Exercise 1. [8 points] Find the singularities of the following functions and the residues of the function at each singularity.

(1) \( f(z) = \frac{1}{z^2 + 5z + 6} \);

(2) \( f(z) = \frac{1}{(z^2 - 1)^2} \).

Solution.

(1) We have

\[ f(z) = \frac{1}{z^2 + 5z + 6} = \frac{1}{(z + 2)(z + 3)}, \]

so \( f \) is analytic on \( \mathbb{C} \setminus \{-2, -3\} \). The singularities of \( f \) are \(-2\) and \(-3\).

At \(-2\), we write, for \(|z + 2| < 1\),

\[ f(z) = \frac{1}{z + 2} \cdot \frac{1}{1 + (z + 2)} = \frac{1}{z + 2} \cdot \sum_{n=0}^{\infty} (-1)^n (z+2)^n = \sum_{n=0}^{\infty} (-1)^n (z+2)^n = \frac{1}{z + 2} + \cdots \]

So \( \text{Res}_{z=-2} f(z) = 1 \).

At \(-3\), we write, for \(|z + 3| < 1\),

\[ f(z) = \frac{-1}{z + 3} \cdot \frac{1}{1 - (z + 3)} = \frac{-1}{z + 3} \cdot \sum_{n=0}^{\infty} (z + 3)^n = -\sum_{n=0}^{\infty} (z + 3)^n = \frac{-1}{z + 3} + \cdots \]

So \( \text{Res}_{z=-3} f(z) = -1 \).

(2) We have

\[ f(z) = \frac{1}{(z^2 - 1)^2} = \frac{1}{(z - 1)^2(z + 1)^2}, \]

so \( f \) is analytic on \( \mathbb{C} \setminus \{-1, 1\} \). The singularities of \( f \) are \(-1\) and \(1\).

At \(-1\), we write, for \(|z + 1| < 2\),

\[ f(z) = \frac{1}{(z + 1)^2} \cdot \frac{1}{(2 - (z + 1))^2} = \frac{1}{4(z + 1)^2} \cdot \frac{1}{(1 - \frac{z+1}{2})^2} \]

\[ = \frac{1}{4(z + 1)^2} \cdot \sum_{n=0}^{\infty} (n + 1) \left( \frac{z + 1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(n + 1)}{2^{n+2}} (z + 1)^{n-2} \]

\[ = \cdots + \frac{1}{2(z + 1)} + \cdots \]

So \( \text{Res}_{z=-1} f(z) = 1/4 \).
At 1, we write, for $|z - 1| < 2$,

$$f(z) = \frac{1}{(z - 1)^2} \cdot \frac{1}{\left(2 - (1 - z)^2\right)^2} = \frac{1}{4(z - 1)^2} \cdot \frac{1}{\left(1 - \frac{1}{4}z^2\right)^2}$$

$$= \frac{1}{4(z - 1)^2} \cdot \sum_{n=0}^{\infty} (n + 1) \left(\frac{1 - z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(n + 1)(-1)^n}{2^{n+2}} (z - 1)^{n-2}$$

$$= \cdots + \frac{(1 + 1)(-1)}{8(z - 1)} + \cdots$$

So $\text{Res}_{z=1} f(z) = -1/4$.

**Exercise 2.** [10 points] Evaluate the integral $\int_{C} f(z) \, dz$ for the following functions and where $C$ is a positively oriented simple closed contour around 0.

1. $f(z) = z^7 \cos\left(\frac{1}{z^2}\right)$
2. $f(z) = \frac{\sinh(2z) - 2z}{z^8}$

**Solution.** In each case, we want to find the residues of $f$ at 0. For this, we expand $f(z)$ as a Laurent series at 0.

1. Let $z \neq 0$. We have

$$f(z) = z^7 \cos\left(\frac{1}{z^2}\right) = z^7 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{z^2}\right)^{2n} = \cdots + \frac{(-1)^2}{(2 \cdot 2)!} \frac{1}{z} + \cdots$$

So $\text{Res}_{z=0} f(z) = 1/24$ and therefore

$$\int_{C} f(z) \, dz = 2i\pi \text{Res}_{z=0} f(z) = \frac{i\pi}{12}.$$

2. Let $z \neq 0$. We have

$$f(z) = \frac{1}{z^8} (\sinh(2z) - 2z) = \frac{1}{z^8} \cdot \sum_{n=0}^{\infty} \frac{(2z)^{2n+1}}{(2n+1)!} - 2z = \frac{1}{z^8} \cdot \sum_{n=1}^{\infty} \frac{(2z)^{2n+1}}{(2n+1)!}$$

$$= \cdots + \frac{2^{2+1}}{(2 \cdot 3 + 1)!} \frac{1}{z} + \cdots$$

So $\text{Res}_{z=0} f(z) = 2^7 / 7! = 8/315$ and therefore

$$\int_{C} f(z) \, dz = 2i\pi \text{Res}_{z=0} f(z) = \frac{16i\pi}{315}.$$

**Exercise 3.** [4 points] Let $f$ be an entire function such that for any $\theta \in [0, \pi]$, $f(i\theta) = e^{\theta}$. Find $f(z)$ for any $z \in \mathbb{C}$. Justify your answer.

