Complex analysis, homework 11, solutions.

Exercise 1. [9 points] Give three Laurent expansions in powers of $z$ for the function

$$f(z) = \frac{i - 3}{(z - i)(z - 3)} = \frac{1}{z - i} - \frac{1}{z - 3}$$

and specify the annular domains in which those expansions are valid.

**Solution.** The function $f$ is analytic on $\mathbb{C} \setminus \{i, 3\}$. So it is analytic on the following annular domains centered at 0

$$D_1 = \{z : |z| < 1\}, \quad D_2 = \{z : 1 < |z| < 3\}, \quad D_3 = \{z : 3 < |z|\}.$$  

Note that in $D_1$ we can actually include 0, because $f$ is analytic at 0, so we actually know that $f$ can be expanded as a Taylor series on $D_1$.

If $|z| < 1$, then $|z/i| < 1$ and therefore

$$\frac{1}{z - i} = \frac{1}{z} \cdot \frac{1}{1 - (z/i)} = -\frac{1}{i} \sum_{n=0}^{\infty} (z/i)^n = -\sum_{n=0}^{\infty} \frac{z^n}{i^{n+1}}.$$  

On the other hand, if $|z| > 1$, then $|i/z| < 1$ and therefore

$$\frac{1}{z - i} = \frac{1}{z} \cdot \frac{1}{1 - (i/z)} = \frac{1}{z} \sum_{n=0}^{\infty} (i/z)^n = \sum_{n=0}^{\infty} i^n z^n = \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n}.$$  

We proceed similarly to get

$$\frac{1}{z - 3} = -\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} \quad \text{if } |z| < 3,$$

$$\frac{1}{z - 3} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{z^n} \quad \text{if } |z| > 3.$$  

Combining this we get

$$f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{3^{n+1} - i^{n+1}} \right) z^n \quad \text{if } z \in D_1,$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} z^n + \sum_{n=1}^{\infty} \frac{i^{n-1}}{z^n} \quad \text{if } z \in D_1,$$

$$f(z) = \sum_{n=1}^{\infty} \left( i^{n-1} - 3^{n-1} \right) \frac{1}{z^n} \quad \text{if } z \in D_3.$$  

Exercise 2. [12 points] Find the radius of convergence of the following power series. Explain your answer.

(1) $\sum_{n=0}^{\infty} (n + 1)^2 n z^n$;

(2) $\sum_{n=0}^{\infty} (n2^n + 3^n) z^n$;
\[ (3) \sum_{n=0}^{\infty} (pe^{i\theta} z)^n, \text{ for some } \theta \in \mathbb{R} \quad \text{and} \quad (4) \sum_{n=0}^{\infty} \frac{n z^{2n}}{(4i)^n}; \]

Solution.

(1) We have \( a_n = (n+1)^2r^n \). Let \( r > 0 \). Then \( |a_n r^n| = ((n+1)^2r)^n \geq 2^n \) for \( n \geq \sqrt{2/r} \). Since \( 2^n \to +\infty \) as \( n \to +\infty \), by comparison we get that \( |a_n r^n| \to +\infty \). In particular \( (a_n r^n) \) is not a bounded sequence for any \( r > 0 \). So the radius of convergence is 0.

(2) We have \( a_n = (n2^n + 3^n) \).

- Let \( r > 1/3 \). Then \( |a_n r^n| = n(2r)^n + (3r)^n \geq (3r)^n \) for any \( n \to +\infty \) because \( 3r > 1 \). So the sequence \( (a_n r^n) \) is not bounded for any \( r > 1/3 \).
- Let \( r < 1/3 \). Then \( (3r)^n \to 0 \) as \( n \to +\infty \). Moreover \( n(2r)^n \to 0 \) because \( 2r < 1 \). And geometric sequences “dominate” polynomial sequences. Therefore \( |a_n r^n| = n(2r)^n + (3r)^n \) is bounded for any \( r < 1/3 \). So the radius of convergence is 1/3.

Note that we don’t need to study the behavior of the sequence when \( r = 1/3 \) to conclude.

