty)$ on the time interval $[-A^{-1}, -A]$, for some $(y, -1) \in Cyl$, and
(2) $l(x, t) \leq 1 + \epsilon$ for all $t \in [-1, -A]$.

Then $l(x, t) < 1 + \epsilon$ for all $t \in [-A^{-1}, -1]$.

Proof. If the lemma fails then for some $\epsilon > 0$ and $A \in (0, 1)$, there is a sequence $\{(\bar{\mathcal{M}}^j, (x_j, -1), l_j)\}$ so that for all j:

- $(\bar{\mathcal{M}}^j, (x_j, -1), l_j)$ is j^{-1}-close to $(Cyl, (y_j, -1), l_\infty)$ on the time interval $[-A^{-1}, -A]$, for some $(x_j, -1) \in Cyl_{-1}$.
- $l_j(x_j, t) \leq 1 + \epsilon$ for all $t \in [-1, -A]$.
- $l_j(x_j, t_j) \geq 1 + \epsilon$ for some $t_j \in [-A^{-1}, -1]$.

Passing to a limit, we obtain $(y_\infty, -1) \in Cyl$ with the property that $l_\infty(y_\infty, -A) \leq 1 + \epsilon$ and $l_\infty(y_\infty, t) \geq 1 + \epsilon$ for some $t \in [-A^{-1}, -1]$. But this contradicts the formula $l_\infty(x, t) = 1 + \frac{z^2}{(-4t)}$. $\square$

6.3. **Stability of cylinders with moving basepoint.** In this subsection we prove a backward stability result for cylindrical regions, initially without fixing the basepoint. We then prove a version fixing the basepoint, which will imply Theorem 6.1.

Proposition 6.11 (Noncompact version)**.** *For all $\epsilon > 0$ and $C < \infty$, there is a $T = T(\epsilon, C) \in (-\infty, 0)$ with the following property. Suppose that $\mathcal{M}$ is a noncompact κ-solution defined on $(-\infty, 0]$ and:*

- *$(x, 0) \in \mathcal{M}_0$ is a point with $R(x, 0) = 1$.*
- *$l : \mathcal{M}_{<0} \to \mathbb{R}$ is the l-function with spacetime basepoint $(x, 0)$.*
- *$t < T$, and $(y, t) \in \mathcal{M}_t$ is a point where the reduced distance satisfies $l(y, t) \leq C$.*

Then one of the following holds:

(1) *The tuple* $(\hat{\mathcal{M}}(-t),(y,-1),l)$ *is* ϵ*-close to a triple* $(Cyl,(y_\infty,-1),l_\infty)$*, where* $y_\infty \in S^2 \times \mathbb{R}$ *and* l_∞ *is the asymptotic* l*-function of (6.8). Note that* y_∞ *need not be at the minimum of* l_∞ *in* Cyl_{-1}*.*
(2) $\mathcal{M}$ *is isometric to a* $\mathbb{Z}_2$*-quotient of a shrinking round cylinder.*

Proof. If the lemma were false then for some $\epsilon > 0$ there would be a sequence $\{(\mathcal{M}^j,(x_j,0))\}$ of pointed κ-solutions, and a sequence $T_j \to -\infty$, such that

- $R(x_j,0)=1$.
- There is a point $(y_j,T_j) \in \mathcal{M}^j_{T_j}$ with $l(x^j_1,T_j) \le C$, such that (1) and (2) fail for $t = T_j$.

From the compactness of the space of pointed normalized κ-solutions and the estimates at the beginning of Subsection 6.2, there is a uniform upper bound on the reduced volume $\tilde{V}_{\mathcal{M}^j}(T_j)$. Using the monotonicity of the reduced volume and the existence of the asymptotic soliton, there is also a uniform positive lower bound. After passing to a subsequence, we can find $t_j \in (T_j,0)$ so that $\frac{T_j}{t_j} \to \infty$, and the reduced volume is constant to within a factor $(1+j^{-1})$ on a time interval $[A_j t_j, A_j^{-1} t_j]$ where $A_j \to \infty$. Then after passing to a subsequence, by Proposition 6.5, the sequence of parabolically rescaled flows $\{(\hat{\mathcal{M}}^j(-t_j),(y_j,-1))\}$ pointed-converges to a gradient shrinking soliton $(\hat{\mathcal{S}}^\infty,(y_\infty,-1))$. Now $\hat{\mathcal{S}}^\infty$ cannot be a round spherical space form, as $\{\mathcal{M}^j_0\}$ is noncompact. Also, $\hat{\mathcal{S}}^\infty$ cannot be a $\mathbb{Z}_2$-quotient of a shrinking round cylinder, because then $\mathcal{M}^j_t$ would contain a one-sided $\mathbb{R}P^2$; by the classification of noncompact κ-solutions this would imply that $\mathcal{M}^j_t$ is isometric to the $\mathbb{Z}_2$-quotient of a round cylinder for all t, contradicting the failure of (2). Therefore $\hat{\mathcal{S}}^\infty$ must be a shrinking round cylinder.

By the same reasoning, the asymptotic soliton of $\mathcal{M}^j$ cannot be a $\mathbb{Z}_2$-quotient of a shrinking round cylinder, so it must be a shrinking round cylinder. It follows that the reduced volume $\tilde{V}_{\mathcal{M}^j}(t)$ is nearly constant in the interval $(-\infty,t_j]$. But this implies that after passing to a subsequence, $\{(\hat{\mathcal{M}}^j(-T_j),(y_j,-1),l_j)\}$ converges to a gradient shrinking soliton $(\hat{\mathcal{M}}^\infty,(y_\infty,-1),l_\infty)$, which is a κ-solution and whose asymptotic reduced volume is that of the shrinking round cylinder. From Lemma A.1, this implies that $\hat{\mathcal{M}}^\infty$ is a shrinking round cylinder, contradicting the failure of (1). □

Although the noncompact case considered in Proposition 6.11 is sufficient for the proof of Theorem 6.1, for the sake of completeness we remark on the extension of Proposition 6.11 to the compact case. A reader who is only interested in the proof of Theorem 6.1 can skip to the paragraph after Proposition 6.12

By way of illustration, consider the setup of Proposition 6.11, except where $\mathcal{M}$ is a κ-solution on S^3. Consider the asymptotic soliton of $\mathcal{M}$. The possibilities are in Lemma A.1. It cannot be a $\mathbb{Z}_2$-quotient of a round shrinking cylinder, since then $\mathcal{M}_t$ would be a connected sum of $\mathbb{R}P^3$ with some other 3-manifold, which contradicts our assumption that it is a 3-sphere. If the asymptotic soliton has constant positive curvature then $\mathcal{M}$ must be a round shrinking 3-sphere. Hence we can assume that the asymptotic soliton is a round shrinking cylinder. As in the proof of Proposition 6.11, there is a long interval on which $\mathcal{M}$ is close to a gradient shrinking soliton $\hat{\mathcal{S}}^\infty$. Then $\hat{\mathcal{S}}^\infty$ is either a round shrinking sphere or a round shrinking cylinder. In the first case, $\mathcal{M}$ is close to a round shrinking sphere when viewed around a time t in a certain interval $[T^*, T]$, and to a round shrinking cylinder when viewed around a time t in a certain interval $(-\infty, AT^*]$. In the second case, $\mathcal{M}$ is close to a round shrinking cylinder when viewed around a time t in a certain interval $(-\infty, T]$. In this second case, it will be convenient to add the vacuous statement that $\mathcal{M}$ is close to a round shrinking sphere when viewed around a time t in an empty interval $\{t : t \leq T, t \geq T^*\}$ with $T < T^*$, and also say that $\mathcal{M}$ is close to a round shrinking cylinder when viewed around a time $t \in (-\infty, \min(T, AT^*)]$, with $T < AT^*$.

The possibilities for all possible topologies are summarized in the following proposition. Note that in the conclusion of the proposition, T^* may be greater than T, in which case any statement about $\{t \mid t \leq T, t \geq T^*\}$ is vacuous.

Proposition 6.12. *For all $\epsilon > 0$, $C < \infty$ there exist $T = T(\epsilon, C) \in (-\infty, 0)$, $A = A(\epsilon, C) < \infty$ with the following property. Suppose $\mathcal{M}$ is a κ-solution defined on $(-\infty, 0]$ and:*

- *$(x, 0) \in \mathcal{M}_0$ is a point with $R(x, 0) = 1$.*
- *$l : \mathcal{M}_{<0} \to \mathbb{R}$ denotes the l-function with spacetime basepoint $(x, 0)$.*

Then one of the following holds:

1. *$\mathcal{M}_0$ is diffeomorphic to $\mathbb{R}^3$, and for every $t \leq T$, and every $(x, t) \in \mathcal{M}_t$ with $l(x, t) < C$, the tuple $(\hat{\mathcal{M}}(-t), (y, -1), l)$ is ϵ-close to $(Cyl, (y_\infty, -1), l_\infty)$.*
2. *$\mathcal{M}$ is isometric to a $\mathbb{Z}_2$-quotient of a shrinking round cylinder, and there is a $T^* \in (-\infty, 0)$ such that and for every t with $t \leq T$ and $t \geq T^*$ (respectively $t \leq AT^*$) and every $(y, t) \in$*

$\mathcal{M}_t$ with $l(y,t) < C$, the tuple $(\hat{M}(-t),(y,-1),l)$ is ϵ-close to $(Cyl,(y_\infty,-1),l_\infty)$ (respectively $(Cyl/\mathbb{Z}_2,(y_\infty,-1),l_\infty)$).

(3) *$\mathcal{M}$ is diffeomorphic to S^3, and there is a $T^* \in (-\infty,0)$ such that for every t with $t \le T$ and $t \ge T^*$ (respectively $t \le AT^*$) and every $(y,t) \in \mathcal{M}_t$ with $l(y,t) < C$, the tuple $(\hat{M}(-t),(y,-1),l)$ is ϵ-close to $(Sphere,(y_\infty,-1),l_\infty)$ (respectively $(Cyl,(y_\infty,-1),l_\infty)$).*

(4) *$\mathcal{M}$ is diffeomorphic to $RP^3 = S^3/\mathbb{Z}_2$, and there are times $-\infty < T^* < T^{**} < 0$ such that for every t with $t \le T$ and $t \ge T^{**}$ (respectively $T^* \le t \le AT^{**}$, respectively $t \le AT^*$) and every $(y,t) \in \mathcal{M}_t$ with $l(y,t) < C$, the tuple $(\hat{M}(-t),(y,-1),l)$ is ϵ-close to $(Sphere/\mathbb{Z}_2,(y_\infty,-1),l_\infty)$ (respectively $(Cyl,(y_\infty,-1),l_\infty)$, respectively $(Cyl/\mathbb{Z}_2,(y_\infty,-1),l_\infty)$).*

(5) *$\mathcal{M}$ is a round spherical space form $Sphere/\Gamma$ where $\Gamma \subset O(4)$, and for all $t \le T$, and every $(y,t) \in \mathcal{M}_t$, the tuple $(\hat{\mathcal{M}}(-t),(y,t),l)$ is ϵ-close to $(Sphere/\Gamma,(y_\infty,-1),l_\infty)$.*

Returning to the proof of Theorem 6.1, we now use the stability result of Proposition 6.11, together with Lemma 6.10, to show that worldlines that start close to necks have a nearly minimal value of l, provided one goes at least a certain controlled amount backward in time.

