Exercise 1. For the following functions, say at which points they are differentiable and find their derivatives. Show your steps.

1. \( f(z) = \frac{z^2}{iz + 1} \)
2. \( f(z) = z(z^2 + iz)^5 \)

Solution.

1. First note that \( f(z) \) is defined iff \( iz + 1 \not= 0 \), which is equivalent to \( z \not= i \). For \( z \not= i \), the numerator and the denominator are differentiable at \( z \) as polynomials and the denominator is non-zero, so by the quotient rule, \( f \) is differentiable at \( z \) and
   \[
   f'(z) = \frac{2z(iz + 1) - iz^2}{(iz + 1)^2} = \frac{iz^2 + 2z}{(iz + 1)^2}.
   \]

2. The function \( f \) is a polynomial so it is differentiable everywhere. Using the product rule and then the chain rule, we get
   \[
   f'(z) = (z^2 + iz)^5 + z \cdot \frac{d}{dz} (z^2 + iz)^5
   = (z^2 + iz)^5 + z \cdot (2z + i) \cdot 5(z^2 + iz)^4
   = (z^2 + iz)^4(2z + i + 5z(2z + i))
   = (z^2 + iz)^4(11z + 6i).z.
   \]

Exercise 2[5 points] Let \( z_0 \in \mathbb{C} \). Let \( f \) be a function differentiable at \( z_0 \). For any \( z \in \mathbb{C} \) such that \( f(z) \) is defined, we set \( g(z) = f(\bar{z}) \). Prove that \( g \) is differentiable at \( \bar{z}_0 \) and express \( g'(\bar{z}_0) \) in terms of \( f'(z_0) \).

Solution. For \( h \in \mathbb{C} \) in a small enough neighborhood of 0, we have
   \[
   g(z_0 + h) - g(z_0) = f(\bar{z}_0 + h) - f(\bar{z}_0) \quad \text{(by definition of } g)\]
   \[
   = \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{(using } \bar{z} = z \text{ and } z_1 \pm z_2 = \bar{z}_1 \pm \bar{z}_2)\]
   \[
   = \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{(using } z_1/z_2 = \bar{z}_1/\bar{z}_2).\]
We know that \( \lim_{h \to 0} h = 0 \) and \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \), so by composition of limits \( \lim_{h \to 0} \frac{f(\bar{z}_0 + h) - f(\bar{z}_0)}{h} = f'(z_0) \). Moreover, using that the limit of the conjugate equals the conjugate of the limit, we get
   \[
   \lim_{h \to 0} g(z_0 + h) - g(z_0) = f'(z_0).\]
So \( g \) is differentiable at \( \bar{z}_0 \) and \( g'(\bar{z}_0) = \bar{f}'(z_0) \).

Exercise 3[8 points] Let \( f(z) = z \operatorname{Im}(z) \) for \( z \in \mathbb{C} \). Find the points \( z \in \mathbb{C} \) where \( f \) is differentiable and find its derivative \( f'(z) \) at these points. For all the other points in the complex plane, prove that
Solution. For any \( z = x + iy \in \mathbb{C} \), we can write \( f(z) = (x + iy)y = xy + iy^2 = u(x, y) + iv(x, y) \), with \( u(x, y) = xy \) and \( v(x, y) = y^2 \). The functions \( u \) and \( v \) have partial derivatives everywhere, which are
\[
\begin{align*}
    u_x(x, y) &= y & u_y(x, y) &= x & v_x(x, y) &= 0 & v_y(x, y) &= 2y.
\end{align*}
\]
Hence, for \((x, y) \in \mathbb{R}^2\), we have
\[
\begin{cases}
    u_x(x, y) = v_y(x, y) \\
    u_y(x, y) = -v_x(x, y)
\end{cases} \quad \Leftrightarrow \quad \begin{cases}
    y = 2y \\
    x = 0
\end{cases} \quad \Leftrightarrow \quad \begin{cases}
    y = 0 \\
    x = 0
\end{cases}
\]
Hence, Cauchy-Riemann equations are only satisfied at \((0, 0)\). So for any \( z \neq 0 \), \( f \) is not differentiable at \( z \).

We now want to prove \( f \) is differentiable at 0.

