**Exercise 1** [5 points] Prove the function defined by \( f(z) = x^2 - y^2 + y + 2 + ix(2y - 1) \) for \( z = x + iy \) is entire and find \( f'(z) \).

**Solution.** We write \( f(z) = u(x, y) + iv(x, y) \) with \( u(x, y) = x^2 - y^2 + y + 2 \) and \( v(x, y) = 2xy - x \). Note that \( u \) and \( v \) are two-variable polynomials so they have partial derivatives everywhere and these partial derivatives are continuous everywhere. Moreover,

\[ u_x(x, y) = 2x = v_y(x, y) \]

\[ u_y(x, y) = -2y + 1 = -(2y - 1) = -v_x(x, y) \]

Therefore, the Cauchy-Riemann Equations are satisfied everywhere. We can apply the theorem in Section 23 to conclude that \( f \) is differentiable on \( \mathbb{C} \) and therefore entire. Moreover,

\[ f'(z) = u_x(x, y) + iv_x(x, y) = 2x + i(2y - 1) = 2z - i. \]

**Exercise 2.** [5 points] Compute the following quantities (that is express them in \( x + iy \) form):

1. \( \exp(2 + i\frac{5\pi}{6}) \);
2. \( \log((-e + ei)/\sqrt{2}) \) and \( \text{Log}((-e + ei)/\sqrt{2}) \).

**Solution.**

1. \( \exp(2 + i\frac{5\pi}{6}) = e^2 \cdot e^{i\frac{5\pi}{6}} = e^2 \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) = e^2 \left( -\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = -e^{\frac{\sqrt{3}}{2}}e^2 + i\frac{1}{2}e^2. \)

2. Let \( z = (-e + ei)/\sqrt{2} \). Then

\[ |z| = \sqrt{\left(-\frac{e}{\sqrt{2}}\right)^2 + \left(\frac{e}{\sqrt{2}}\right)^2} = \sqrt{\frac{e^2}{2} + \frac{e^2}{2}} = e. \]

Therefore, \( z = |z|(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = |z|(\cos(\frac{3\pi}{4}) + i \sin(\frac{3\pi}{4})) = e \cdot e^{i\frac{3\pi}{4}} \). We get

\[ \log(z) = \ln|z| + i \arg(z) = 1 + i \left(\frac{3\pi}{4} + 2k\pi\right), \quad k \in \mathbb{Z}. \]

Since \( \frac{3\pi}{4} \in (-\pi, \pi] \), we have \( \text{Arg}(z) = \frac{3\pi}{4} \) and therefore

\[ \text{Log}(z) = 1 + i\frac{3\pi}{4}. \]

**Exercise 3.** [3 points] Let \( z \in \mathbb{C} \). Prove that \( \overline{\text{exp}(z)} = \text{exp}(\overline{z}) \).

**Solution.** Write \( z = x + iy \), with \( x, y \in \mathbb{R} \). Then

\[ \overline{\text{exp}(z)} = e^{\overline{x}} \cdot e^{\overline{y}} = e^x \cdot e^{iy}. \]
Since \( e^x \) is real, we have \( \overline{e^x} = e^x \). On the other hand, since \( e^{iy} = \cos y + i \sin y \), we have \( \overline{e^{iy}} = \cos(-y) + i \sin(-y) = e^{-iy} \) (This is a useful formula!). So we get
\[
\overline{\exp(z)} = e^x \cdot e^{-iy} = e^{x-iy} = \exp(\overline{z}).
\]

**Exercise 4.** [4 points] Solve the equation \( e^{2z} + 1 = i \).

**Solution.** Let \( z = x + iy \in \mathbb{C} \). We have
\[
e^{2z} + 1 = i \iff e^{2z} = -1 + i
\]
\[
\iff e^{2x} \cdot e^{-2yi} = -1 + i
\]
\[
\iff 2x = \ln \sqrt{2} \iff 2y = \frac{3\pi}{4} + 2k\pi, \text{ for some } k \in \mathbb{Z}
\]
\[
\iff \begin{cases} x = \frac{1}{2} \ln 2 \\ y = \frac{3\pi}{8} + k\pi, \text{ for some } k \in \mathbb{Z} \end{cases}
\]
\[
\iff z = \frac{1}{4} \ln 2 + i \left( \frac{3\pi}{8} + k\pi \right), \text{ for some } k \in \mathbb{Z}.
\]

So the set of solutions to the equation is \( \{ \frac{1}{4} \ln 2 + i \left( \frac{3\pi}{8} + k\pi \right) : k \in \mathbb{Z} \} \).

**Exercise 5.** [6 points] Prove that

1. \( \log((1 - i)^2) = 2 \log(1 - i) \);
2. \( \log((1 + i\sqrt{3})^4) \neq 4 \log(1 + i\sqrt{3}) \).

**Solution.**

1. First note that \( 1 - i = \sqrt{2} e^{-i\frac{\pi}{4}} \), where \(-\frac{\pi}{4}\) is its principal argument. So we have
\[
\log(1 - i) = \ln \sqrt{2} - i\frac{\pi}{4} = \frac{1}{2} \left( \ln 2 - i\frac{\pi}{2} \right).
\]
Moreover, \((1 - i)^2 = (\sqrt{2} e^{-i\frac{\pi}{4}})^2 = 2e^{-i\frac{\pi}{2}}\), where \(-\frac{\pi}{2}\) is its principal argument. So we have
\[
\log((1 - i)^2) = \ln 2 - i\frac{\pi}{2}.
\]
This proves that \( \log((1 - i)^2) = 2 \log(1 - i) \).