Lemma 6.13 (Basepoint stability). *For all $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ and $T = T(\epsilon) \in (-\infty,0)$ such that if $(\mathcal{M},(x,0))$ is a pointed κ-solution with $R(x,0) = 1$, and $(x,0)$ is a δ-neck, then $l(x,t) < 1+\epsilon$ for all $t < T$.*

Proof. Suppose the lemma were false. Then for some $\epsilon > 0$, there would be a sequence $\{(\mathcal{M},(x_j,0))\}$ of pointed κ-solutions so that

(1) $R(x_j,0) = 1$ and
(2) $(x_j,0)$ is a $\frac{1}{j}$-neck, but
(3) $l_j(x_j,\bar{t}_j) \ge 1+\epsilon$ for some $\bar{t}_j \le -j$.

We can assume that $\epsilon < 1$. Let $\mu_1 > 0$ be a parameter to be determined later.

By Proposition 6.11, there is a $T_1 \in (-\infty,0)$ such that for large j and every $(y,t) \in \mathcal{M}^j_{\le T_1}$ with $l_j(y,t) < 2$, the tuple $(\hat{\mathcal{M}}^j(-t),(y,-1),l_j)$ is μ_1-close to $(Cyl,(y',-1),l_\infty)$, for some $(y',-1) \in Cyl_{-1}$.

Since $(\mathcal{M}^j,(x_j,0))$ converges to the pointed round cylinder by assumption, Lemma 6.7(3) implies there is a $T_2 \in (-\infty,T_1]$ such that for large j, we have $l_j(x_j,T_2) < 1+\frac{\epsilon}{4}$.

Put $t_j = \max\{t \in (-\infty,T_2] \,:\, l_j(x_j,t) \ge 1+\epsilon\}$. Since $l_j(x_j,T_2) < 1+\epsilon$, we know that $t_j \ne T_2$. Now $l_j(x_j,t_j) = 1+\epsilon$. Note that there is an $A \in (0,1)$ independent of μ_1 such that $\limsup_{j \to \infty} \frac{T_2}{t_j} < \frac{A}{2}$, since

$$l_j(x_j, T_2) < 1 + \frac{\epsilon}{4} < 1 + \epsilon = l_j(x_j, t_j) \tag{6.14}$$

in view of the time-derivative bound on l of (A.5).

For large j, in $\hat{\mathcal{M}}^j(-t_j)$ we have $l_j(x_j, t) \leq 1+\epsilon$ for $t \in [-1, -A]$, but $l_j(x_j, -1) = 1+\epsilon$. Since $1+\epsilon < 2$, we know that $(\hat{\mathcal{M}}^j(-t_j), (x_j, -1), l_j)$ is μ_1-close to $(Cyl, (y', -1), l_\infty)$ for some $(y', -1) \in Cyl_{-1}$. If $\bar{\mu}$ is the constant from Lemma 6.10, and $\mu_1 < \bar{\mu}$, then that lemma gives a contradiction. □

Proof of Theorem 6.1. This follows by combining Proposition 6.11 with Lemma 6.13.

□

7. Finiteness of points with bad worldlines

In this section we study bad worldlines; recall from Definition 1.10 that a worldline $\gamma : I \to \mathcal{M}$ in a singular Ricci flow $\mathcal{M}$ is bad if $\inf I > 0$. In Theorem 7.1 we prove that only finitely many bad worldlines intersect a given connected component in a given time slice. We then give some applications.

The main result of this section is the following theorem.

Theorem 7.1. *Let $\mathcal{M}$ be a singular Ricci flow. For $T < \infty$, every connected component C_T of $\mathcal{M}_T$ intersects at most N bad worldlines, where $N = N(T, \mathrm{vol}(\mathcal{M}_0))$. In particular, the set of bad worldlines in $\mathcal{M}$ is at most countable.*

We begin with a lemma.

Lemma 7.2. *For all $D < \infty$ there exist $\hat{\epsilon} = \hat{\epsilon}(\kappa, D) > 0$ and $A = A(\kappa, D) < \infty$ such that if $m \in \mathcal{M}_t$ and $(\mathcal{M}, m)$ is $\hat{\epsilon}$-modelled (see Appendix A.2) on a κ-solution of diameter $\leq D$, then:*

1. *The connected component N_t of $\mathcal{M}_t$ containing m is compact and has $\mathrm{Rm} > 0$.*
2. *Let $g'(\cdot)$ be the Ricci flow on the (smooth manifold underlying) N_t defined on the time interval $[t, T')$, with initial condition at time t given by $g'(t) = g\big|_{N_t}$, and blow-up time T'. Let $\mathcal{N}$ be the corresponding Ricci flow spacetime. Then $\mathcal{N}_{\leq t'}$ is compact for all $t' < T'$, and if $\tilde{\mathcal{M}}$ is the connected component of $\mathcal{M}_{\geq t}$ containing N_t, then $\tilde{\mathcal{M}}$ is isomorphic to $\mathcal{N}$.*
3. *$R \geq AR(m)$ on $\tilde{\mathcal{M}}$.*

Proof. (1). Let $\mathrm{Sol}_\kappa^{\mathrm{D}}$ be the collection of pointed κ-solutions $(\mathcal{M}', (x,0))$ such that $R(x,0) = 1$, and $\mathrm{Diam}(\mathcal{M}'_0) \leq D$. Then every $(\mathcal{M}', (x,0)) \in \mathrm{Sol}_\kappa^{\mathrm{D}}$ has $\mathrm{Rm} > 0$, and since $\mathrm{Sol}_\kappa^{\mathrm{D}}$ is compact, there is a $\lambda > 0$ such that $\mathrm{Rm} \geq \lambda$ in $\mathcal{M}'_0$ for all $(\mathcal{M}', (x,0)) \in \mathrm{Sol}_\kappa^{\mathrm{D}}$. Part (1) now follows.

(2). Let $\mathcal{N}$ be as above. Then $\mathcal{N}_{[t,t']}$ is compact for all $t' \in [t, T')$, and by Hamilton's theorem for manifolds with $\mathrm{Ric} > 0$, we know that

$$\min\{R(m') \,:\, m' \in \mathcal{N}_{t'}\} \to \infty \quad \text{as} \quad t' \to T'\,. \tag{7.3}$$

Consider an isometric embedding of Ricci flow spacetimes $\mathcal{N}_J \hookrightarrow \mathcal{M}_J$ that extends the isometric embedding $\mathcal{N}_t \to \mathcal{M}_t$, and which is defined on a maximal time interval J starting at time t. Then J cannot be a closed interval $[t, \hat{t}]$, since then the embedding could extended to a larger time interval using uniqueness for Ricci flows. If $J = [t, \hat{t})$ with $\hat{t} < T'$, then since R is bounded on $\mathcal{N}_{[t,\hat{t}]}$, by the properness of R on $\mathcal{M}_{\leq T'}$ we may extend the embedding to an isometric embedding $\mathcal{N}_{[t,\hat{t}]} \to \mathcal{M}_{[t,\hat{t}]}$, contradicting the maximality of J. Therefore there exists an isometric embedding $\mathcal{N}_{[t,T')} \to \mathcal{M}_{[t,T')}$ of Ricci flow spacetimes, as asserted. The image is clearly an open subset of $\mathcal{M}_{\geq t}$; it is closed by (7.3). This proves (2).

(3). From the proof of (1) above, for every $(\mathcal{M}', (x,0)) \in \mathrm{Sol}_\kappa^{\mathrm{D}}$, we have $R \geq 6\lambda$ on $\mathcal{M}'_0$. Taking $A = 3\lambda$, and $\hat{\epsilon}$ sufficiently small, we get that $R \geq AR(m)$ in N_t, and therefore in $\mathcal{N}_t$ as well. By the maximum principle applied to the scalar curvature evolution equation, we have $R \geq AR(m)$ on $\mathcal{N}$, and hence on $\tilde{\mathcal{M}}$ as well. □

Proof of Theorem 7.1. In the proof below, we take $\kappa = \kappa(T)$.

Let $\epsilon_1, \Delta > 0$ be constants, to be determined later. During the course of the argument below, we will state a number of inequalities involving ϵ_1 and Δ; these will be treated as a cumulative set of constraints imposed on ϵ_1 and Δ, i.e. we will be assuming that each inequality is satisfied.

We recall that by Proposition 5.32, C_T determines a connected component C_t of $\mathcal{M}_t$ for all $t \leq T$. Let Bad be the collection of bad worldlines intersecting C_T.

Choose $0 < t_- < t_+ \leq T$ such that $t_+ - t_- < \Delta$, and let $\mathrm{Bad}_{[t_-,t_+)}$ be the set of $\gamma : I \to \mathcal{M}$ which belong to Bad, where $\inf I \in [t_-, t_+)$. We will show that if $t_+ - t_- < \Delta = \Delta(\kappa, T, \epsilon)$, then $|\,\mathrm{Bad}_{[t_-,t_+)}\,|$ is bounded by a function of κ, T and $\mathrm{vol}(\mathcal{M}_0)$; the theorem then follows immediately.

Step 1. If $\Delta < \bar{\Delta}(\epsilon_1, \kappa, \epsilon, T)$, then there exists $\hat{T} \in [0, T]$ such that:

(a) For every $\gamma : I \to \mathcal{M}$ in Bad *we have* $\inf I < \hat{T}$.

(b) *For every* $\gamma : I \to \mathcal{M}$ *in* $\mathrm{Bad}_{[t_-,t_+)}$*, and every* $t \in [t_-, t_+]$ *with* $t \le \hat{T}$*, the pair* $(\mathcal{M}, \gamma(t))$ *is* ϵ_1*-modelled on a noncompact* κ*-solution.*

By the compactness of the space of pointed normalized κ-solutions (see Appendix A.5) there exist $\epsilon_2 = \epsilon_2(\epsilon_1) > 0$ and $D = D(\epsilon_1) < \infty$ such that if $(\mathcal{M}, \gamma(t))$ is ϵ_2-modelled on a pointed κ-solution with diameter greater than D, then it is ϵ_1-modelled on a noncompact κ-solution. Put $\epsilon_3 = \min(\hat{\epsilon}(\kappa, D), \epsilon_2)$, where $\hat{\epsilon}(\kappa, D)$ is as in Lemma 7.2.