**Solution.** For $z = i\theta$ with $\theta \in [0, \pi]$, we have

$$f(z) = f(i\theta) = e^{\theta} = e^{-iz}.$$ 

Therefore, the functions $f(z)$ and $e^{-iz}$ coincide on the segment line from 0 to $i\pi$. Moreover, there are both analytic on the domain $\mathbb{C}$, which contains this line segment. Hence, by the theorem of Section 28, we have $f(z) = e^{-iz}$ for any $z \in \mathbb{C}$.
Exercise 4. [8 points] Let $f$ be the function defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where we assume that these series converge on the annular domain $D = \{ R_1 < |z - z_0| < R_2 \}$ for some $0 \leq R_1 < R_2 \leq +\infty$. The goal of this exercise is to prove the following result seen in class: $f$ is analytic on $D$ and

$$f'(z) = \sum_{n=1}^{\infty} na_n (z - z_0)^{n-1} + \sum_{n=1}^{\infty} (-n)b_n (z - z_0)^{-n-1}, \quad z \in D.$$

For this, you are allowed to apply the theorem for power series seen in Section 71 (involving only non-negative powers of $z - z_0$), but not the theorem for Laurent series that we are trying to prove.

(1) Let $f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, for $z \in D$. Explain why this power series has a radius of convergence larger than or equal to $R_2$. Deduce that $f_1$ is analytic on $D$ and

$$f_1'(z) = \sum_{n=1}^{\infty} na_n (z - z_0)^{n-1}, \quad z \in D.$$

(2) Let $f_2(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$, for $z \in D$. We introduce the following power series

$$g(w) = \sum_{n=1}^{\infty} b_n w^n.$$

Noting that for $z \in D$, $f_2(z) = g\left(\frac{1}{z - z_0}\right)$, show that the power series $g(z)$ has a radius of convergence larger than or equal to $1/R_1$ and therefore is analytic on $\{ w : |w| < 1/R_1 \}$. Find $g'(w)$.

(3) Deduce from question (b) that $f_2$ is analytic on $D$ and

$$f_2'(z) = \sum_{n=1}^{\infty} (-n)b_n (z - z_0)^{-n-1}, \quad z \in D.$$

(4) Using questions (a) and (c), conclude the exercise.

Solution.

(1) Let $R$ be the radius of convergence of the power series $f_1(z)$. We have seen in the theorem of Section 69 that for any $z$ such that $|z - z_0| > R$, the power series diverges. But we know by the assumption of the exercise that the power series converges at any $z \in D$. Therefore, $D$ has to be included in the set $\{ z : |z - z_0| \leq R \}$. This implies $R_2 \leq R$.

Let $z \in D$. We have $|z - z_0| < R_2 \leq R$, so $z$ is in the open disk of convergence and therefore by the theorem of Section 71, $f$ is analytic at $z$ and

$$f_1'(z) = \sum_{n=1}^{\infty} na_n (z - z_0)^{n-1}.$$

In particular, $f_1$ is analytic on $D$. 

(2) Let $R'$ be the radius of convergence of the power series $g(z)$. Let $w \neq 0$. We set $z = z_0 + \frac{1}{w}$, so that $w = \frac{1}{z-z_0}$ and therefore $g(w) = f_2(z)$. Note that $z \in D \iff R_1 < |z-z_0| < R_2 \iff \frac{1}{R_2} < |w| < \frac{1}{R_1}$.

So, for any $\frac{1}{R_2} < |w| < \frac{1}{R_1}$, the power series $g(w)$ converges (because $z \in D$ so the series $f_2(z)$ converges by the assumption of the exercise). As in (a), this implies, by the theorem of Section 69, that $R' \geq 1/R_1$. Therefore, by the theorem of Section 71, $g$ is analytic on \{ $w : |w| < 1/R_1$ \} and

$$g'(w) = \sum_{n=1}^{\infty} b_n w^{n-1}.$$  

(3) For $z \in D$, we have $f_2(z) = g\left(\frac{1}{z-z_0}\right)$ and $|\frac{1}{z-z_0}| < 1/ R_1$ so by the chain rule $f_2$ is differentiable at $z$ and

$$f_2'(z) = \frac{-1}{(z-z_0)^2} g'\left(\frac{1}{z-z_0}\right) = \frac{-1}{(z-z_0)^2} \sum_{n=1}^{\infty} b_n n (z-z_0)^{n-1} = \sum_{n=1}^{\infty} (-n) b_n (z-z_0)^{-n-1}.$$  

In particular, $f_2$ is analytic on $D$.

(4) We have $f = f_1 + f_2$ and we know that $f_1$ and $f_2$ are analytic on $D$. So $f$ is analytic on $D$ and

$$f'(z) = f_1'(z) + f_2'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} + \sum_{n=1}^{\infty} (-n) b_n (z-z_0)^{-n-1}, \quad z \in D.$$  