Analytic number theory, homework 1.

Exercise 1. Let q be a positive integer. Show that if $\sigma > 1$ then

$$\sum_{n\geq 1, n\wedge q=1} n^{-s} = \zeta(s)\prod_{p|q}(1-p^{-s}),$$

where the product is over primes dividing q.

Exercise 2. Let $G(s) = \sum p^{-s}$ be the prime zeta function. Prove that

$$G(s) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log \zeta(ds)$$

for any if $\sigma > 1$. Show that G can be extended to $\sigma > 0$, the extension having a countable number of (logarithmic) singularities on this domain.

Exercise 3. For a given $k \in \mathbb{N}^*$, let $\sigma_k(x)$ be the number of integers in $[\![2, x]\!]$ such that $\Omega(n) = k$. Prove that

$$\sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}$$

as $x \to \infty$.

Exercise 4. Prove that, for any $|z| < 2$ and $\sigma > 1$,

$$\sum_{n\geq 1} \frac{z^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{z}{p^s - 1}\right),$$

$$\sum_{n\geq 1} \frac{z^{\Omega(n)}}{n^s} = \prod_p \frac{1}{1-\frac{z}{p^s}}.$$

Exercise 5. Prove that for a small enough constant c the following holds, uniformly on $|t| > 1$, $\sigma > 1 - \frac{c}{\log \tau}$ ($\tau = |t| + 4$):

$$\frac{\zeta'}{\zeta}(s) \ll \log \tau,$$

$$\log \zeta(s) \ll \log\log \tau + \mathrm{O}(1),$$

$$\frac{1}{\zeta(s)} \ll \log \tau.$$

Exercise 6. Let $\alpha(s) = \sum a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c, and $\mathrm{si}(x) = -\int_x^\infty \frac{\sin u}{u}\mathrm{d}u$. Prove the following quantitative version of Perron's formula: for any $\sigma_0 > \max\{0, \sigma_c\}$, uniformly in $x > C$, C large enough, we have

$$\sum_{n<x} a_n = \frac{1}{2\pi\mathrm{i}}\int_{\sigma_0-\mathrm{i}T}^{\sigma_0+\mathrm{i}T} \alpha(s)\frac{x^s}{s}\mathrm{d}s + R,$$

where

$$R = \frac{1}{\pi}\sum_{x/2<n<x} a_n \mathrm{si}\left(T\log(x/n)\right) - \frac{1}{\pi}\sum_{x<n<2x} a_n \mathrm{si}\left(T\log(n/x)\right) + \mathrm{O}\left(\frac{x^{\sigma_0}}{T}\sum_{n\geq 1}\frac{|a_n|}{n^{\sigma_0}}\right).$$