Let $\mathcal{W}$ be the set of $m \in \bigcup_{t \le T} C_t$ such that the pair $(\mathcal{M}, m)$ is ϵ_3-modelled on a pointed κ-solution with diameter at most D. Suppose first that $\mathcal{W}$ is nonempty. Lemma 7.2 implies that $R(m) \le A^{-1} \inf_{C_T} R$, for all $m \in \mathcal{W}$; therefore by the properness of $R : \mathcal{M}_{\le T} \to \mathbb{R}$, the time function $\mathfrak{t}$ attains a minimum value $\hat{T}$ on $\mathcal{W}$. Pick $m \in \mathcal{W} \cap \mathcal{M}_{\hat{T}}$. Then by Lemma 7.2, the connected component of $\mathcal{M}_{[\hat{T},T]}$ containing m is isomorphic to the $[\hat{T}, T]$-time slab of the spacetime $\mathcal{N}$ of a Ricci flow on a compact manifold with positive Ricci curvature. In particular, it also coincides with $\bigcup_{t \in [\hat{T},T]} C_t$, and therefore the curvature is bounded on the latter. Hence for every $\gamma : I \to \mathcal{M}$ in Bad, we have $\inf I < \hat{T}$. If $\mathcal{W}$ is empty then we put $\hat{T} = T$; then the conclusion of part (a) of Step 1 still holds, so we continue.

By Proposition 5.30, there exists $\hat{R} = \hat{R}(\epsilon_3, \kappa, r(T))$, such that for all $m \in \mathcal{M}_{\le T}$ with $R(m) \ge \hat{R}$, the pair $(\mathcal{M}, m)$ is ϵ_3-modelled on a pointed κ-solution. By Lemma 5.29, if $\Delta < \bar{\Delta}(\hat{R}, \kappa)$, $\gamma : I \to \mathcal{M}$ belongs to $\mathrm{Bad}_{[t_-,t_+)}$ and $t \in [t_-, t_+] \cap I \cap [0, \hat{T})$, then we have $R(\gamma(t)) > \hat{R}$. Hence either $(M, \gamma(t))$ is (A) ϵ_1-modelled on a noncompact κ-solution, or (B) ϵ_3-modelled on a κ-solution of diameter at most D; but in case (B) we would have $\gamma(t) \in \mathcal{W}$, which is impossible because $t < \hat{T}$. This proves that part (b) of Step 1 holds when $t < \hat{T}$. The borderline case $t = \hat{T}$ follows by applying the previous arguments to times t slightly less than $\hat{T}$ and taking the limit as $t \nearrow \hat{T}$, using the fact that being ϵ_1-modelled on a noncompact κ-solution is a closed condition. This completes Step 1.

Hereafter we assume that $\Delta < \bar{\Delta}(\epsilon_1, \kappa, \epsilon, T)$. By part (a) of Step 1, the set Bad is the same as the set of bad worldlines intersecting $C_{\hat{T}}$. Hence we may replace T by $\hat{T}$; then by part (b) of Step 1, for every $\gamma : I \to \mathcal{M}$ in $\mathrm{Bad}_{[t_-,t_+)}$, and every $t \in [t_-, t_+]$, the pair $(\mathcal{M}, \gamma(t))$ is ϵ_1-modelled on a noncompact κ-solution.

Step 2. Provided that $\epsilon_1 < \epsilon_1(\kappa)$, for all $\gamma : I \to \mathcal{M}$ belonging to $\mathrm{Bad}_{[t_-,t_+)}$ *and every $t \in I \cap [t_-, t_+)$, the pair $(\mathcal{M}, \gamma(t))$ is not a generalized $\frac{\delta_{neck}}{2}$-neck, the latter being in the sense of Appendix A.2.*

Suppose that $\gamma : I \to \mathcal{M}$ belongs to $\mathrm{Bad}_{[t_-,t_+)}$, and $(\mathcal{M}, \gamma(\hat{t}_0))$ is a generalized $\frac{\delta_{neck}}{2}$-neck for some $\hat{t}_0 \in I \cap [t_-, t_+)$. By Step 1 we know that $(\mathcal{M}, \gamma(\hat{t}_0))$ is ϵ_1-modelled on a noncompact pointed κ-solution $(\mathcal{M}^1, (x_0, 0))$. If $\epsilon_1 < \bar{\epsilon}_1(\frac{\delta_{neck}}{2})$, then $(\mathcal{M}^1, (x_0, 0))$ will be a generalized δ_{neck}-neck. Let $T_1 = T(\delta_{neck}, \frac{\delta_{neck}}{4}) \in (-\infty, 0)$ be as in Corollary 6.2. Then $(\mathcal{M}^1, (x_0, T_1))$ is a generalized $\frac{\delta_{neck}}{4}$-neck. If $\epsilon_1 < \bar{\epsilon}_1(T_1, \frac{\delta_{neck}}{2})$, then we get that:

- γ is defined at $\hat{t}_1 = \hat{t}_0 + R^{-1}(\gamma(\hat{t}_0))T_1$.
- $(\mathcal{M}, \gamma(\hat{t}_1))$ is a generalized $\frac{\delta_{neck}}{2}$-neck.
- $R(\gamma(\hat{t}_1)) < \frac{1}{2}R(\gamma(\hat{t}_0))$.

Thus we may iterate this to produce a sequence $\{\hat{t}_0, \hat{t}_1, \ldots\} \subset I$ such that $\hat{t}_i \le \hat{t}_{i-1} + R^{-1}(\gamma(\hat{t}_0))T_1$ for all i. This contradicts the fact that $\inf I \in [t_-, t_+)$, and completes Step 2.

Hereafter we assume that $\epsilon_1 < \epsilon_1(\kappa)$. Let $D_0 < \infty$ be such that if $\mathcal{M}'$ is a noncompact κ-solution, $m_1, m_2 \in \mathcal{M}'_t$, and neither m_1 nor m_2 is a $\frac{\delta_{neck}}{4}$-neck, then

$$d_t(m_1, m_2) < D_0 R(m_1)^{-\frac{1}{2}} . \tag{7.4}$$

Let $D_1 \in (2D_0, \infty)$ be a constant, to be determined in Step 4.

Step 3. Provided that $\epsilon_1 < \bar{\epsilon}'_1(\kappa, D_1)$, if $\gamma_1, \gamma_2 \in$ $\mathrm{Bad}_{[t_-,t_+)}$*, and*

$$d_{t_+}(\gamma_1(t_+), \gamma_2(t_+)) < D_1 R(\gamma_1(t_+)))^{-\frac{1}{2}} , \tag{7.5}$$

then $\gamma_1 = \gamma_2$.

Suppose that $t \in I_1 \cap I_2 \cap [t_-, t_+]$. By Steps 1 and 2, $(\mathcal{M}, \gamma_1(t_+))$ is ϵ_1-modelled on a noncompact κ-solution and neither $\gamma_1(t_+)$ nor $\gamma_2(t_+)$ is a $\frac{\delta_{neck}}{2}$-neck. If $\epsilon_1 < \bar{\epsilon}'_1(D_1, \delta_{neck})$ then using the ϵ_1-closeness to a noncompact κ-solution and (7.4), we can say that

$$d_t(\gamma_1(t), \gamma_2(t)) < D_1 R(\gamma_1(t)))^{-\frac{1}{2}} , \tag{7.6}$$

implies

$$d_t(\gamma_1(t), \gamma_2(t)) < 2D_0 R(\gamma_1(t)))^{-\frac{1}{2}} \tag{7.7}$$

Since we are assuming (7.5), a continuity argument shows that (7.6) holds for all $t \in I_1 \cap I_2 \cap [t_+, t_-]$. If $\inf I_1 \ge \inf I_2$, then $\lim_{t \to \inf I_1} R(\gamma_2(t)) = \infty$, so $\inf I_1 = \inf I_2$; similar reasoning holds if $\inf I_2 \le \inf I_1$. Thus $\inf I_1 = \inf I_2$. Moreover, if $\epsilon_1 < \bar{\epsilon}''_1(\kappa)$ then any geodesic from $\gamma_1(t)$

to $\gamma_2(t)$ in C_t will lie in the set with $\mathrm{Ric} > 0$, so $d_t(\gamma_1(t), \gamma_2(t))$ is a decreasing function of t. Since (7.6) implies that $d_t(\gamma_1(t), \gamma_2(t)) \to 0$ as $t \to \inf I_1$, it follows that $\gamma_1 = \gamma_2$. This completes Step 3.

Hereafter we assume that $\epsilon_1 < \bar{\epsilon}_1'(\kappa, D_1)$.

Step 4. Provided that $\Delta < \bar{\Delta}(\kappa, T)$, the cardinality of $\mathrm{Bad}_{[t_-,t_+)}$ *is at most $N = N(\kappa, T, \mathrm{vol}(\mathcal{M}_0))$.*

Take $\hat{\epsilon} = \frac{\delta_{neck}}{2}$, and let $C_1 = C_1(\hat{\epsilon}, T)$, $\overline{R} = \overline{R}(\hat{\epsilon}, T)$, and $N_1, \ldots, N_k \subset \mathcal{M}_{t_+}$ be as in Proposition 5.31. With reference to Step 3, take $D_1 = C_1$. Let $\Delta < \overline{\Delta}(\overline{R}, \kappa, r(T))$ be such that if $\gamma : I \to \mathcal{M}$ belongs to $\mathrm{Bad}_{[t_-,t_+)}$, and $t \in [t_-, t_+] \cap I$, then $R(\gamma(t)) \geq \overline{R}$; c.f. the proof of Step 1.

Then the set

$$S = \{\gamma(t_+) \ : \ \gamma \in \mathrm{Bad}_{[t_-,t_+)}\} \tag{7.8}$$

is contained in $\{m \in \mathcal{M}_{t_+} : R(m) \geq \overline{R}\} \subset \bigcup_i N_i$. By Step 3, for any two distinct elements $\gamma_1, \gamma_2 \in \mathrm{Bad}_{[t_-,t_+)}$ we have

$$d_t(\gamma_1(t_+), \gamma_2(t_+)) \geq D_1 R^{-\frac{1}{2}}(\gamma_1(t_+)) = C_1 R^{-\frac{1}{2}}(\gamma_1(t_+))\,. \tag{7.9}$$

By Proposition 5.31, for all $i \in \{1, \ldots, k\}$ we have $|S \cap N_i| \leq 2$. Therefore $|S| \leq 2k$. This proves that $\mathrm{Bad}_{[t_-,t_+)}$ is finite, and hence a weaker version of the theorem, namely that the set of all bad worldlines is countable.

Since the set of bad worldlines is countable, their union has measure zero in spacetime. Therefore we may apply Proposition 5.11, to conclude that

$$\mathrm{vol}(C_{t+}) \leq \mathrm{vol}(\mathcal{M}_{t+}) \leq \mathcal{V}(0)\,(1+2t)^{\frac{3}{2}}\,. \tag{7.10}$$

If $k \geq 2$, then each N_i has nonempty boundary. Hence part (5) of Proposition 5.31 gives a bound $k < k(\mathcal{V}(0), T)$.

This proves Theorem 7.1. □

Corollary 7.11. *If $\mathcal{M}$ is a singular Ricci flow, then the volume function $\mathcal{V}(t) = \mathrm{vol}(\mathcal{M}_t)$ is absolutely continuous.*

Proof. By Theorem 7.1 the set of bad worldlines is countable, and hence has measure zero. Combining this with Lemma 5.23, we may apply Proposition 5.11(4), to conclude that $\mathcal{V}(t)$ is absolutely continuous. □

Theorem 7.1 also has the following topological implications.