2. First note that \( 1 + i\sqrt{3} = 2\left( \frac{1}{2} + i\frac{\sqrt{3}}{2} \right) = 2e^{i\frac{\pi}{3}} \), where \( \frac{\pi}{3} \) is its principal argument. Therefore,
\[
\log(1 + i\sqrt{3}) = \ln 2 + i\frac{\pi}{3}
\]
Moreover, \((1 + i\sqrt{3})^4 = 2^4 e^{i\frac{4\pi}{3}} = 2^4 e^{-i\frac{2\pi}{3}}\), where \(-\frac{2\pi}{3}\) is its principal argument. So we get
\[
\log((1 + i\sqrt{3})^4) = \ln(2^4) - i\frac{2\pi}{3} = 4 \ln 2 - i\frac{2\pi}{3} \neq 4 \ln 2 + i\frac{4\pi}{3} = 4 \log(1 + i\sqrt{3}).
\]

**Exercise 6.** [7 points] Recall that for any \( z \neq 0 \), we define \( \log(z) = \ln |z| + i \arg(z) \). Let \( D = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \).
(1) Using a geometric argument, express Arg(z) for \( z = x + iy \in D \) in terms of \( \cos^{-1}, x \) and \( y \). Explain why this formula does not work for all \( z \neq 0 \).

(2) Using the theorem of Section 23, prove that Log is analytic on \( D \) and that \( \text{Log}'(z) = 1/z \) for any \( z \in D \).

Reminder: \( \frac{d}{dt} \cos^{-1}(t) = -\frac{1}{\sqrt{1 - t^2}} \).

Solution.

(1) Let \( z \neq 0 \). Write \( z = x + iy = re^{i\theta} \), with \( r > 0 \) and \( \theta \in (-\pi, \pi] \) (so that \( \theta = \text{Arg}(z) \)). Then \( x = r \cos \theta \) and therefore \( \cos \theta = \frac{x}{r} \) (this is always true).

Now assume that \( z \in D \). In that case \( \theta \in (0, \pi) \), so we have (because \( \cos: (0, \pi) \to (-1, 1) \) is bijective with inverse function \( \text{arccos} \))

\[
\cos \theta = \frac{x}{r} \iff \theta = \text{arccos} \frac{x}{r} \iff \text{Arg}(z) = \text{arccos} \left( \frac{x}{\sqrt{x^2 + y^2}} \right).
\]

This formula is not true if \( z \) is in the lower half plane, because then \( \theta \in (-\pi, 0) \), but the function \( \text{arccos} \) only takes values in \([0, \pi]\).

\[
\begin{align*}
\text{Log}(z) &= u(x, y) + iv(x, y) \\
u(x, y) &= \ln(\sqrt{x^2 + y^2}) = \frac{1}{2} \ln(x^2 + y^2) \\
v(x, y) &= \text{arccos} \left( \frac{x}{\sqrt{x^2 + y^2}} \right)
\end{align*}
\]

Note that \( u \) and \( v \) have partial derivatives everywhere in \( D \): for \( u \), note that \( x^2 + y^2 \) is always positive and \( \ln \) is differentiable on \((0, \infty)\), and for \( v \) note that, since \( x^2 + y^2 \) is always positive, \( x/\sqrt{x^2 + y^2} \) has partial derivatives and \( x/\sqrt{x^2 + y^2} \) takes values only in \((0, 1)\), where \( \text{arccos} \) is differentiable. Since \( D \) is open, for any point in \( D \), the partial derivatives exist in a neighborhood of this point (because there is a neighborhood of this point included in \( D \)).

The partial derivatives of \( u \) are

\[
\begin{align*}
u_x(x, y) &= \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} \\
u_y(x, y) &= \frac{y}{x^2 + y^2}.
\end{align*}
\]
The partial derivatives of $v$ are

\begin{align*}
v_x(x, y) &= \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \cdot \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}}\right) = -\frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \cdot \frac{\sqrt{x^2 + y^2} - x \frac{2x}{2\sqrt{x^2 + y^2}}}{x^2 + y^2} \\
&= -\frac{1}{\sqrt{\frac{y^2}{x^2 + y^2}}} \cdot \frac{(x^2 + y^2) - x^2}{(x^2 + y^2)^{3/2}} = -\frac{\sqrt{x^2 + y^2} - y^2}{y (x^2 + y^2)^{3/2}} = -\frac{y}{x^2 + y^2}
\end{align*}

\begin{align*}
v_y(x, y) &= \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \cdot \frac{\partial}{\partial y} \left(\frac{x}{\sqrt{x^2 + y^2}}\right) = -\frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \cdot \frac{-x \frac{2y}{2\sqrt{x^2 + y^2}}}{x^2 + y^2} \\
&= \frac{1}{\sqrt{\frac{y^2}{x^2 + y^2}}} \cdot \frac{x y}{(x^2 + y^2)^{3/2}} = \frac{x}{x^2 + y^2}.
\end{align*}

Therefore, note that the Cauchy-Riemann Equations are satisfied at any point in $D$. Finally note that these partial derivatives are continuous on $D$, because $x^2 + y^2$ is never 0. So, we can apply the theorem in Section 23 to conclude that Log is differentiable on $D$. Moreover,

\begin{align*}
\text{Log}'(z) &= u_x(x, y) + iv_x(x, y) = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = \frac{x - iy}{|z|^2} = \frac{\pi}{z} = \frac{1}{z}.
\end{align*}