

### Analytic number theory, homework 3.

**Exercise 1.** Let  $q(s) = \sum \mu(n)^2/n^s$ . Where does  $q$  converge absolutely? Does it have an Euler product, and if yes in which region does it converge absolutely? Can you express  $q$  in terms of  $\zeta$ ? What is the largest domain in which you can find an analytic continuation for  $(s-1)q(s)$ ?

**Exercise 2.** Let  $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$ . Let  $K$  be a given positive integer and  $\alpha_1, \dots, \alpha_K$  real numbers. Prove that for any  $N \geq 1$  there exists  $1 \leq n \leq N^K$  such that for any  $j \in \llbracket 1, K \rrbracket$  we have  $\|n\alpha_j\| \leq \frac{1}{N}$ .

Let  $\sigma > 1$  and  $\varepsilon > 0$  be given. Prove that there exists  $T > 0$  such that for any  $t \in \mathbb{R}$  we have

$$|\zeta(\sigma + it + iT) - \zeta(\sigma + it)| \leq \varepsilon.$$

**Exercise 3.** Reproduce a complete proof of the prime number theorem along arithmetic sequences in the special case (mod 4), simplifying the general proof as much as you can. In particular, show directly that, if  $\chi_1$  is the non-principal character (mod 4), then

$$L(1, \chi_1) = \frac{\pi}{4} \neq 0.$$