Corollary 7.12. *Let $\mathcal{M}$ be a singular Ricci flow.*

(1) *If $T \geq 0$ and $W \subset \mathcal{M}_T$ is an open subset that does not contain any compact connected components of $\mathcal{M}_T$, then there is a smooth time-preserving map $\Gamma : W \times [0,T] \to \mathcal{M}$ that is a "weak isotopy", in the sense that it maps $W \times \{t\}$ diffeomorphically onto an open subset of $\mathcal{M}_t$, for all $t \in [0,T]$.*
(2) *For all $T \geq 0$, the pair $(\mathcal{M}, \mathcal{M}_{\leq T})$ is k-connected for $k \leq 2$.*

Proof. (1). Let $\mathcal{C}$ be the collection of connected components of $\mathcal{M}_T$. Pick $C \in \mathcal{C}$. Let B be the set of bad worldlines intersecting C. By Theorem 7.1 the set B is finite, so its intersection with C is contained in a 3-disk D^3. There is a $t_C < T$ such that the worldline of every $m \in C$ is defined in the interval $[t_C, T]$. Hence we get a time-preserving map $F_C : C \times [t_C, T] \to \mathcal{M}$ that is a diffeomorphism onto its image.

By assumption, either C is noncompact, or C is compact and $W \not\supset C$. Therefore there is a smooth homotopy $\{H_t : W \cap C \to C\}_{t \in [t_C, T]}$ (purely in the time-T slice) such that $H_T : W \cap C \to C$ is the inclusion map, $H_t : W \cap C \to C$ is a diffeomorphism onto its image for all $t \in [t_C, T]$, and $H_{t_C}(W \cap C) \cap D^3 = \emptyset$. We define Γ on $(W \cap C) \times [t_C, T]$ by $\Gamma(m,t) = F_C(H_t(m), t)$, and extend this to $(W \cap C) \times [0, t_C]$ by following worldlines. Note that if $C_1, C_2 \in \mathcal{C}$ are distinct components of $\mathcal{M}_T$ then $F_{C_1}(C_1 \cap W)$ is disjoint from $F_{C_2}(C_2 \cap W)$, so the resulting map Γ has the property that $\Gamma(\cdot, t) : W \to \mathcal{M}_t$ is an injective local diffeomorphism for every $t \in [0,T]$.

(2). Suppose that $0 \leq k \leq 2$, and $f : (D^k, \partial D^k) \to (\mathcal{M}, \mathcal{M}_{\leq T})$ is a map of pairs, where $\partial D^k = S^{k-1}$ if $k \geq 1$ and $\partial D^0 = \emptyset$. By Theorem 7.1, the Hausdorff dimension of the bad worldlinesis at most one. Then after making a small homotopy we may assume that f is smooth, and that its image is disjoint from the bad worldlines. We can now find a homotopy through maps of pairs by using the backward flow of the time vector field, which is well-defined on $f(D^k)$. □

8. Curvature and volume estimates

In this section we establish further curvature and volume estimates for singular Ricci flows. If η is the constant in (A.9) then we show in Proposition 8.1 that for any t, the scalar curvature is in $L^{\frac{1}{\eta}}$ on the time slice $\mathcal{M}_t$. Proposition 8.15, when combined with part (5) of Proposition 5.11, shows that the volume $\mathcal{V}(t)$ of the time-t slice is a $\frac{1}{\eta}$-Hölder function of t.

Proposition 8.1. *Let $\mathcal{M}$ be a singular Ricci flow. Let η be the constant from (A.9). We can assume that $\eta \geq 1$. Then for any $t > \frac{1}{100\eta}$,*

$$\int_{\mathcal{M}_t} |R|^{\frac{1}{\eta}} \, \mathrm{dvol}_{g(t)} \leq 2r(t)^{-\frac{2}{\eta}}(1+2t)^{\frac{3}{2}}\mathcal{V}(0). \tag{8.2}$$

Proof. We use the notation of the proof of Proposition 5.11. Let $X_3 \subset \mathcal{M}_t$ be the complement of the set of points in $\mathcal{M}_t$ with a bad worldline. From Theorem 7.1, it has full measure in $\mathcal{M}_t$. Given $x \in X_3$, let $\gamma_x : [0,t] \to \mathcal{M}_{[0,t]}$ be the restriction of its worldline to the interval $[0,t]$. Define $J_t(x)$ as in (5.16), with $t_1 = 0$. From (5.17), we have

$$J_t(x) = e^{-\int_0^t R(\gamma_x(u))\, du}. \tag{8.3}$$

Put

$$X_3^{>r(t)^{-2}} = \{x \in X_3 \,:\, R(x) > r(t)^{-2}\} \tag{8.4}$$

and

$$X_3^{\leq r(t)^{-2}} = \{x \in X_3 \,:\, R(x) \leq r(t)^{-2}\} \tag{8.5}$$

Given $x \in X_3^{>r(t)^{-2}}$, the gradient bound (A.9) implies that

$$\frac{1}{R(\gamma_x(u))} \leq \frac{1}{R(x)} + \eta(t-u), \tag{8.6}$$

as long as the right-hand side is at most $r(t)^2$, i.e. as long as $u \geq u_0$, where u_0 is defined by

$$\frac{1}{R(x)} + \eta(t-u_0) = r(t)^2. \tag{8.7}$$

From our assumptions,

$$u_0 = t - \frac{1}{\eta}r(t)^2 + \frac{1}{\eta}\frac{1}{R(x)} \geq t - \frac{1}{\eta}r(0)^2 \geq t - \frac{1}{100\eta} > 0. \tag{8.8}$$

Now

$$\int_{u_0}^t R(\gamma_x(u))\, du \geq \int_{u_0}^t \frac{1}{\frac{1}{R(x)} + \eta(t-u)}\, du = \frac{1}{\eta}\log(R(x)r(t)^2), \tag{8.9}$$

and

$$\begin{aligned} \int_0^{u_0} R(\gamma_x(u))\, du &\geq -\int_0^{u_0} \frac{3}{1+2u}\, du \geq -\int_0^t \frac{3}{1+2u}\, du \\ &= -\frac{3}{2}\log(1+2t), \end{aligned} \tag{8.10}$$

so

$$\frac{\mathrm{dvol}_{g(t)}}{\mathrm{dvol}_{g(0)}}(x) = J_t(x) \leq (1+2t)^{\frac{3}{2}}(R(x)r(t)^2)^{-\frac{1}{\eta}}. \tag{8.11}$$

Then

$$\begin{aligned}(8.12)\qquad \int_{X_3^{>r(t)^{-2}}} R^{\frac{1}{\eta}}\ \mathrm{dvol}_{g(t)} &\le r(t)^{-\frac{2}{\eta}}(1+2t)^{\frac{3}{2}} \int_{X_3^{>r(t)^{-2}}} \mathrm{dvol}_{g(0)} \\ &\le r(t)^{-\frac{2}{\eta}}(1+2t)^{\frac{3}{2}}\mathcal{V}(0).\end{aligned}$$

Next, for $x \in X_3$, we have $-R(x) \le 3 < r(t)^{-2}$, so

$$(8.13)\qquad \int_{X_3^{\le r(t)^{-2}}} |R|^{\frac{1}{\eta}}\ \mathrm{dvol}_{g(t)}(x) \le r(t)^{-\frac{2}{\eta}}\mathcal{V}(t) \le r(t)^{-\frac{2}{\eta}}(1+2t)^{\frac{3}{2}}\mathcal{V}(0).$$

The proposition follows. □

Remark 8.14. From Proposition 5.11, the scalar curvature is L^1 on almost all time slices. It seems conceivable that Proposition 8.1 could be improved to say that for any $p \in (0,1)$, the scalar curvature on the time-t slice is L^p for all t. Note that in the case of a shrinking round cylinder, the constant η of (A.9) is exactly one.

Proposition 8.15. *Let $\mathcal{M}$ be a singular Ricci flow. Let η be the constant from (A.9). We can assume that $\eta \ge 1$. Then whenever $0 \le t_1 \le t_2 < \infty$ satisfies $t_2 - t_1 < \frac{1}{\eta} r(t_2)^2$ and $t_1 > \frac{1}{100\eta}$, we have*

$$\begin{aligned}(8.16)\qquad &\mathcal{V}(t_2) - \mathcal{V}(t_1) \ge \\ &- \eta^{\frac{1}{\eta}} \left(2\int_{\mathcal{M}_{t_1}} |R|^{\frac{1}{\eta}}\ \mathrm{dvol}_{g(t_1)} + r(t_2)^{-\frac{2}{\eta}}\mathcal{V}(t_1)\right)(t_2-t_1)^{\frac{1}{\eta}} \ge \\ &- 5\eta^{\frac{1}{\eta}} r(t_2)^{-\frac{2}{\eta}}(1+2t_1)^{\frac{3}{2}}\mathcal{V}(0)\cdot(t_2-t_1)^{\frac{1}{\eta}}.\end{aligned}$$

Proof. Let $X_1 \subset \mathcal{M}_{t_1}$ be the set of points $x \in \mathcal{M}_{t_1}$ whose worldline γ_x extends forward to time t_2 and let $X_2 \subset \mathcal{M}_{t_1}$ be the points x whose worldline γ_x does not extend forward to time t_2. Put

$$(8.17)\qquad X_1' = \left\{ x \in X_1 \ :\ R(x) > \frac{1}{\eta(t_2-t_1)} \right\},$$

$$(8.18)\qquad X_1'' = \left\{ x \in X_1 \ :\ r(t_2)^{-2} < R(x) \le \frac{1}{\eta(t_2-t_1)} \right\}$$

and

$$(8.19)\qquad X_1''' = \left\{ x \in X_1 \ :\ R(x) \le r(t_2)^{-2} \right\}.$$

Then

$$\begin{aligned}\text{(8.20)}\quad \mathrm{vol}(\mathcal{M}_{t_2}) - \mathrm{vol}(\mathcal{M}_{t_1}) \geq\ & \mathrm{vol}_{t_2}(X_1') - \mathrm{vol}_{t_1}(X_1') + \\ & \mathrm{vol}_{t_2}(X_1'') - \mathrm{vol}_{t_1}(X_1'') + \\ & \mathrm{vol}_{t_2}(X_1''') - \mathrm{vol}_{t_1}(X_1''') - \mathrm{vol}_{t_1}(X_2) \\ \geq\ & \mathrm{vol}_{t_2}(X_1'') - \mathrm{vol}_{t_1}(X_1'') + \mathrm{vol}_{t_2}(X_1''') - \\ & \mathrm{vol}_{t_1}(X_1''') - \mathrm{vol}_{t_1}(X_2) - \mathrm{vol}_{t_1}(X_1').\end{aligned}$$

Lemma 8.21. *Given $x \in X_2$, let $[t_1, t_x)$ be the domain of the forward extension of γ_x, with $t_x < t_2$. For all $u \in [t_1, t_x)$, we have*

$$R(\gamma_x(u)) \geq \frac{1}{\eta(t_x - u)}. \tag{8.22}$$

Proof. If the lemma is not true, put

$$u' = \sup \left\{ u \in [t_1, t_x) \ : \ R(\gamma_x(u)) < \frac{1}{\eta(t_x - u)} \right\}. \tag{8.23}$$

