Complex analysis, homework 5 due February 23rd.

Exercise 1.[5 points] Prove the function defined by $f(z) = x^2 - y^2 + y + 2 + ix(2y-1)$ for $z = x + iy$ is entire and find $f'(z)$.

Exercise 2.[5 points] Compute the following quantities (that is express them in $x + iy$ form):

(1) $\exp(2 + i\frac{5\pi}{6})$;
(2) $\log((-e + ei)/\sqrt{2})$ and $\operatorname{Log}((-e + ei)/\sqrt{2})$.

Exercise 3.[3 points] Let $z \in \mathbb{C}$. Prove that $\overline{\exp(z)} = \exp(\overline{z})$.

Exercise 4.[4 points] Solve the equation $e^{2z} + 1 = i$.

Exercise 5.[6 points] Prove that

(1) $\operatorname{Log}((1 - i)^2) = 2 \operatorname{Log}(1 - i)$;
(2) $\operatorname{Log}((1 + i\sqrt{3})^4) \neq 4 \operatorname{Log}(1 + i\sqrt{3})$;

Exercise 6.[7 points] Recall that for any $z \neq 0$, we define $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$. Let $D = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

(1) Using a geometric argument, express $\operatorname{Arg}(z)$ for $z = x + iy \in D$ in terms of $\cos^{-1}$, x and y. Explain why this formula does not work for all $z \neq 0$.
(2) Using the theorem of Section 23, prove that Log is analytic on D and that $\operatorname{Log}'(z) = 1/z$ for any $z \in D$.

Reminder: $\dfrac{\mathrm{d}}{\mathrm{d}t} \cos^{-1}(t) = -\dfrac{1}{\sqrt{1 - t^2}}$.