(3) We have \( a_n = (pe^{i\theta})^n \). Therefore \( |a_n r^n| = (pr)^n \) so the sequence \( (a_n r^n) \) is bounded if and only if \( r \leq 1/|p| \). So the radius of convergence is \( 1/|p| \).

(4) We have \( a_{2n} = \frac{2^n}{(4n)^n} \) and \( a_{2n+1} = 0 \) for \( n \geq 0 \). The sequence \( (a_{2n+1} r^{2n+1}) \) is bounded for any \( r \geq 0 \). The sequence \( (a_{2n} r^{2n}) \) is bounded whenever the sequence \( (a_{2n} r^{2n}) \) is bounded. But we have \( |a_{2n} r^{2n}| = n(r/2)^{2n} \). If \( r < 2 \), this sequence converges to 0 and therefore is bounded. If \( r > 2 \), this sequence tends to +\infty and therefore is not bounded. So the radius of convergence is 2.

Exercise 3. [5 points] Show that the following function is entire

\[ f(z) = \begin{cases} \frac{\sin(z)}{z-\pi} & \text{if } z \neq \pi, \\ -1 & \text{if } z = \pi. \end{cases} \]

Solution. The function \( \sin \) is analytic on \( \mathbb{C} \), so by Taylor’s theorem, it is equal to its Taylor series at \( \pi \) on the whole complex plane:

\[ \sin(z) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(\pi)}{n!} (z-\pi)^n, \quad z \in \mathbb{C}. \]

Note that \( \sin(\pi) = 0 \) so the first term is 0. Therefore, for any \( z \neq 0 \),

\[ f(z) = \frac{1}{z-\pi} \sum_{n=1}^{\infty} \frac{\sin^{(n)}(\pi)}{n!} (z-\pi)^n = \sum_{k=0}^{\infty} \frac{\sin^{(k+1)}(\pi)}{(k+1)!} (z-\pi)^k. \]
Since, \( \sin'(\pi) = \cos(\pi) = -1 \), this last power series equals \(-1 = f(\pi)\) when \( z = \pi \). So we conclude that, for any \( z \in \mathbb{C} \),
\[
f(z) = \sum_{k=0}^{\infty} \frac{\sin((k+1)\pi)}{(k+1)!} (z - \pi)^k.
\]

In particular, by proving this formula, we showed that this last series is convergent for any \( z \in \mathbb{C} \) that is its radius of convergence is infinite. By the theorem of Sec. 71, this power series is analytic on \( \mathbb{C} \) and therefore \( f \) is entire.

Note that we did not need to calculate all the coefficients of the series (even if we could have). This means that we can replace \( \sin(z) \) by any entire function \( g(z) \) such that \( g(\pi) = 0 \) and \( g'(\pi) = -1 \) and the result would have been true as well.

**Exercise 4.** [4 points] Recall \( \log z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n} \), for \( |z-1| < 1 \). For \( |z-1| < 1 \), let \( C_z \) be a contour from 1 to \( z \) included in the open disk centered at 1 with radius 1. Write the following quantity as a power series in \( z \) around 1:
\[
\int_{C_z} \log(w) \, dw.
\]
Justify your answer.

**Solution.** We have, for \( |z-1| < 1 \),
\[
\log z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}.
\]
Since this power series is convergent on the whole open disk centered at 1 with radius 1 and the contour \( C_z \) is included in this disk, we can integrate it term by term (by the theorem of Sec. 71) and we get
\[
\int_{C_z} \log(w) \, dw = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{C_z} (w-1)^n \, dw.
\]
Now, note that \( (w-1)^n \) is continuous and has antiderivative \( \frac{(w-1)^{n+1}}{n+1} \) on the whole complex plane, so we have
\[
\int_{C_z} (w-1)^n \, dw = \left[ \frac{(w-1)^{n+1}}{n+1} \right]_1^z = \frac{(z-1)^{n+1}}{n+1} - 0.
\]
Therefore, we get
\[
\int_{C_z} \log(w) \, dw = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} (z-1)^{n+1} = \sum_{k=2}^{\infty} \frac{(-1)^k}{(k-1)k} (z-1)^k.
\]