Then $u' > t_1$. From the gradient estimate (A.9) and the fact that $\lim_{u \to t_x} R(\gamma_x(u)) = \infty$, we know that $u' < t_x$. Whenever $u \geq u'$, we have

$$R(\gamma_x(u)) \geq \frac{1}{\eta(u_x - u')} \geq \frac{1}{\eta(t_2 - t_1)} > r(t_2)^{-2}, \tag{8.24}$$

so there is some $\mu > 0$ so that the gradient estimate (A.9) holds on the interval $(u' - \mu, t_x)$. This implies that (8.22) holds for all $u \in (u' - \mu, t_x)$, which contradicts the definition of u'. This proves the lemma. □

Hence

$$(X_2 \cup X_1') \subset \left\{ x \in \mathcal{M}_{t_1} \ : \ R(x) \geq \frac{1}{\eta(t_2 - t_1)} \right\} \tag{8.25}$$

and

$$\begin{aligned}\text{(8.26)}\quad \mathrm{vol}_{t_1}(X_2) + \mathrm{vol}_{t_1}(X_1') &\leq \mathrm{vol} \left\{ x \in \mathcal{M}_{t_1} \ : \ R(x) \geq \frac{1}{\eta(t_2 - t_1)} \right\} \\ &\leq \eta^{\frac{1}{\eta}} (t_2 - t_1)^{\frac{1}{\eta}} \int_{\mathcal{M}_{t_1}} |R|^{\frac{1}{\eta}} \ \mathrm{dvol}_{g(t_1)},\end{aligned}$$

since $\eta^{\frac{1}{\eta}} (t_2 - t_1)^{\frac{1}{\eta}} |R|^{\frac{1}{\eta}} \geq 1$ on the set $\{x \in \mathcal{M}_{t_1} : R(x) \geq \frac{1}{\eta(t_2 - t_1)}\}$.

Suppose now that $x \in X_1''$.

Lemma 8.27. *For all $u \in [t_1, t_2]$, we have*

$$R(\gamma_x(u)) \leq \frac{1}{\frac{1}{R(x)} - \eta(u - t_1)} < \infty. \tag{8.28}$$

Proof. If the lemma is not true, put

$$u'' = \inf \left\{ u \in [t_1, t_2] \ : \ R(\gamma_x(u)) > \frac{1}{\frac{1}{R(x)} - \eta(u - t_1)} \right\}. \tag{8.29}$$

Then $u'' < t_2$ and the gradient estimate (A.9) implies that $u'' > t_1$. Now

$$R(\gamma_x(u'')) = \frac{1}{\frac{1}{R(x)} - \eta(u'' - t_1)} > R(x) > r(t_2)^{-2}. \tag{8.30}$$

Hence there is some $\mu > 0$ so that $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for $u \in [u'', u'' + \mu]$. If $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for all $u \in [t_1, u'']$ then (A.9) implies that (8.28) holds for $u \in [t_1, u'' + \mu]$, which contradicts the definition of u''. On the other hand, if it is not true that $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for all $u \in [t_1, u'']$, put

$$v'' = \sup \left\{ u \in [t_1, u''] \ : \ R(\gamma_x(u)) < r(t_2)^{-2} \right\}. \tag{8.31}$$

Then $v'' > t_1$ and $R(\gamma_x(v'')) = r(t_2)^{-2}$. Equation (A.9) implies that

$$R(\gamma_x(u'')) \leq \frac{1}{r(t_2)^2 - \eta(u'' - v'')} < \frac{1}{\frac{1}{R(x)} - \eta(u'' - t_1)}, \tag{8.32}$$

which contradicts (8.30). This proves the lemma. □

Hence if $x \in X_1''$ then

$$\begin{aligned} \int_{t_1}^{t_2} R(\gamma_x(u))\, du &\leq \int_{t_1}^{t_2} \frac{1}{\frac{1}{R(x)} - \eta(u - t_1)}\, du \\ &= -\frac{1}{\eta} \log\left(1 - \eta R(x) \cdot (t_2 - t_1)\right), \end{aligned} \tag{8.33}$$

so

$$\frac{\mathrm{dvol}_{g(t_2)}}{\mathrm{dvol}_{g(t_1)}}(x) = J_{t_2}(x) \geq \left(1 - \eta R(x) \cdot (t_2 - t_1)\right)^{\frac{1}{\eta}}. \tag{8.34}$$

Thus

$$\begin{aligned} &\mathrm{vol}_{t_2}(X_1'') - \mathrm{vol}_{t_1}(X_1'') \geq \\ &\int_{X_1''} \left(\left(1 - \eta R \cdot (t_2 - t_1)\right)^{\frac{1}{\eta}} - 1 \right) \mathrm{dvol}_{g(t_1)}. \end{aligned} \tag{8.35}$$

Since $\eta \geq 1$, if $z \in [0, 1]$ then $\left(z^{\frac{1}{\eta}}\right)^{\eta} + \left(1 - z^{\frac{1}{\eta}}\right)^{\eta} \leq 1$, so

$$(1 - z)^{\frac{1}{\eta}} - 1 \geq -z^{\frac{1}{\eta}}. \tag{8.36}$$

Then

$$\mathrm{vol}_{t_2}(X_1'') - \mathrm{vol}_{t_1}(X_1'') \geq -\eta^{\frac{1}{\eta}} (t_2 - t_1)^{\frac{1}{\eta}} \int_{X_1''} R^{\frac{1}{\eta}} \ \mathrm{dvol}_{g(t_1)} . \tag{8.37}$$

Now suppose that $x \in X_1'''$.

Lemma 8.38. *For all $u \in [t_1, t_2]$, we have*

$$R(\gamma_x(u)) \leq \frac{1}{r(t_2)^2 - \eta(u - t_1)} < \infty. \tag{8.39}$$

Proof. If the lemma is not true, put

$$u''' = \inf \left\{ u \in [t_1, t_2] \ : \ R(\gamma_x(u)) > \frac{1}{r(t_2)^2 - \eta(u - t_1)} \right\}. \tag{8.40}$$

Then $u''' < t_2$. If $R(x) < r(t_2)^{-2}$ then clearly $u''' > t_1$. If $R(x) = r(t_2)^{-2}$ then since $r(t_1) > r(t_2)$, there is some $\nu > 0$ so that $R(\gamma_x(u)) > r(u)^{-2}$ for $u \in [t_1, t_1 + \nu]$; then (A.9) gives the validity of (8.39) for $u \in [t_1, t_1 + \nu]$, which implies that $u''' > t_1$. In either case, $t_1 < u''' < t_2$. Now

$$R(\gamma_x(u''')) = \frac{1}{r(t_2)^2 - \eta(u''' - t_1)} > r(t_2)^{-2}. \tag{8.41}$$

Hence there is some $\mu > 0$ so that $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for $u \in [u''', u''' + \mu]$. If $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for all $u \in [t_1, u''']$ then (A.9) implies that (8.39) holds for $u \in [t_1, u''' + \mu]$, which contradicts the definition of u'''. On the other hand, if it is not true that $R(\gamma_x(u)) \geq r(t_2)^{-2}$ for all $u \in [t_1, u''']$, put

$$v''' = \sup \left\{ u \in [t_1, u'''] \ : \ R(\gamma_x(u)) < r(t_2)^{-2} \right\}. \tag{8.42}$$

Then $v''' > t_1$ and $R(\gamma_x(v''')) = r(t_2)^{-2}$. The gradient estimate (A.9) implies that

$$R(\gamma_x(u''') \leq \frac{1}{r(t_2)^2 - \eta(u''' - v''')} < \frac{1}{r(t_2)^2 - \eta(u''' - t_1)}, \tag{8.43}$$

which contradicts (8.41). This proves the lemma. □

Hence if $x \in X_1'''$ then

$$\begin{aligned} \int_{t_1}^{t_2} R(\gamma_x(u))\, du &\leq \int_{t_1}^{t_2} \frac{1}{r(t_2)^2 - \eta(u - t_1)}\, du \\ &= -\frac{1}{\eta} \log \left(1 - \eta r(t_2)^{-2} \cdot (t_2 - t_1)\right), \end{aligned} \tag{8.44}$$

so

$$\frac{\mathrm{dvol}_{g(t_2)}}{\mathrm{dvol}_{g(t_1)}}(x) = J_{t_2}(x) \geq \left(1 - \eta r(t_2)^{-2} \cdot (t_2 - t_1)\right)^{\frac{1}{\eta}}. \tag{8.45}$$

Thus

$$\text{vol}_{t_2}(X_1''') - \text{vol}_{t_1}(X_1''') \geq \int_{X_1'''} \left(\left(1 - \eta r(t_2)^{-2} \cdot (t_2 - t_1)\right)^{\frac{1}{\eta}} - 1 \right) \, \text{dvol}_{g(t_1)} . \tag{8.46}$$

Since $\eta r(t_2)^{-2} \cdot (t_2 - t_1) \in [0, 1]$, we can apply (8.36) to conclude that

$$\begin{aligned} \text{vol}_{t_2}(X_1''') - \text{vol}_{t_1}(X_1''') &\geq - \eta^{\frac{1}{\eta}} r(t_2)^{-\frac{2}{\eta}} \cdot (t_2 - t_1)^{\frac{1}{\eta}} \text{vol}_{t_1}(X_1''') \\ &\geq - \eta^{\frac{1}{\eta}} r(t_2)^{-\frac{2}{\eta}} \mathcal{V}(t_1) \cdot (t_2 - t_1)^{\frac{1}{\eta}}. \end{aligned} \tag{8.47}$$

Combining (8.20), (8.26), (8.37) and (8.47) gives (8.16). □

Appendix A. Background material

In this section we collect some needed facts about Ricci flows and Ricci flows with surgery. More information can be found in [32].

A.1. **Notation and terminology.** Let $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ be a Ricci flow spacetime (Definition 1.5). For brevity, we will often write $\mathcal{M}$ for the quadruple. In a Ricci flow with surgery, we will sometimes loosely write a point $m \in \mathcal{M}_t$ as a pair (x, t).

Given $t > 0$, the rescaled Ricci flow spacetime is $\hat{\mathcal{M}}(t) = (\mathcal{M}, \frac{1}{t}\mathfrak{t}, t\partial_\mathfrak{t}, \frac{1}{t}g)$.

Given $m \in M_t$, we write $B(m, r)$ for the open metric ball of radius r in $\mathcal{M}_t$. We write $P(m, r, \Delta t)$ for the parabolic neighborhood, i.e. the set of points m' in $\mathcal{M}_{[t,t+\Delta t]}$ if $\Delta t > 0$ (or $\mathcal{M}_{[t+\Delta t,t]}$ if $\Delta t < 0$) that lie on the worldline of some point in $B(m, r)$. We say that $P(m, r, \Delta t)$ is *unscathed* if $B(m, r)$ has compact closure in $\mathcal{M}_t$ and for every $m' \in P(m, r, \Delta t)$, the maximal worldline γ through m' is defined on a time interval containing $[t, t + \Delta t]$ (or $[t + \Delta t, t]$). We write $P_+(m, r)$ for the forward parabolic ball $P(m, r, r^2)$ and $P_-(m, r)$ for the backward parabolic ball $P(m, r, -r^2)$.