Approach 1 (most efficient): Since \( u \) and \( v \) have partial derivatives in a neighborhood of \((0, 0)\), they are all continuous at \((0, 0)\) and Cauchy-Riemann equations are satisfied at \((0, 0)\), by the theorem of Section 23, we get that \( f \) is differentiable at 0 and
\[
f'(0) = u_x(0, 0) + iv_x(0, 0) = 0.
\]

Approach 2 (using only results of past week): We use the definition of differentiability. For \( h \in \mathbb{C} \),
\[
\frac{f(0 + h) - f(0)}{h} = \frac{h \text{Im}(h) - 0}{h} = \text{Im}(h).
\]
Hence
\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = 0.
\]
So \( f \) is differentiable at 0 and \( f'(0) = 0 \).

Exercise 4: Let \( f \) be a function differentiable on \( \mathbb{C} \).

1. Prove that if \( \text{Re}(f) \) is constant on \( \mathbb{C} \), then \( f \) is constant on \( \mathbb{C} \).
2. Prove that if \( |f| \) is constant on \( \mathbb{C} \), then \( f \) is constant on \( \mathbb{C} \).

Hint: Use the Cauchy-Riemann equations. You can use the following fact: if a real-valued function on \( \mathbb{R}^2 \) has its both partial derivatives that are zero on \( \mathbb{R}^2 \), then this function is constant on \( \mathbb{R}^2 \). For (b), you can start by squaring the modulus and differentiate either with respect to \( x \) or with respect to \( y \).

Solution For both parts, we write \( f(z) = u(x, y) + iv(x, y) \) for any \( z = x + iy \in \mathbb{C} \). Since \( f \) is differentiable on \( \mathbb{C} \), \( u \) and \( v \) have partial derivatives on \( \mathbb{R}^2 \) and the Cauchy-Riemann equations are true: \( u_x = v_y \) and \( u_y = -v_x \).

1. We assume \( \text{Re}(f) \) is constant on \( \mathbb{C} \), that is \( u \) is constant on \( \mathbb{R}^2 \). Since \( u \) is constant, \( u_x = 0 \) and \( u_y = 0 \). It follows from the Cauchy-Riemann equations that \( v_x = 0 \) and \( v_y = 0 \). By the fact in the hint, we deduce that \( v \) is constant on \( \mathbb{R}^2 \). Therefore \( f \) is constant on \( \mathbb{C} \).
2. We assume \( |f| \) is constant on \( \mathbb{C} \). Hence \( |f|^2 = u^2 + v^2 \) is also constant on \( \mathbb{C} \). So differentiating with respect to \( x \) and with respect to \( y \), we get
\[
2u_x u + 2v_x v = 0 \quad \text{and} \quad 2u_y u + 2v_y v = 0.
\]
Using the Cauchy-Riemann equations in the second equation, we get
\[
\begin{align*}
    u_x u + v_x v &= 0 \\
    -v_x u + u_x v &= 0.
\end{align*}
\]
Multiplying the first equation by \( u \) and the second one by \( v \) and then summing them, we get
\[
(u^2 + v^2)u_x = 0.
\]
If for some \((x, y) \in \mathbb{R}^2\), \( u^2(x, y) + v^2(x, y) = 0 \), then this means that \(|f(x + iy)| = 0\), but \(|f| \) is constant on \( \mathbb{C} \) so for any \( z' \in \mathbb{C} \), we have \(|f(z')| = 0 \) and therefore \( f(z') = 0 \). In particular \( f \) is constant on \( \mathbb{C} \).
Otherwise \( u^2 + v^2 \) does not vanish on \( \mathbb{R}^2 \) so we can deduce from (0.2) that \( u_x = 0 \) everywhere. But, back to (0.1), multiplying the first equation by \( v \) and the second one by \( u \) and then subtracting them, we get \( (u^2 + v^2)v_x = 0 \) which implies that \( v_x = 0 \) everywhere. And using the Cauchy-Riemann equations, we also have \( u_y = 0 \) and \( v_y = 0 \). By the fact on the hint, we deduce that \( u \) and \( v \) are constant on \( \mathbb{R}^2 \). Therefore \( f \) is constant on \( \mathbb{C} \).