Complex analysis, homework 10 due April 20th.

Exercise 1. [8 points] For \( n \geq 0 \), let
\[
z_n = \frac{(n + i)^2 - 2in^2}{n^2}.
\]
Prove that \( \lim_{n \to \infty} z_n = 1 - 2i \) using the definition of the limit.

Exercise 2. [6 points] Let \((z_n)_{n \geq 0}\) be a sequence of complex numbers. Let \( S \in \mathbb{C} \).
Prove that
\[
\sum_{n=0}^{\infty} z_n = S \Rightarrow \sum_{n=0}^{\infty} \overline{z_n} = \overline{S}.
\]

Exercise 3. [10 points] Prove that the Taylor series of \( \log \) at \( i \) is
\[
\log(z) = \frac{i\pi}{2} + \sum_{k=1}^{\infty} \frac{-i^k}{k} (z - i)^k.
\]
Precise the complex numbers \( z \) for which this formula applies.

Exercise 4. [6 points] Find the Taylor series at \( 0 \) of
\[
f(z) = \frac{\sin(z) - z}{z^2}.
\]