We write *Cyl* for the standard Ricci flow on $S^2 \times \mathbb{R}$ that terminates at time zero, with $g(t) = (-2t)g_{S^2} + dz^2$. We write *Sphere* for the standard round shrinking 3-sphere that terminates at time zero.

A.2. **Closeness of Ricci flow spacetimes.** Let $\mathcal{M}^1$ and $\mathcal{M}^2$ be two Ricci flow spacetimes in the sense of Definition 1.1. Consider a time interval $[a, b]$. Suppose that $m_1 \in \mathcal{M}^1$ and $m_2 \in \mathcal{M}^2$ have $\mathfrak{t}_1(m_1) = \mathfrak{t}_2(m_2) = b$. We say that $(\mathcal{M}^1, m_1)$ and $(\mathcal{M}^2, m_2)$ are *ϵ-close* on the time interval $[a, b]$ if there are open subsets $U_i \subset \mathcal{M}^i$ with $P(m_i, \epsilon^{-1}, a - b) \subset U_i$, $i \in \{1, 2\}$, and there is a pointed diffeomorphism $\Phi : (U_1, m_1) \to (U_2, m_2)$ so that

- $B(m_i, \epsilon^{-1})$ has compact closure in $\mathcal{M}^i_b$,

- $P(m_i, \epsilon^{-1}, a-b)$ is unscathed,
- Φ is time-preserving, i.e. $\mathfrak{t}_2 \circ \Phi = \mathfrak{t}_1$,
- $\Phi_* \partial_{\mathfrak{t}_1} = \partial_{\mathfrak{t}_2}$ and
- $\Phi^* g_2 - g_1$ has norm less than ϵ in the $C^{[1/\epsilon]+1}$-topology on U_1.

Now consider an open time interval $(-\infty, b)$. Suppose that $\mathfrak{t}_1(m_1) = \mathfrak{t}_2(m_2) = c \in (-\infty, b)$. After time shift and parabolic rescaling, we can assume that $c = -1$ and $b = 0$. In this case, we say that $(\mathcal{M}^1, m_1)$ and $(\mathcal{M}^2, m_2)$ are ϵ-close on the time interval $(-\infty, 0)$ if there are open sets $U_i \subset \mathcal{M}^i$ with $(P(m_i, \epsilon^{-1}, 1-\epsilon) \cup P(m_i, \epsilon^{-1}, -\epsilon^{-2})) \subset U_i$, $i \in \{1, 2\}$, and there is a pointed diffeomorphism $\Phi : (U_1, m_1) \to (U_2, m_2)$ so that

- $B(m_i, \epsilon^{-1})$ has compact closure in $\mathcal{M}^i_{-1}$,
- $P(m_i, \epsilon^{-1}, 1-\epsilon)$ and $P(m_i, \epsilon^{-1}, -\epsilon^{-2})$ are unscathed,
- Φ is time-preserving, i.e. $\mathfrak{t}_2 \circ \Phi = \mathfrak{t}_1$,
- $\Phi_* \partial_{\mathfrak{t}_1} = \partial_{\mathfrak{t}_2}$ and
- $\Phi^* g_2 - g_1$ has norm less than ϵ in the $C^{[1/\epsilon]+1}$-topology on U_1.

We define ϵ-closeness similarly on other time intervals, whether open or half-open.

If $(\mathcal{M}_1, m_1)$ and $(\mathcal{M}_2, m_2)$ are Ricci flow spacetimes then we say that $(\mathcal{M}_1, m_1)$ is *ϵ-modelled* on $(\mathcal{M}_2, m_2)$ if after shifts in the time parameters so that $\mathfrak{t}_1(m_1) = \mathfrak{t}_2(m_2) = 0$, and parabolic rescaling by $R(m_1)$ and $R(m_2)$ respectively, the resulting Ricci flow spacetimes are ϵ-close to each other on the time interval $[-\epsilon^{-1}, 0]$. (It is implicit in the definition that $R(m_1) > 0$ and $R(m_2) > 0$; this will be the case for us since we are interested in modelling regions of high scalar curvature.) A point m in a Ricci flow spacetime $\mathcal{M}$ is a *generalized ϵ-neck* if $(\mathcal{M}, m)$ is ϵ-modelled on $(\mathcal{M}', m')$, where $\mathcal{M}'$ is either a shrinking round cylinder or the $\mathbb{Z}_2$-quotient of a shrinking round cylinder.

A.3. **Necks, horns and caps.** We say that $(\mathcal{M}, m)$ is a (strong) *δ-neck* if after time shifting and parabolic rescaling, it is δ-close on the time interval $[-1, 0]$ to the product Ricci flow which, at its final time, is isometric to the product of $\mathbb{R}$ with a round 2-sphere of scalar curvature one. The basepoint is taken at time 0. In this case, we also say that m is the *center of a δ-neck*.

If I is an open interval then a metric on an embedded copy of $S^2 \times I$ in $\mathcal{M}_t$, such that each point is contained in an δ-neck, is called a *δ-tube*, or a *δ-horn*, or a *double δ-horn*, if the scalar curvature stays bounded on both ends, stays bounded on one end and tends to infinity on the other, or tends to infinity on both ends, respectively. (Our definition differs slightly from that in [32, Definition 58.2], where the definition is in terms of the "δ-necks" of that paper, as opposed to the "strong δ-necks" that we are using now.)

A metric on B^3 or $B^3 - \mathbb{R}P^3$, such that each point outside some compact set is contained in a δ-neck, is called a *δ-cap* or a *capped δ-horn*, if the scalar curvature stays bounded or tends to infinity on the end, respectively.

A.4. **κ-noncollapsing.** Let $\mathcal{M}$ be an $(n+1)$-dimensional Ricci flow spacetime. Let $\kappa : [0, \infty) \to (0, \infty)$ be a decreasing function. We say that $\mathcal{M}$ is *κ-noncollapsed at scales below* ϵ if for each $\rho < \epsilon$ and all $m \in \mathcal{M}$ with $\mathfrak{t}(m) \geq \rho^2$, whenever $P(m, \rho, -\rho^2)$ is unscathed and $|\operatorname{Rm}| \leq \rho^{-2}$ on $P(m, \rho, -\rho^2)$, then we also have $\operatorname{vol}(B(m, \rho)) \geq \kappa(\mathfrak{t}(m))\rho^n$. In the application to Ricci flow with surgery, ϵ will be taken to be the global parameter.

We refer to [32, Section 15] for the definitions of the l-function $l(m)$ and the reduced volume $\tilde{V}(\tau)$. For notation, we recall that the l-function is defined in terms of $\mathcal{L}$-geodesics going backward in time from a basepoint $m' \in \mathcal{M}$. The parameter τ is backward time from m'; e.g. $\tau(m) = \mathfrak{t}(m') - \mathfrak{t}(m)$.

A.5. **κ-solutions.** Given $\kappa \in \mathbb{R}^+$, a *κ-solution* $\mathcal{M}$ is a smooth Ricci flow solution defined on a time interval of the form $(-\infty, C)$ (or $(-\infty, C]$) such that

- The curvature is uniformly bounded on each compact time interval, and each time slice is complete.
- The curvature operator is nonnegative and the scalar curvature is everywhere positive.
- The Ricci flow is κ-noncollapsed at all scales.

We will sometimes talk about κ-solutions without specifying κ. Unless other specified, it is understood that $C = 0$. If $(\mathcal{M}, m)$ is a pointed κ-solution then we will sometimes understand it to be defined on the interval $(-\infty, \mathfrak{t}(m)]$.

Examples of κ-solutions are *Cyl* and *Sphere*.

Any pointed κ-solution $(\mathcal{M}, m)$ has an *asymptotic soliton*. It is obtained by constructing the l-function using $\mathcal{L}$-geodesics emanating backward from m. For any $t < \mathfrak{t}(m)$, there is some point $m'_t \in \mathcal{M}_t$ where $l(m) \leq \frac{n}{2}$. Put $\tau = \mathfrak{t}(m) - t$. Then the parabolic rescaling $\left(\hat{\mathcal{M}}(\tau), m'_t\right)$ subconverges as $\tau \to \infty$ to a nonflat gradient shrinking soliton called the asymptotic soliton [32, Proposition 39.1]. (In the cited reference, the convergence is shown on the (rescaled) time interval $\left[-1,-\frac{1}{2}\right]$, but using the estimates of Subsection A.7 one easily gets pointed convergence on the time interval $(-\infty,0)$.)

Hereafter, we suppose that the spacetime $\mathcal{M}$ of the κ-solution is four-dimensional. A basic fact is that the space of pointed κ-solutions $(\mathcal{M},m)$, with $R(m)=1$, is compact [32, Theorem 46.1].

Given $\delta>0$, let $\mathcal{M}_\delta$ denote the points in $\mathcal{M}$ that are not centers of δ-necks. We call these *cap points*. Put $\mathcal{M}_{t,\delta}=\mathcal{M}_t\cap\mathcal{M}_\delta$. From [32, Corollary 47.2], if δ is small enough then there is a $C=C(\delta,\kappa)>0$ such that if $\mathcal{M}_t$ is noncompact then

- $\mathcal{M}_{t,\delta}$ is compact with $\mathrm{Diam}(\mathcal{M}_{t,\delta})\le CQ^{-\frac{1}{2}}$ and
- $C^{-1}Q\le R(m)\le CQ$ whenever $m\in M_{t,\delta}$,

where $Q=R(m')$ for some $m'\in\partial M_{t,\delta}$.

If $\mathcal{M}$ is noncompact, and not a round shrinking cylinder, then $\mathcal{M}_{t,\delta}\neq\emptyset$. A version of the preceding paragraph that also holds for compact κ-solutions can be found in [32, Corollary 48.1].

A compact κ-solution is either a quotient of the round shrinking sphere, or is diffeomorphic to S^3 or $\mathbb{R}P^3$ [32, Lemma 59.3].

There is some $\kappa_0>0$ so that any κ-solution is a κ_0-solution or a quotient of the round shrinking S^3 [32, Proposition 50.1]

A.6. **Gradient shrinking solitons.**

Lemma A.1. *Let $\mathcal{M}$ be a three-dimensional gradient shrinking soliton that is a κ-solution and blows up as $t\to0$. For $t<0$ and a point $(y,t)\in\mathcal{M}$, let $l_{y,t}\in C^\infty(\mathcal{M}_{<t})$ be the l-function on $\mathcal{M}$ constructed using $\mathcal{L}$-geodesics going backward in time from (y,t). Then there is a function $l_\infty\in C^\infty(\mathcal{M})$ so that the limit $\lim_{t\to0^-}l_{y,t}=l_\infty$ exists, independent of y, with continuous convergence on compact subsets on $\mathcal{M}$. Define $\tilde{V}_\infty:(-\infty,0)\to(0,\infty)$ as in (6.6). Then one of the following holds:*

1. *$\mathcal{M}$ is the shrinking round cylinder solution Cyl on $S^2\times\mathbb{R}$, with $R(x,t)=(-t)^{-1}$, $l_\infty((x,z),t)=1+\frac{z^2}{-4t}$ and $\tilde{V}_\infty(t)=\frac{16\pi^{\frac{3}{2}}}{e}$.*
2. *$\mathcal{M}$ is the $\mathbb{Z}_2$-quotient of the cylinder in (2), with $\tilde{V}_\infty(t)=\frac{8\pi^{\frac{3}{2}}}{e}$.*
3. *$\mathcal{M}$ is a shrinking round spherical space form $Sphere/\Gamma$, where $\Gamma\subset\mathrm{SO}(4)$, $R(x,t)=\frac{3}{2}(-t)^{-1}$, $l_\infty(x,t)=\frac{3}{2}$ and $\tilde{V}_\infty(t)=\frac{16\pi^2}{|\Gamma|}$.*

Proof. The classification of the solitons follows from [32, Corollary 51.22]. Let f be a potential for the soliton, i.e.

$$\mathrm{Ric}+\mathrm{Hess}(f)=-\frac{1}{2t}g \tag{A.2}$$

and

$$\frac{\partial f}{\partial t} = |\nabla f|^2. \tag{A.3}$$

There is a constant C so that $R + |\nabla f|^2 + \frac{1}{t} f = -\frac{C}{t}$.

From [17, Theorem 3.7] and [39, Proposition 2.5], for any sequence $t_i \to 0^-$, after passing to a subsequence there is a limit $\lim_{i\to\infty} l_{y,t_i}$ with continuous convergence on compact subsets of $\mathcal{M}$. In the case of a gradient shrinking soliton, [13, Chapter 7.7.3] implies that $\lim_{i\to\infty} l_{y,t_i} = f + C$; c.f. [17, Example 3.3]. Thus the limit $\lim_{t\to 0^-} l_{y,t}$ exists and equals $f + C$, independent of y. In our case, the formulas for l_∞ and $\tilde{V}_\infty$ now follow from straightforward calculation. □

A.7. **Estimates on l-functions.** We recall some estimates on the l-function that hold for κ-solutions, taken from [54]

The letter C will denote a generic universal constant. From [54, (2.53)],

$$R \leq \frac{Cl}{\tau}. \tag{A.4}$$

From [54, (2.54),(2.56)],

$$|\nabla l|^2, |l_\tau| \leq \frac{Cl}{\tau}. \tag{A.5}$$

From [54, (2.55)],

$$|\sqrt{l}(q_1, \tau) - \sqrt{l}(q_2, \tau)| \leq \sqrt{\frac{C}{4\tau}}\, d(q_1, q_2, \tau). \tag{A.6}$$

From [54, (2.57)],

$$(\frac{\tau_1}{\tau_2})^C \leq \frac{l(q, \tau_1)}{l(q, \tau_2)} \leq (\frac{\tau_2}{\tau_1})^C. \tag{A.7}$$

From [54, (3.7)],
(A.8)

$$- l(q_1, \tau) - 1 + C_1 \frac{d^2(q_1, q_2, \tau)}{\tau} \leq l(q_2, \tau) \leq 2l(q_1, \tau) + C_2 \frac{d^2(q_1, q_2, \tau)}{\tau}.$$

A.8. **Canonical neighborhoods.** In this section we recall the notion of a canonical neighborhood for a Ricci flow with surgery, and define the notion of a canonical neighborhood in a singular Ricci flow. We mention that this rather complicated looking definition is motivated by the structure of κ-solutions and the standard (postsurgery) solution.

Let $r : [0, \infty) \to (0, \infty)$ be a decreasing function. Let $\epsilon > 0$ be small enough so that the bulletpoints at the end of Subsection A.5

hold (with $\delta = \epsilon$). Let $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ be the constants in [32, Definition 69.1].

As in [32, Definition 69.1], a Ricci flow with surgery $\mathcal{M}$ defined on the time interval $[a,b]$ satisfies the *r-canonical neighborhood assumption* if every $(x,t) \in \mathcal{M}_t^{\pm}$ with scalar curvature $R(x,t) \geq r(t)^{-2}$ has a canonical neighborhood in the corresponding forward/backward time slice, in the following sense. There is an $\hat{r} \in (R(x,t)^{-\frac{1}{2}}, C_1 R(x,t)^{-\frac{1}{2}})$ and an open set $U \subset \mathcal{M}_t^{\pm}$ with $\overline{B^{\pm}(x,t,\hat{r})} \subset U \subset B^{\pm}(x,t,2\hat{r})$ that falls into one of the following categories :

(a) $U \times [t-\Delta t, t] \subset \mathcal{M}$ is a strong ϵ-neck for some $\Delta t > 0$. (Note that after parabolic rescaling the scalar curvature at (x,t) becomes 1, so the scale factor must be $\approx R(x,t)$, which implies that $\Delta t \approx R(x,t)^{-1}$.)

(b) U is an ϵ-cap which, after rescaling, is ϵ-close to the corresponding piece of a κ_0-solution or a time slice of a standard solution.

(c) U is a closed manifold diffeomorphic to S^3 or RP^3.

(d) U is ϵ-close to a closed manifold of constant positive sectional curvature.

Moreover, the scalar curvature in U lies between $C_2^{-1} R(x,t)$ and $C_2 R(x,t)$. In cases (a), (b), and (c), the volume of U is greater than $C_2^{-1} R(x,t)^{-\frac{3}{2}}$. In case (c), the infimal sectional curvature of U is greater than $C_2^{-1} R(x,t)$.

Finally, we require that

$$|\nabla R(x,t)| < \eta R(x,t)^{\frac{3}{2}}, \qquad \left| \frac{\partial R}{\partial t}(x,t) \right| < \eta R(x,t)^2, \tag{A.9}$$

where η is the constant from [32, (59.5)]. Here the time dervative $\frac{\partial R}{\partial t}(x,t)$ should be interpreted as a one-sided derivative when the point (x,t) is added or removed during surgery at time t.

We use a slightly simpler definition of canonical neighborhood in the case of singular Ricci flows, for Definition 1.5. We do not need to consider forward/backward time slices, and in case (b), we do not need to consider the case that U is close to a time slice of a standard solution.

Remark A.10. Alternatively, for a singular Ricci flow, one could replace the above definition of canonical neighborhood with the requirement that every point with $R \geq r(t)^{-2}$ is ϵ-modelled on a $\kappa(t)$-solution. This is quantitatively equivalent to the definition above, as follows from Proposition 5.30 and [32, Lemma 59.7].

A.9. **Ricci flow with surgery.** We recall that there are certain parameters in the definition of Ricci flow with surgery, namely a number $\epsilon > 0$ and positive nonincreasing functions $r, \kappa, \delta : [0,\infty) \to (0,\infty)$. The function r is the canonical neighborhood scale; c.f. Appendix A.8.

The function κ is the noncollapsing parameter; c.f. Appendix A.4. The parameter $\epsilon > 0$ is a global parameter in the definition of a Ricci flow with surgery [7, Remark 58.5].

The function $\delta : [0, \infty) \to (0, \infty)$ is a surgery parameter. There is a further parameter $h(t) < \delta^2(t) r(t)$ so that if a point (x, t) lies in an ϵ-horn and has $R(x, t) \geq h(t)^{-2}$, then (x, t) lies in a $\delta(t)$-neck [32, Lemma 71.1]. One can then perform surgery on such cross-sectional 2-spheres [32, Sections 72 and 73]. Perelman showed that there are positive nonincreasing step functions r_P, κ_P and $\overline{\delta}_P$ so that if the (positive nonincreasing) function δ satisfies $\delta(t) < \overline{\delta}_P(t)$ then there is a well-defined Ricci flow with surgery, with a discrete set of surgery times [32, Sections 77-80].

In particular, we can assume that δ is strictly decreasing. If $r \leq r_P$ and $\kappa \leq \kappa_P$ are positive functions then the r_P-canonical neighborhood assumption implies the r-canonical neighborhood assumption, and κ_P-noncollapsing implies κ-noncollapsing. Hence Ricci flow with surgery also exists in terms of the parameters (r, κ, δ). Consequently, we can assume that r, κ and δ are strictly decreasing.

As in [32, Section 68], the formal structure of a *Ricci flow with surgery* is given by

- A collection of Ricci flows $\{(M_k \times [t_k^-, t_k^+), g_k(\cdot))\}_{1 \leq k \leq N}$, where $N \leq \infty$, M_k is a compact (possibly empty) manifold, $t_k^+ = t_{k+1}^-$ for all $1 \leq k < N$, and the flow g_k goes singular at t_k^+ for each $k < N$. We allow t_N^+ to be ∞.
- A collection of limits $\{(\Omega_k, \bar{g}_k)\}_{1 \leq k \leq N}$, in the sense of [32, Section 67], at the respective final times t_k^+ that are singular if $k < N$. (Here Ω_k is an open subset of M_k.)
- A collection of isometric embeddings $\{\psi_k : X_k^+ \to X_{k+1}^-\}_{1 \leq k < N}$ where $X_k^+ \subset \Omega_k$ and $X_{k+1}^- \subset M_{k+1}$, $1 \leq k < N$, are compact 3-dimensional submanifolds with boundary. The $X_k^{\pm}$'s are the subsets which survive the transition from one flow to the next, and the ψ_k's give the identifications between them.

We will say that t is a *singular time* if $t = t_k^+ = t_{k+1}^-$ for some $1 \leq k < N$, or $t = t_N^+$ and the metric goes singular at time t_N^+.

A Ricci flow with surgery does not necessarily have to have any real surgeries, i.e. it could be a smooth nonsingular flow.

We now describe the Ricci flow spacetime associated with a Ricci flow with surgery. We begin with the time slab $M_k \times [t_k^-, t_k^+]$ for $1 \leq k \leq N$, which has a time function $\mathfrak{t} : M_k \times [t_k^-, t_k^+] \to [t_k^-, t_k^+]$ given by projection onto the second factor, and a time vector field $\partial_{\mathfrak{t}}$ inherited from the coordinate vector field on the factor $[t_k^-, t_k^+]$.

For every $1 < k \leq N$, put $W_k^- = (M_k \setminus \mathrm{Int}(X_k^-)) \times \{t_k^-\}$ and for $1 \leq k < N$, let $W_k^+ = (M_k \setminus \mathrm{Int}(X_k^+)) \times \{t_k^+\}$. Since $W_k^\pm$ is a closed subset of the 4-manifold with boundary $M_k \times [t_k^-, t_k^+]$, the complement $Z_k = (M_k \times [t_k^-, t_k^+]) \setminus (W_k^- \cup W_k^+)$ is a 4-manifold with boundary, where $\partial Z_k = (M_k \times \{t_k^-, t_k^+\}) \setminus (W_k^- \cup W_k^+)$. Note that the Ricci flow $g_k(\cdot)$ with singular limit $\bar{g}_k$ defines a smooth metric $\hat{g}_k$ on the subbundle $\ker d\mathfrak{t} \subset TZ_k$ that satisfies $\mathcal{L}_{\partial_\mathfrak{t}} \hat{g}_k = -2 \operatorname{Ric}(\hat{g}_k)$.

For every $1 \leq k < N$, we glue Z_k to Z_{k+1} using the identification $\mathrm{Int}(X_k^+) \stackrel{\psi_k}{\to} \mathrm{Int}(X_{k+1}^-)$, to obtain a smooth 4-manifold with boundary $\mathcal{M}$, where $\partial \mathcal{M}$ is the image of $W_1^- \cup W_N^+$ under the quotient map $\coprod_k Z_k \to \mathcal{M}$. The time functions, time vector fields, and metrics descend to $\mathcal{M}$, yielding a tuple $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ which is a Ricci flow spacetime in the sense of Definition 1.1.

Recall the notion of a normalized Riemannian manifold from the introduction. Our convention is that the trace of the curvature operator is the scalar curvature. From [32, Appendix B], if a smooth three-dimensional Ricci flow $\mathcal{M}$ has normalized initial condition then the scalar curvature satisfies

$$R(x,t) \ \geq \ - \frac{3}{1+2t}. \tag{A.11}$$

If follows that the volume satisfies

$$\mathcal{V}(t) \leq (1+2t)^{\frac{3}{2}} \mathcal{V}(0). \tag{A.12}$$

These estimates also hold for a Ricci flow with surgery.

Let A be a symmetric 3×3 real matrix. Let λ_1 denote its smallest eigenvalue. For $t \geq 0$, put

$$\begin{aligned} K(t) = \{ A \, : \, \mathrm{tr}(A) \geq -\frac{3}{1+t}, \text{ and if } \lambda_1 \leq -\frac{1}{1+t} \text{ then} \\ \mathrm{tr}(A) \geq -\lambda_1 \left(\log(-\lambda_1) + \log(1+t) - 3 \right) \}. \end{aligned} \tag{A.13}$$

Then $\{K(t\}_{t \geq 0}$ is a family of $O(3)$-invariant convex sets which is preserved by the ODE on the space of curvature operators [14, Pf. of Theorem 6.44]. If a smooth three-dimensional Ricci flow has normalized initial conditions then the time-zero curvature operators lie in $K(0)$. Using (A.11), we obtain the Hamilton-Ivey estimate that whenever the lowest eigenvalue $\lambda_1(x,t)$ of the curvature operator satisfies $\lambda_1 \leq -\frac{1}{1+t}$, we have

$$R \geq -\lambda_1 \left(\log(-\lambda_1) + \log(1+t) - 3 \right). \tag{A.14}$$

The surgery procedure is designed to ensure that (A.14) also holds for Ricci flows with surgery.

(Perelman's definition of a normalized Riemannian manifold is slightly different; he requires that the sectional curvatures be bounded by one in absolute value [41, Section 5.1]. With his convention, $R(x,t) \geq -\frac{6}{1+4t}$ and $\mathcal{V}(t) \leq (1+4t)^{\frac{3}{2}} \mathcal{V}(0)$.)

References

[1] S. Angenent, C. Caputo and D. Knopf, "Minimally invasive surgery for Ricci flow singularities", J. Reine Angew. Math. (Crelle) 672, p. 39-87 (2012)

[2] S. Angenent and D. Knopf, "Precise asymptotics of the Ricci flow neckpinch", Comm. Anal. Geom. 15, p. 773-844 (2007)

[3] W. Ballmann, Lectures on spaces of nonpositive curvature, DMV Seminar Band 25, Birkhäuser, Basel (1995)

[4] K. Brakke, The motion of a surface by its mean curvaure, Mathematical Notes 20, Princeton University Press, Princeton (1978)

[5] S. Brendle and G. Huisken, "Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$", preprint, http://arxiv.org/abs/1309.1461 (2013)

[6] M. Gromov, J. Cheeger and M. Taylor, "Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds", J. Diff. Geom. 17, p. 15-53 (1982)

[7] Y. G. Chen, Y. Giga, S. Goto, "Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations", J. Diff. Geom. 33, p. 749-786 (1991)

[8] B.-L. Chen, S.-H. Tang and X.-P. Zhu, "Complete classification of compact four-manifolds with positive isotropic curvature", J. Diff. Geom. 91, p. 41-80 (2012)

[9] X. X. Chen, G. Tian and Z. Zhang, "On the weak Kähler-Ricci flow", Trans. Amer. Math. Soc. 363, p. 2849-2863 (2011)

[10] X. X. Chen and Y. Wang, "Bessel functions, heat kernel and the conical Kähler-Ricci flow", preprint, http://arxiv.org/abs/1305.0255 (2013)

[11] X. X. Chen and Y. Wang, "On the long time behavior of the conical Kähler-Ricci flows", http://arxiv.org/abs/1402.6689 (2014)

[12] B. Chow, "On the entropy estimate for the Ricci flow on compact 2-orbifolds", J. Diff. Geom. 33, p. 597-600 (1991)

[13] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther and J. Isenberg, The Ricci flow : techniques and applications: geometric aspects, part I, Amer. Math. Soc., Providence (2007)

[14] B. Chow, P. Lu and L. Ni, Hamilton's Ricci flow, Amer. Math. Soc., Providence (2006)

[15] B. Chow and L.-F. Wu, "The Ricci flow on compact 2-orbifolds with curvature negative somewhere", Comm. Pure Appl. Math. 44, p. 275-286 (1991)

[16] J. F. Colombeau, "Infinite dimensional C^∞ mappings with a given sequence of derivatives at a given point", J. of Math. Analysis and Applications 31, p. 95-104 (1979)

[17] J. Enders, "Reduced distance based at singular time in the Ricci flow", preprint, http://arxiv.org/abs/0711.0558 (2007)

[18] L. C. Evans and J. Spruck, "Motion of level sets by mean curvature I", J. Diff. Geom. 33, p. 635-681 (1991)

[19] P. Eyssidieux, V. Guedj and A. Zeriahi, "Weak solutions to degenerate complex Monge-Ampère Flow II", preprint, http://arxiv.org/abs/1407.2504 (2014)
[20] H. Federer and W. Fleming, "Normal and integral currents", Ann. Math. 72, p. 428-520 (1960)
[21] M. Feldman, T. Ilmanen and D. Knopf, "Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons", J. Diff. Geom. 65, p. 169-209 (2003)
[22] G. Giesen and P. Topping, "Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics", J. Eur. Math. Soc. 12, p. 1429-1451 (2010)
[23] G. Giesen and P. Topping, "Existence of Ricci flows of incomplete surfaces", Comm. Partial Differential Equations 36, p. 1860-1880 (2011)
[24] G. Giesen and P. Topping, "Ricci flows with unbouded curvature", Math. Z. 273, p. 449-460 (2013)
[25] R Hamilton, "A compactness property for solutions of the Ricci flow", Amer. J.. Math. 117, p. 545-572 (1995)
[26] R. Hamilton, "Four-manifolds with positive isotropic curvature", Comm. Anal. Geom. 5, p. 1-92 (1997)
[27] R. Hamilton, "Three-orbifolds with positive Ricci curvature", in Collected papers on Ricci flow, Ser. Geom. Top. 27, International Press, Somerville (2003)
[28] R. Haslhofer and B. Kleiner, "Mean curvature flow with surgery", preprint, http://arxiv.org/abs/1404.2332 (2014)
[29] J. Head, "On the mean curvature evolution of two-convex hypersurfaces", J. Diff. Geom. 94, p. 241-266 (2013)
[30] T. Ilmanen, "Elliptic regularization and partial regularity for motion by mean curvature", Mem. Amer. Math. Soc. 108, no. 520 (1994)
[31] T. Ivey, "Ricci solitons on compact three-manifolds", Diff. Geom. and its Applications 3, p. 301-307 (1993)
[32] B. Kleiner and J. Lott, "Notes on Perelman's papers", Geom. and Top. 12, 2587-2858 (2008)
[33] B. Kleiner and J. Lott, "Geometrization of three-dimensional orbifolds via Ricci flow", to appear, Astérisque
[34] H. Koch and T. Lamm, "Geometric flows with rough initial data", Asian J. Math. 16, p. 209-235 (2012)
[35] J. Lauer, "Convergence of mean curvature flows with surgery", Comm. Anal. and Geom. 21, p. 355-363 (2013)
[36] J. Liu and X. Zhang, "The conical Kähler-Ricci flow on Fano manifolds", preprint, http://arxiv.org/abs/1402.1832 (2014)
[37] P. Lu, "A compactness property for solutions of the Ricci flow on orbifolds", Amer. J. Math. 123, p. 1103-1134 (2001)
[38] R. Mazzeo, Y. Rubinstein and N. Sesum, "Ricci flow on surfaces with conic singularities", preprint, http://arxiv.org/abs/1306.6688 (2013)
[39] A. Naber, "Noncompact shrinking four solitons with nonnegative curvature", J. für die reine und angew. Math. (Crelle's Journal) 645, p. 125-153 (2010)
[40] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, http://arxiv.org/abs/math/0211159 (2002)
[41] G. Perelman, "Ricci flow with surgery on three-manifolds", preprint, http://arxiv.org/abs/math/0303109 (2003)

[42] D. Phong, J. Song, J. Sturm and X. Wang, "The Ricci flow on the sphere with marked points", preprint, http://arxiv.org/abs/1407.1118 (2014)

[43] O. Schnürer, F. Schulze and M. Simon, "Stability of Euclidean space under Ricci flow", Comm. Anal. Geom. 16, p. 127-158 (2008)

[44] L. Simon, Lectures on geometric measure theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra (1983)

[45] M. Simon, "Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature", Comm. Anal. Geom. 10, p. 1033-1074 (2002)

[46] M. Simon, "Ricci flow of almost non-negatively curved three manifolds", J. Reine Angew. Math. 630, p. 177-217 (2009)

[47] M. Simon, "Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below", J. Reine Angew. Math. 662, p. 59-94 (2012)

[48] J. Song and G. Tian, "The Kähler-Ricci flow through singularities", preprint, http://lanl.arxiv.org/abs/0909.4898 (2009)

[49] P. Topping, Lectures on the Ricci flow, Cambridge University Press, Cambridge (2006)

[50] P. Topping, "Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics", J. Eur. Math. Soc. 12, p. 1429-1451 (2010)

[51] P. Topping, "Uniqueness and nonuniqueness for Ricci flow on surfaces", Int. Math. Res. Not. IMRN 2012, p. 2356-2376 (2012)

[52] Y. Wang, "Smooth approximations of the conical Kähler-Ricci flows", preprint, http://arxiv.org/abs/1401.5040 (2014)

[53] L.-F. Wu, "The Ricci flow on 2-orbifolds with positive curvature", J. Diff. Geom. 33, p. 575-596 (1991)

[54] R. Ye, "On the l-function and the reduced volume of Perelman I", Trans. Amer. Math. Soc. 360, p. 507-531 (2008)

[55] H. Yin, "Ricci flow on surfaces with conical singularities", J. Geom. Anal. 20, p. 970-995

[56] H. Yin, "Ricci flow on surfaces with conical singularities II", preprint, http://arxiv.org/abs/1305.4355 (2013)

Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012

E-mail address: bkleiner@cims.nyu.edu

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720

E-mail address: lott@berkeley.edu